Hecke L-functions and Artin L-functions

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This is an expository piece giving an overview of material that we will develop more fully later in the course. In a nutshell, we will take as our motivating question the problem of unifying two theories of L-functions.

1 The Problem

By an *L-function* we mean a Dirichlet series with an *Euler product* expressing the Dirichlet series as a product over primes, and a *functional equation*. We will start by defining two overlapping classes of L-functions, then showing that although they overlap, neither is contained in the other. Thus one seeks to find a larger class of L-functions which contains both, unifying the two theories. This was accomplished by Weil [8] who introduced a new group, now known as the *Weil group*, which can be regarded as a modification of the Galois group.

Example 1 (Dirichlet L-functions) Let χ be a primitive Dirichlet character modulo N. Let

$$L(s,\chi) = \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s} = \prod_p \left(1 - \frac{1}{p^s}\right)^{-1} \tag{1}$$

be its L-function. Then $L(s,\chi)$ has both an Euler product and a functional equation. To describe the functional equation let

$$\epsilon = \begin{cases} 0 & \text{if } \chi(-1) = 1; \\ 1 & \text{if } \chi(-1) = -1. \end{cases}$$

It is known that $L(s,\chi)$ has meromorphic continuation to all s. If $\chi = 1$, so $L(s,\chi) = \zeta(s)$ is the Riemann zeta function, it has a simple pole at $s = 1$; otherwise, it is entire. If

$$\Lambda(s,\chi) = |N|^{s/2} \pi^{-(s+\epsilon)/2} \, \Gamma\left(\frac{s+\epsilon}{2}\right) \, L(s,\chi),$$

then the functional equation states that

$$\Lambda(s,\chi) = W_\chi \, \Lambda(1-s,\bar{\chi}). \tag{2}$$

where the constant

$$W_\chi = \frac{(-i)^\varepsilon}{\sqrt{|N|}} \sum_{a \bmod N} \chi(a) \, e^{2\pi i a/N}$$

has absolute value 1. (See Bump [2], Theorem 1.1.1.)

In this example, (1) is the Euler product, and (2) is the functional equation.

Example 2 (Hecke L-functions) Let F be a global field, and let $\mathbb{A}$ be its adele ring. Let $\chi : \mathbb{A}^\times / F^\times \longrightarrow \mathbb{C}^\times$ be a continuous unitary character. Since $\mathbb{A}^\times / F^\times$ is called the *idele class group*, such a χ is called an idele class character. Let v be a nonarchimedean place of F, and let $\mathfrak{p} = \mathfrak{p}_v$ be the corresponding prime of the ring $\mathfrak{o}_v$ of integers in F_v. Let ϖ_v be the image in $\mathbb{A}^\times$ of a generator of the maximal ideal of $\mathfrak{o}_v$ under the inclusion

$$F_v^\times \longrightarrow \mathbb{A}^\times.$$

We say that χ is *unramified* at v if $\mathfrak{o}_v^\times$ is contained in the kernel of the restriction of χ to $F_v^\times \subset \mathbb{A}^\times$. In this case we define $\chi(\mathfrak{p}) = \chi(\varpi_v)$. The *Hecke L-function* is

$$L(s,\chi) = \prod_{\mathfrak{p}} \left(I - \frac{1}{\mathbb{N}\mathfrak{p}^s}\right)^{-1},$$

where the product is over all unramified primes. The Hecke L-functions have analytic continuation and functional equations.

Example 3 (Artin L-functions) Before we can define Artin L-functions, we review some basic algebraic number theory. For proofs, see any text, for example Lang [5], particularly Section 5 of Chapter I. Let F be a global field, and let K/F be a finite Galois extension. Let v be a place of F which is nonarchimedean and unramified in K. Then there is associated with v a unique conjugacy class of *Frobenius elements* of $\mathrm{Gal}(K/F)$. To define a Frobenius element, let $\mathfrak{p} = \mathfrak{p}_v$ be the prime of the ring $\mathfrak{o}_F$ of integers in F associated with the place v, and let $\mathfrak{P}$ be a prime of $\mathfrak{o}_K$ such that $\mathfrak{P} \cap \mathfrak{o}_F = \mathfrak{p}$. Let w be the place of K associated with $\mathfrak{P}$. Then $\mathfrak{o}_K/\mathfrak{P}$ is a finite extension of the finite field and $\mathfrak{o}_F/\mathfrak{p}$ and we can write $\mathfrak{o}_F/\mathfrak{p} \cong \mathbb{F}_q$ and $\mathfrak{o}_K/\mathfrak{P} \cong \mathbb{F}_{q^f}$ for some $q = q_v$ and f. Now let $H \subseteq \mathrm{Gal}(K/F)$ be the stabilizer of $\mathfrak{P}$ in the Galois action on primes of $\mathfrak{o}_K$, and let E be the fixed field of H. Thus $H = \mathrm{Gal}(K/E)$. An element of $\sigma \in H$ induces an automorphism $\bar{\sigma} \in \mathrm{Gal}(\mathbb{F}_{q^f}/\mathbb{F}_q)$ and the map $\sigma \longmapsto \bar{\sigma}$ is an isomorphism $H \longrightarrow \mathrm{Gal}(\mathbb{F}_{q^f}/\mathbb{F}_q)$. The latter group is cyclic, and generated by the Frobenius automorphism $x \longmapsto x^q$. Let $\phi_{\mathfrak{P}}$ be the corresponding element of $H \subseteq \mathrm{Gal}(K/F)$. Given $\mathfrak{p}$, the element $\phi_{\mathfrak{P}}$ depends on the choice of $\mathfrak{P}$, but only up to conjugacy since $\mathrm{Gal}(K/F)$ acts transitively on the primes above $\mathfrak{p}$.

We may now define the Artin L-functions. Let $\rho : \mathrm{Gal}(K/F) \longrightarrow \mathrm{GL}(n, \mathbb{C})$ be any representation. Since $\rho(\phi_{\mathfrak{P}})$ is determined up to conjugation, the value of

$$\det(I_n - \rho(\phi_{\mathfrak{P}})\mathbb{N}\mathfrak{p}_v^{-s})^{-1} = \prod_{i=1}^{n} (1 - \alpha_i q_v^{-s})^{-1}$$

depends only on $\mathfrak{p}$. Here $\mathbb{N}\mathfrak{p}_v$ is the cardinality of the residue field, previously denoted q_v, and $\alpha_1, \cdots, \alpha_n$ are the eigenvalues of $\rho(\phi_{\mathfrak{P}})$. The *Artin L-function* of π is defined to be

$$L_v(s, \pi) = \prod_{\mathfrak{p}} \det(I_n - \rho(\phi_{\mathfrak{P}})\, \mathbb{N}\mathfrak{p}_v^{-s})^{-1},$$

where the product is over unramified places of F. It is actually desirable to extend this definition to all places, and this can be done. At an archimedean places, $L_v(s, \pi)$ will be a product of Gamma functions.

It is a theorem of Artin and Brauer that $L(s, \pi) = \prod_v L_v(s, \pi)$ has meromorphic continuation to all s. This definition can be extended to all places v, and in particular if v is archimedean, $L_v(s, \pi)$ is a product of Gamma functions, and there is a functional equation. The *Artin conjecture* asserts that $L(s, \pi)$ is entire, except possibly for a pole at $s = 1$ with multiplicity equal to the multiplicity with which π contains the trivial representation. Unfortunately the method of Artin and Brauer does not exclude the possibility of unwanted poles in the critical strip.

Let us show that Examples 2 and 3 both include the class of Dirichlet L-functions, in different ways. Let χ be a primitive Dirichlet character modulo N. Let $F = \mathbb{Q}$, and let $\mathbb{A}$ be its adele ring.

If N is a positive integer, let

$$U(N) = \mathbb{R}_+^\times \prod_{p|N} U_p(N) \prod_{p\nmid N} \mathbb{Z}_p^\times,$$

where $U_p(N) = \{x \in \mathfrak{o}_p^\times \mid x \equiv 1 \bmod N\}$. Also, let

$$V(N) = \mathbb{R}_+^\times \prod_{p|N} U_p(N) \prod_{p\nmid N} \mathbb{Q}_p^\times.$$

Proposition 1 *We have*

$$\mathbb{A}^\times / U(N)\mathbb{Q}^\times \cong (\mathbb{Z}/N\mathbb{Z})^\times.$$

The isomorphism may be chosen so that if χ is a character of $\mathbb{A}^\times/\mathbb{Q}^\times$ which is trivial on $U(N)$, and if χ_0 is the Dirichlet character obtained by composing χ with this isomorphism, then

$$L(s, \chi) = L(s, \chi_0).$$

Proof It follows from Theorem 1 on p. 35 of Lang [5] applied to the finite number of places corresponding to primes dividing N that $\mathbb{A}^\times = \mathbb{Q}^\times V(N)$. Thus

$$\mathbb{A}^\times/\mathbb{Q}^\times = \mathbb{Q}^\times V(N)/\mathbb{Q}^\times \cong V(N)/(\mathbb{Q}^\times \cap V(N)),$$

and

$$\mathbb{A}^\times/U(N)\mathbb{Q}^\times = \mathbb{Q}^\times V(N)/U(N)\mathbb{Q}^\times \cong V(N)/U(N)(\mathbb{Q}^\times \cap V(N)).$$

We observe that

$$V(N)/U(N)(\mathbb{Q}^\times \cap V(N)) \cong I_N/P_N,$$

where I_N is the group of fractional ideals of $\mathbb{Q}$ prime to N, and P_N is the subgroup consisting of fractional ideals (α) with $\alpha \in \mathbb{Q}^\times \cap V(N)$. Indeed, we have a surjective homomorphism $\theta : V(N) \longrightarrow I_N$ in which we associate with an idele $a = (a_v)$ the fractional ideal generated by

$$\theta(a) = \prod_{v \text{ nonarchimedean}} p_v^{\mathrm{ord}_v(a_v)}.$$

Clearly only primes dividing not N will appear in $\theta(a)$. The kernel of the composite

$$V(N) \xrightarrow{\theta} I_N \longrightarrow I_N/P_N$$

is easily seen to be $U(N)(\mathbb{Q}^\times \cap V(N))$, so

$$\mathbb{A}^\times/U(N)\mathbb{Q}^\times \cong I_N/P_N.$$

On the other hand, we claim $I_N/P_N \cong (\mathbb{Z}/N\mathbb{Z})^\times$. Indeed, if $a \in \mathbb{Z}$ is not divisible by N, then the fractional ideal generated by a is in I_N, and its image in I_N/P_N depends only on the class of a modulo N, so we have a homomorphism

$$(\mathbb{Z}/N\mathbb{Z})^\times \longrightarrow I_N/P_N$$

which is easily seen to be an isomorphism. It is that easily seen that if p is a prime not dividing N, then $\chi_0(p) = \chi(\varpi_p)$, where ϖ_p is the image of p in $\mathbb{A}^\times$ under the inclusions

$$\mathbb{Q}^\times \longrightarrow \mathbb{Q}_p^\times \longrightarrow \mathbb{A}^\times,$$

and the equality of L-functions follows. $\square$

Proposition 2 *(i) If $\chi : \mathbb{A}^\times/\mathbb{Q}^\times \longrightarrow \mathbb{C}$ is any unitary character, then χ has finite order if and only if its restriction to $\mathbb{R}_+^\times$ is trivial.*
(ii) there exists a unique $\nu \in \mathbb{R}$ such that $\chi\tau_\nu$ has finite order, where the character $\tau_\nu(x) = |x|^{i\nu}$.

Proof For (i), if χ has finite order, then the image of $\mathbb{R}_+^\times$ is connected subgroup of the finite group $\mathrm{im}(\chi)$, hence trivial. On the other hand, suppose that χ is trivial on $\mathbb{R}_+^\times$. Let Ω be an open neighborhood of the identity in $\mathbb{C}^\times$ which does not contain any nontrivial subgroup of $\mathbb{C}^\times$. The preimage of Ω in $\mathbb{A}^\times$ contains both $\mathbb{R}_+^\times$ and an open neighborhood of the identity, so it contains $U(N)$ for some N. Since $\mathbb{A}^\times/\mathbb{Q}^\times U(N)$ is finite, it follows that χ has finite order.

As for (ii), since the restriction of χ to $\mathbb{R}_+^\times$ is unitary, there is a unique ν such that $\chi\tau_\nu$ is trivial on $\mathbb{R}_+^\times$, and the statement follows. $\square$

Combining the last two Propositions shows in what sense Dirichlet L-functions are the same as Hecke L-functions when the ground field is $\mathbb{Q}$: every Dirichlet L-function is the same as the L-function of the Hecke L-function of an idele class character of finite order, and every Hecke L-function is a shift (by a purely imaginary ν) of such an L-function.

We claim also that the class of Artin L-functions contains the class of Dirichlet characters. For this, let χ be a primitive Dirichlet character modulo N, and let $\zeta = \zeta_N = e^{2\pi i/N}$. We observe that there is an isomorphism

$$\theta : \mathrm{Gal}(\mathbb{Q}(\zeta)/\mathbb{Q}) \cong (\mathbb{Z}/N\mathbb{Z})^\times$$

in which the residue class $\bar{a} \in (\mathbb{Z}/N\mathbb{Z})^\times$ of $a \in \mathbb{Z}$ relatively prime to N corresponds to the element $\sigma_a \in \mathrm{Gal}(\mathbb{Q}(\zeta)/\mathbb{Q})$, characterized by

$$\sigma_a(\zeta) = \zeta^a.$$

Proposition 3 *The composition*

$$\mathrm{Gal}(\mathbb{Q}(\zeta)/\mathbb{Q}) \cong (\mathbb{Z}/N\mathbb{Z})^\times \xrightarrow{\chi} \mathbb{C}^\times$$

gives a one-dimensional representation of the abelian group $\mathrm{Gal}(\mathbb{Q}(\zeta)/\mathbb{Q})$.

Proof If p is a prime not dividing N, then σ_p is clearly the Frobenius element of $\mathrm{Gal}(\mathbb{Q}(\zeta)/\mathbb{Q})$ corresponding to p, and the statement follows. $\square$

We have now obtained two substantial generalizations of the theory of Dirichlet L-functions: the classes of Hecke and Artin L-functions. Yet neither class is contained in the other.

To give an example of a Hecke L-function which is not an Artin L-function, let $F = \mathbb{Q}(\sqrt{D})/\mathbb{Q}$ be a real quadratic field. For simplicity, let us assume that the class number of F is one. Let ν be a purely imaginary number. If $\alpha \in \mathfrak{o}$, the ring of integers of F, let

$$\chi_0(\alpha) = \left|\frac{\alpha}{\alpha'}\right|^{i\nu},$$

where α' is the Galois conjugate of α. If ν is an integer multiple of $\pi \log|\varepsilon_0|$, where ε_0 is the fundamental unit, then χ_0 is trivial on the units, and hence depends only on the ideal (α). Thus we obtain a character of ideals.

Proposition 4 *In this setting, there exists an idele class character* χ *of* $\mathbb{A}^\times/F^\times$ *such that if* $P \in \mathfrak{o}$ *is a prime element corresponding to the nonarchimedean place* v *of* F*, and if* ϖ_v *is the image of* P *under the canonical embedding*

$$F_v^\times \longrightarrow \mathbb{A}^\times,$$

then $\chi(\varpi_v) = \chi_0(P)$*.*

Proof Let ∞_1 be the real place corresponding to the standard inclusion of F in $\mathbb{R}$, and let ∞_2 be the other real place. Let I be the group of fractional ideals of F. We show that

$$\mathbb{A}^\times = F^\times\, F_\infty^\times V. \tag{3}$$

We have a homomorphism $\theta : \mathbb{A}^\times \longrightarrow I$ which maps an idele $a = (a_v)$ to

$$\prod_{v \text{ nonarchimedean}} \mathfrak{p}_v^{\operatorname{ord}(a_v)},$$

where $\mathfrak{p}_v$ is the ideal of $\mathfrak{o}$ associated with the place v. Since we are assuming that the class number of F is 1, the composition $F^\times \longrightarrow \mathbb{A}^\times \xrightarrow{\theta} I$ is surjective, and its kernel is $F^\times \cap F_\infty^\times V$ where

$$V = \prod_{v \neq \infty_1, \infty_2} \mathfrak{o}_v^\times, \qquad F_\infty^\times = F_{\infty_1}^\times F_{\infty_2}^\times.$$

Now to prove (3), let a be an arbitrary idele. Choose $\alpha \in F^\times$ such that $\sigma(a) = \sigma(\alpha)$, and then note that $a = \alpha\beta$ where $\beta = a\alpha^{-1} \in F^\times \cap F_\infty^\times V$. We see that $a \in F^\times F_\infty^\times V$ and (3) is proved.

Thus

$$\mathbb{A}^\times/F^\times V = F_\infty^\times\, F^\times V/F^\times V \cong F_\infty^\times/(F_\infty^\times \cap F^\times V).$$

Note that $F_\infty^\times \cap F^\times V$ is the image of $\mathfrak{o}^\times$ in $F_\infty^\times$, because if a is an idele which can be written $a = \alpha\xi$ with $\alpha \in F^\times$ and $\xi \in V$, then for all finite places v of F, we have $\alpha_v \xi_v = \alpha_v = 1$, so α_v is a unit in $\mathfrak{o}_v^\times$ for all places v, and therefore $\alpha \in \mathfrak{o}^\times$. Thus the map $\chi_\infty : F_\infty^\times \longrightarrow \mathbb{C}$ defined by

$$(a_{\infty_1}, a_{\infty_2}) \longmapsto \left| \frac{\alpha_{\infty_1}}{\alpha_{\infty_2}} \right|^{i\nu}$$

is trivial on $F_\infty^\times \cap F^\times V$. We define χ to be the character of $\mathbb{A}^\times/F^\times$ which is the composition

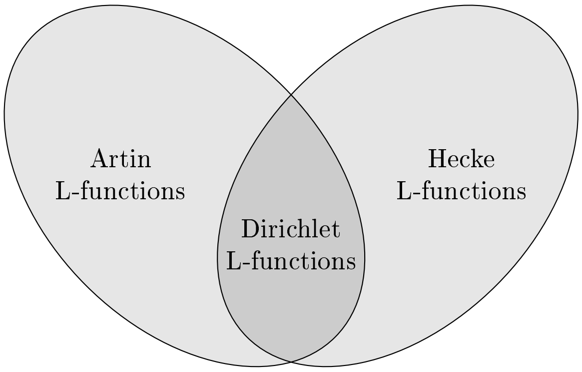
$$\mathbb{A}^\times/F^\times \longrightarrow \mathbb{A}^\times/F^\times V \cong F_\infty^\times/(F_\infty^\times \cap F^\times V) \xrightarrow{\chi_\infty} \mathbb{C}^\times.$$

It is easy to check that the requirements of the Proposition are satisfied. $\square$

Provided $\nu \neq 0$, the idele class character of Proposition 4 is a Hecke character which is not an Artin L-function. (This is easy to see since the character has infinite order.)

On the other hand, examples abound of Artin L-function which is not a Hecke L-function. Let K/F be any nonabelian Galois extension of global fields, and let $\rho : \mathrm{Gal}(K/F) \longrightarrow \mathbb{C}$ be any irreducible representation. Then it may be shown that $L(s, \rho)$ agrees with a Hecke L-function if and only if ρ is one-dimensional. Clearly, this is not always true. *Explaining this point requires class field theory* so we will return to it in a later lecture.

The fundamental problem that was solved by Weil was the unification of the two theories of Hecke and Artin L-functions. We have seen that these two classes overlap, since both contain the example of Dirichlet L-functions as a special case, but neither theory contains the other. This situation may be described by the following "Venn diagram:"



Can these two divergent theories be unified in a larger class?

2 The Solution

Class Field Theory is a fully satisfactory theory of abelian extensions of local and global fields, encompassing the reciprocity laws of number theory.

Gauss first proved the quadratic reciprocity law, leading to further developments of reciprocity laws by Eisenstein, Jacobi and Kummer. The notion of a class field had its origin in work of Kronecker, and the term *class field* was coined by Weber. However it was Hilbert who greatly clarified the notion, beginning with his profound analysis of Kummer's work on the reciprocity law for Kummer fields in his 1897 *Zahlbericht* [4]. The main theorems of Class Field Theory were proved before 1920 by Takagi. Artin supplied a missing insight by elucidating the special role of the Frobenius elements of Galois groups, thereby giving a *canonical* isomorphism between the Galois group of an abelian extension and an ideal class group. Meanwhile in 1932 Hasse, Brauer and Noether determined the structure of division algebras over local and global fields, allowing new foundations for class field theory. Ideles and Adeles were introduced by Chevalley and Weil, allowing a more streamlined treatment of the global theory. Finally Weil introduced the Weil group in 1951, and the role of Galois cohomology was clarified by Hochschild, Artin and Tate. See Lang [5] and Hasse [3] for further history of class field theory.

In class field theory, one obtains an isomorphism of the Galois group $\mathrm{Gal}(K/F)$ of an abelian extension of local or global fields with an object associated just with F. As an example, if K is the maximal unramified extension of F – the so-called *Hilbert class field*, then $\mathrm{Gal}(K/F)$ is isomorphic to the class group of F. In the older language, Galois groups of abelian extensions are related to generalized ideal class groups. In modern language, the ideal class groups are replaced by groups of *idele classes*, that is, quotients of $\mathbb{A}^\times/F^\times$. There is a corresponding local theory, which can be treated in parallel.

Thus to treat the local and global cases together, let F be a local or global field. We associate with F a group C_F as follows.

- If F is a local field then $C_F = F^\times$.
- If F is a global field, then $C_F = \mathbb{A}_F^\times/F^\times$.

If K/F is a finite extension, then there are natural maps:

$$C_F \underset{\text{norm map } N}{\overset{\text{inclusion}}{\rightleftarrows}} C_K$$

The norm map in particular is very important because its role in the reciprocity laws. Specifically, there is a *reciprocity law isomorphism*

$$C_F/\boldsymbol{N}C_K \cong \mathrm{Gal}(K/F)^{\mathrm{ab}}, \tag{4}$$

where if G is any finite group, then $G^{\mathrm{ab}} = G/G'$ is the abelianization, with G' the derived group. Here $\boldsymbol{N} : C_K \longrightarrow C_F$ is the *norm map*, an analog for idele class groups of the familiar norm map $\boldsymbol{N} : K^\times \longrightarrow F^\times$ of field theory.

Understood correctly, this *Artin reciprocity law* embodied in the isomorphism (4) encompasses all the earlier reciprocity laws of Gauss, Jacobi, Eisenstein, Kummer and Hilbert. The group C_F/NC_K is a generalization of the ideal class group.

As a first example, let $F = \mathbb{Q}$ and $K = \mathbb{Q}(\zeta)$, where $\zeta = e^{2\pi i/N}$. Combining Propositions 1 and 3 which underlie the fact that Dirichlet L-functions can be realized as Hecke and Artin L-functions, we have isomorphisms

$$\mathbb{A}^\times/U(N)\mathbb{Q}^\times \cong (\mathbb{Z}/N\mathbb{Z})^\times \cong \mathrm{Gal}(\mathbb{Q}(\zeta)/\mathbb{Q}).$$

Since $\mathbb{A}^\times/U(N)\mathbb{Q}^\times$ is a quotient of $\mathbb{A}^\times/\mathbb{Q}^\times = C_F$, and the resulting homomorphism

$$C_F \longrightarrow \mathrm{Gal}(K/F)$$

is the same as (4).

Class Field Theory contains other important results in addition to the reciprocity law isomorphism. For example, the classification of division algebras over a local or global field by Albert, Hasse, Brauer and Noether belongs to the Class Field Theory.

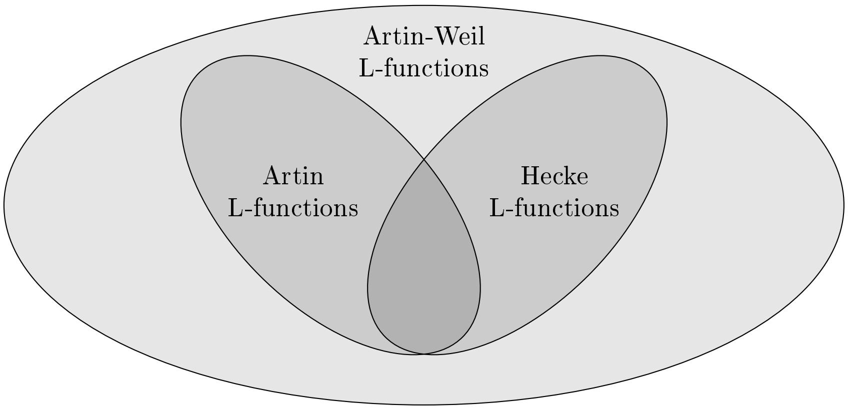
The *Weil group* $W_{K/F}$ is defined when K/F is a finite extension of local or global fields. It fits into an exact sequence

$$1 \longrightarrow C_K \longrightarrow W_{K/F} \longrightarrow \mathrm{Gal}(K/F) \longrightarrow 1$$

The Weil group is thus a modification of the Galois group. An axiomatization of the Weil group may be found in Tate [7]. After showing (based on Artin and Tate [1]) that Weil groups as defined in [7] exist and are unique, Tate observes that the existence and uniqueness of Weil groups involves and is indeed equivalent to the totality of the main theorems in Class Field Theory. He writes:

> Thus, implicit in the existence of Weil groups is all the cohomology of class field theory.

Weil showed how to associate an L-function with any representation of the global Weil group. For example, given a Galois representation $\rho : \mathrm{Gal}(K/F) \longrightarrow \mathrm{GL}(n, \mathbb{C})$, there is a Weil group representation. After the work of Weil, the two distinct generalizations of Hecke and Artin L-functions can again be unified in a single theory:



Although we will stop here, there is a great deal more to the story. The unification achieved by Weil in 1951 is thus a part of a larger unification (still conjectural) proposed by Langlands [6]. Weil groups are therefore essential in understanding the local Langlands correspondence and the global Langlands conjectures.

References

[1] E. Artin and J. Tate. Class field theory. W. A. Benjamin, Inc., New York-Amsterdam, 1968.

[2] Daniel Bump. Automorphic forms and representations, volume 55 of Cambridge Studies in Advanced Mathematics. Cambridge University Press, Cambridge, 1997.

[3] Helmut Hasse. History of class field theory. In Algebraic Number Theory (Proc. Instructional Conf., Brighton, 1965), pages 266–279. Thompson, Washington, D.C., 1967.

[4] David Hilbert. The theory of algebraic number fields. Springer-Verlag, Berlin, 1998. Translation by Iain Adamson of Hilbert's 1897 *Zahlbericht*.

[5] Serge Lang. Algebraic number theory, volume 110 of Graduate Texts in Mathematics. Springer-Verlag, New York, second edition, 1994.

[6] R. P. Langlands. Problems in the theory of automorphic forms. In Lectures in modern analysis and applications, III, pages 18–61. Lecture Notes in Math., Vol. 170. Springer, Berlin, 1970.

[7] J. Tate. Number theoretic background. In Automorphic forms, representations and L-functions (Proc. Sympos. Pure Math., Oregon State Univ., Corvallis, Ore., 1977), Part 2, Proc. Sympos. Pure Math., XXXIII, pages 3–26. Amer. Math. Soc., Providence, R.I., 1979.

[8] André Weil. Sur la thorie du corps de classes. J. Math. Soc. Japan, 3:1–35, 1951.