Appendix to
“A Summation Formula for Divisor Functions Associated to Lattices”

Jennifer Beineke and Daniel Bump

July 26, 2004

Two results in the paper [1] were quoted from work of F. Sato [4]. Since his paper is not yet in print, we give here *ad hoc* proofs for the special case $n=2$. (The case $n=1$ is also easily checked.)

The two results are Theorems 2 and 3 of [1]. We refer to the paper for notation.

Theorem 1 *Let*

$$h(g) = |\det(g)|^{-n/2} H \begin{pmatrix} g & \\ & I \end{pmatrix}, \qquad g \in \mathrm{GL}_n(F). \tag{1}$$

Then

$$h(g) = \begin{cases} \tau_{n(s-1/2)}(g) & \text{if } g \in \mathrm{Mat}_n(\mathfrak{o}), \\ 0 & \text{otherwise}. \end{cases}$$

This theorem was proved (from a classical viewpoint) by F. Sato [4]. Since his results are not yet in print, we will give an *ad hoc* proof when $n \leqslant 2$.

Proof (Assuming $n \leqslant 2$.) In the case $n=1$ the function H is a GL_2 Whittaker function and the result is well-known. We assume $n=2$. Defining $h(g)$ by (1), it follows from

$$H\left(\begin{pmatrix} I & X \\ & I \end{pmatrix}\begin{pmatrix} g & \\ & g \end{pmatrix} x\right) = \psi(\mathrm{tr}\, X) H(x)$$

and the right invariance of H by $\mathrm{GL}_{2n}(\mathfrak{o})$ that $h : \mathrm{GL}_n(F) \longrightarrow \mathbb{C}$ is $\mathrm{GL}_n(\mathfrak{o})$ bi-invariant and supported in $\mathrm{Mat}_n(\mathfrak{o})$. Moreover H is determined by the function h. All we have to check is that if we define

$$H\left(\begin{pmatrix} I & X \\ & I \end{pmatrix}\begin{pmatrix} g_1 & * \\ & g_2 \end{pmatrix} k\right) = \psi(\mathrm{tr}\, X) |\det(g_1)/\det(g_2)|^{n/2} \tau_{n(s-1/2)}(g_2^{-1} g_1)$$

then this function H is an eigenfunction of the Hecke algebra with the same eigenvalues as the spherical vector in π_s, since this requirement imposes recursion relations on h that determine its values. We will check this when $n=2$.

Actually we do not need the entire Hecke algebra since a subalgebra will suffice to determine the values of h.

Let ϖ be a prime element of $\mathfrak{o}$. By counting lattices we get the following explicit formula, with $q = |\mathfrak{o}/(\varpi)|$:

$$\tau_{2s-1}\begin{pmatrix} \varpi^k & \\ & \varpi^l \end{pmatrix} = \sum_{u=0}^{l}\left[\sum_{j=1}^{k+l+1-2u} q^{(2s-1)(k+l+2-2u-2j)}\right] q^u =$$
$$\sum_{u=0}^{l}\left(\frac{q^{(2s-1)(k+l+1-2u)} - q^{-(2s-1)(k+l+1-2u)}}{q^{2s-1} - q^{-(2s-1)}}\right) q^u. \tag{2}$$

If k, l are given let $\Phi_{k,l}$ be the characteristic function of the set of $g \in \mathrm{Mat}_4(\mathfrak{o})$ with elementary divisors $\varpi^k, \varpi^l, 1, 1$. Let $T_{k,l}f(g) = \int_{\Phi_{k,l}} f(gk)dk$. The eigenvalues of $T_{1,0}$ and $T_{1,1}$ on the spherical vector in π_s are

$$\lambda_1 = q^{3-2s} + q^{2-2s} + q^{1+2s} + q^{2s}$$

and

$$\lambda_2 = q^{4-4s} + q^3 + 2q^2 + q + q^{4s}$$

We must have $\lambda_1 H(g)$ equal to

$$\sum_{\alpha,\beta,\gamma} H\left(g\begin{pmatrix} \varpi & \alpha & \beta & \gamma \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix}\right) + \sum_{\alpha,\beta} H\left(g\begin{pmatrix} 1 & & & \\ & \varpi & \alpha & \beta \\ & & 1 & \\ & & & 1 \end{pmatrix}\right)$$
$$+ \sum_{\alpha} H\left(g\begin{pmatrix} 1 & & & \\ & 1 & & \\ & & \varpi & \alpha \\ & & & 1 \end{pmatrix}\right) + H\left(g\begin{pmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & \varpi \end{pmatrix}\right).$$

Denoting

$$h(k_1, k_2) = q^{k_1+k_2} H\begin{pmatrix} \varpi^{k_1} & & & \\ & \varpi^{k_2} & & \\ & & 1 & \\ & & & 1 \end{pmatrix}$$

this gives a recursion (under the assumption that $k_1 \geqslant k_2 \geqslant 0$):

$$\lambda_1 h(k_1, k_2) = A + B + C + D$$

where

$$A = q^2 h(k_1 + 1, k_2),$$

$$B = \begin{cases} qh(k_1 + 1, k_2) & \text{if } k_1 = k_2, \\ qh(k_1, k_2 + 1) & \text{if } k_1 > k_2, \end{cases}$$

$$C = \begin{cases} 0 & \text{if } k_1 = 0, \\ qh(k_1 - 1, k_2) & \text{if } k_1 > 0, k_2 = 0, \\ q^2 h(k_1, k_2 - 1) & \text{if } k_1 = k_2 > 0, \\ qh(k_1 - 1, k_2) + q(q-1)h(k_1, k_2 - 1) & \text{if } k_1 > k_2 > 0. \end{cases}$$

$$D = \begin{cases} 0 & \text{if } k_2 = 0, \\ qh(k_1, k_2 - 1) & \text{if } k_2 > 0. \end{cases}$$

Similarly

$$\lambda_2 h(k_1, k_2) = E + F + G + H + I + J$$

where

$$E = q^2 h(k_1 + 1, k_2 + 1),$$

$$F = \begin{cases} q^2 & \text{if } k_1 = k_2 = 0, \\ q^2 h(k_1, k_2) & \text{if } k_1 > k_2 = 0, \\ q^2 h(k_1, k_2) + q^2(q-1)h(k_1 + 1, k_2 - 1) & \text{if } k_1 \geqslant k_2 > 0, \end{cases}$$

$$G = \begin{cases} q^2 h(k_1 + 1, k_2 - 1) & \text{if } k_2 > 0, \\ 0 & \text{if } k_2 = 0, \end{cases}$$

$$H = \begin{cases} 0 & \text{if } k_1 = 0, \\ qh(k_1 - 1, k_2 + 1) + q(q-1)h(k_1, k_2) & \text{if } k_1 > k_2 + 1, \\ q^2 h(k_1, k_2) & \text{if } k_1 = k_2 + 1, \\ q^2 h(k_1 + 1, k_2 - 1) & \text{if } k_1 = k_2 > 0, \end{cases}$$

$$I = qh(k_1, k_2)$$

and

$$J = \begin{cases} q^2 h(k_1 - 1, k_2 - 1) & \text{if } k_1 \geqslant k_2 > 0, \\ 0 & \text{if } k_2 = 0. \end{cases}$$

It may be checked that (2) satisfies these recursion relations, and that these determine the values of $h(k_1, k_2)$ so (1) is true when $n = 2$. $\square$

Theorem 2 *We have*

$$\int_{\mathrm{Mat}_n(F)} f^\circ \left(\begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix} \begin{pmatrix} I & X \\ & I \end{pmatrix} \right) \psi(\mathrm{tr}\, X) dX = \prod_{1 \leqslant j \leqslant n} \frac{1}{\zeta_F(2ns + 1 - j)} \tag{3}$$

The integral is absolutely convergent if $\mathrm{re}(s) > 1 - \frac{1}{2n}$.

As with Theorem 1, Theorem 2 follows from results of Sato [4]. Since his results are not yet in print, we will give an *ad hoc* proof when $n = 2$.

Proof (Assuming $n=2$.) We note the matrix identity

$$\begin{pmatrix} & & -1 & \\ & & & -1 \\ 1 & & & \\ & 1 & & \end{pmatrix}\begin{pmatrix} 1 & & x & y \\ & 1 & z & w \\ & & 1 & \\ & & & 1 \end{pmatrix} =$$

$$\begin{pmatrix} 1 & & & \\ & & -1 & \\ & 1 & & \\ & & & 1 \end{pmatrix}\begin{pmatrix} 1 & & & \\ & 1 & y & \\ & & 1 & \\ & & & 1 \end{pmatrix}\begin{pmatrix} 0 & -1 & & \\ 1 & 0 & & \\ & & 0 & 1 \\ & & -1 & 0 \end{pmatrix}\times$$

$$\begin{pmatrix} 1 & x & & \\ & 1 & & \\ & & 1 & -w \\ & & & 1 \end{pmatrix}\begin{pmatrix} 1 & & & \\ & & 1 & \\ & -1 & & \\ & & & 1 \end{pmatrix}\begin{pmatrix} 1 & & & \\ & 1 & z & \\ & & 1 & \\ & & & 1 \end{pmatrix}.$$

By the formula of Gindikin and Karpelevich (Langlands [3] or Casselman [2], Theorem 3.1 and Remark 3.3),

$$\int_F f^\circ\left(\begin{pmatrix} 1 & & & \\ & & -1 & \\ & 1 & & \\ & & & 1 \end{pmatrix}\begin{pmatrix} 1 & & & \\ & 1 & y & \\ & & 1 & \\ & & & 1 \end{pmatrix} g\right) dy = \frac{\zeta_F(4s-1)}{\zeta_F(4s)} f_1^\circ(g)$$

where f_1° is the spherical vector in the induced representation satisfying

$$f_1^\circ\left(\begin{pmatrix} y_1 & * & * & * \\ & y_2 & * & * \\ & & y_3 & * \\ & & & y_4 \end{pmatrix} k\right) = |y_1|^{2s}|y_2|^{1-2s}|y_3|^{2s-1}|y_4|^{-2s},$$

normalized so $f_1^\circ(I)=1$.

The integral over z splits into two cases. If $|z|\leqslant 1$ then we may drop the last two matrices and the contribution is

$$\frac{\zeta_F(4s-1)}{\zeta_F(4s)}\int_{F^2} f_1^\circ\left(\begin{pmatrix} 0 & -1 & & \\ 1 & 0 & & \\ & & 0 & 1 \\ & & -1 & 0 \end{pmatrix}\begin{pmatrix} 1 & x & & \\ & 1 & & \\ & & 1 & -w \\ & & & 1 \end{pmatrix}\right)\psi(x+w)dxdw.$$

The integral splits into separate factors for x and w, one of which is

$$1+\int_{|x|>1} f_1^\circ\begin{pmatrix} x^{-1} & & & \\ & x & & \\ & & 1 & \\ & & & 1 \end{pmatrix}\psi(x)dx = 1-q^{1-4s} = \zeta_F(4s-1)^{-1}.$$

The w factor is also $\zeta_F(4s-1)^{-1}$. Multiplying we get a contribution of $\zeta_F(4s)^{-1}\zeta_F(4s-1)^{-1}$.

If $|z| > 1$ the contribution is $\zeta_F(4s-1)/\zeta_F(4s)$ times

$$\int_{F^2} f_1^\circ\left(\begin{pmatrix} 0 & -1 & & \\ 1 & 0 & & \\ & & 0 & 1 \\ & & -1 & 0 \end{pmatrix}\begin{pmatrix} 1 & xz^{-1} & & \\ & z^{-1} & * & \\ & & z & -w \\ & & & 1 \end{pmatrix}\right)\psi(x+w)\,dx\,dw.$$

Pulling the z out left and changing $x \longrightarrow zx$, $w \longrightarrow zw$ gives

$$\frac{\zeta_F(4s-1)}{\zeta_F(4s)}|z|^{4s}\times$$

$$\int_{F^2} f_1^\circ\left(\begin{pmatrix} 0 & -1 & & \\ 1 & 0 & & \\ & & 0 & 1 \\ & & -1 & 0 \end{pmatrix}\begin{pmatrix} 1 & x & & \\ & 1 & & \\ & & 1 & -w \\ & & & 1 \end{pmatrix}\right)\psi(z(x+w))\,dx\,dw.$$

The value of f_1° is constant if x is restricted to a single coset of $\mathfrak{o}$, and the integral over such a coset of $\psi(zx)$ vanishes. So this contribution is zero. $\square$

References

[1] J. Beineke and D. Bump A Summation Formula for Divisor Functions Associated to Lattices *To appear*, 2004.

[2] W. Casselman. The unramified principal series of $\mathfrak{p}$-adic groups. I. The spherical function. *Compositio Math.*, 40(3):387–406, 1980.

[3] R. Langlands. *Euler Products.* Yale University Press, New Haven, Conn., 1971. A James K. Whittemore Lecture in Mathematics given at Yale University, 1967, Yale Mathematical Monographs, 1.

[4] F. Sato. Enumeration of subgroups of finite abelian p-groups, local densities of square matrices and Fourier coefficients of Eisenstein series of GL_n. *To appear*, 2004.