INTEGRAL MOMENTS AND BOUNDS FOR AUTOMORPHIC L–FUNCTIONS

by

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and

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1918: Hardy–Littlewood

$$\int_0^T \left|\zeta\left(\tfrac{1}{2}+it\right)\right|^2 \, dt \sim T \log T$$

1926: Ingham

$$\int_0^T \left|\zeta\left(\tfrac{1}{2}+it\right)\right|^4 \, dt \sim \frac{1}{2\pi^2} T (\log T)^4$$

1979: Heath-Brown

$$\int_0^T \left|\zeta\left(\tfrac{1}{2}+it\right)\right|^4 \, dt$$

$$= T \cdot P_4(\log T) + \mathcal{O}\left(T^{\frac{7}{8}+\epsilon}\right)$$

Let

$$f(z) = \sum_{n=1}^{\infty} a(n) e^{2\pi i n z}$$

be a cusp form of weight κ for $SL_2(\mathbb{Z})$ with associated L–function $L_f(s)$.

1982: Good

$$\int_0^T \left| L_f\left(\frac{\kappa}{2} + it\right)\right|^2 \, dt$$

$$= \ aT(\log T + b) + \mathcal{O}\left(\left(T \log T\right)^{\frac{2}{3}}\right)$$

for certain constants a, b.

Good's result was extended to fourth moment of the Riemann zeta-function by Motohashi in 1993.

Let k be a number field

$$G = GL_2 \text{ over } k$$

and define the standard subgroups

$$P = \left\{ \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \right\}$$

$$N = \left\{ \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix} \right\}$$

$$M = \left\{ \begin{pmatrix} * & 0 \\ 0 & * \end{pmatrix} \right\}$$

$$Z = \text{center of } G$$

For any place ν of k, let $K_\nu^{\max}$ be the standard maximal compact subgroup; for finite ν, we take

$$K_\nu^{\max} = GL_2(\mathfrak{o}_\nu)$$

at real places

$$K_\nu^{\max} = O(2)$$

at complex places

$$K_\nu^{\max} = U(2)$$

One of our goals is to form a suitable Poincaré series, and then unwind a corresponding global integral to express it as an inverse Mellin transform of an Euler product. The Poincaré series will be of the form

$$\mathrm{P\acute{e}}(g) \;=\; \sum_{\gamma\in M_k\backslash G_k} \varphi(\gamma g) \;\; (g \in G_{\mathbb{A}})$$

for suitable *monomial* vector

$$\varphi = \bigotimes_{\nu} \varphi_\nu$$

For *finite* primes ν, the local component φ_ν is defined as follows. For a character $\chi_{0,\nu}$ of M_ν ($K_\nu = K_\nu^{\max}$), put

$$\varphi_\nu(g) = \begin{cases} \chi_{0,\nu}(m) & g = mk \in M_\nu \cdot K_\nu \\ 0 & g \notin M_\nu \cdot K_\nu \end{cases}$$

For ν *infinite,* we need not entirely specify the local component φ_ν, only requiring the left equivariance

$$\varphi_\nu(mnk) = \chi_{0,\nu}(m) \cdot \varphi_\nu(n)$$

for $n \in N_\nu$ and $m \in M_\nu$. Require also that $\chi_0 = \bigotimes_\nu \chi_{0,\nu}$ be M_k–invariant.

For the sake of clarity, we shall assume that

$$\chi_{0,\nu}(m) = \left|\frac{y_1}{y_2}\right|_\nu^v \quad m = \begin{pmatrix} y_1 & 0 \\ 0 & y_2 \end{pmatrix}, \quad v \in \mathbb{C}$$

Thus, φ has trivial central character. Then φ is left $M_{\mathbb{A}}$–equivariant by the character χ_0. Also, for ν infinite, our assumptions imply that the local component

$$x \to \varphi_\nu \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}$$

is a function of $|x|$ only.

Let f_1 and f_2 be cuspforms on $G_{\mathbb{A}}$.

The integral under consideration is

$$I(\chi_0) = \int_{Z_{\mathbb{A}} G_k \backslash G_{\mathbb{A}}} \text{Pé}(g)\, f_1(g)\, \bar{f}_2(g)\, dg$$

We assume *compatibility* among central characters so that the integrand is genuinely well-defined on $Z_{\mathbb{A}} \backslash G_{\mathbb{A}}$.

The integral immediately unwinds to

$$\int_{Z_{\mathbb{A}} M_k \backslash G_{\mathbb{A}}} \varphi(g)\, f_1(g)\, \bar{f}_2(g)\, dg$$

Using the Fourier expansion

$$f_1(g) \;=\; \sum_{\xi \in Z_k \backslash M_k} W_1(\xi\, g)$$

this further unwinds to

$$\int_{Z_{\mathbb{A}}\backslash G_{\mathbb{A}}} \varphi(g)\, W_1(g)\, \bar{f}_2(g)\, dg$$

Let $C = GL_1(\mathbb{A})/GL_1(k)$ be the idele class group, and $\widehat{C}$ its dual. More explicitly, by Fujisaki's Lemma, C is a product of a copy of $\mathbb{R}^+$ and a compact group C_0. By Pontryagin duality, $\widehat{C} \approx \mathbb{R} \times \widehat{C}_0$ with $\widehat{C}_0$ discrete. Also, for any compact open subgroup U_{fin} of the finite-prime part in C_0, the dual of C_0/U_{fin} is finitely generated with rank $[k : \mathbb{Q}] - 1$.

With these notations, the general Mellin transform and inversion are

$$f(x) = \int_{\widehat{C}} \int_C f(y)\chi(y)\,dy\,\chi^{-1}(x)\,d\chi$$

$$= \sum_{\chi' \in \widehat{C}_0} \frac{1}{2\pi i} \int_{\Re(s)=\sigma} \int_C f(y)\chi'(y)|y|^s\,dy$$

$$\cdot\, {\chi'}^{-1}(x)|x|^{-s}\,ds$$

for a suitably normalized Haar measure on C.

For ν *infinite* and $s \in \mathbb{C}$,

$$\begin{aligned}
&\mathcal{K}_\nu(s,\, \chi_{0,\nu},\, \chi_\nu) = \\
&\int_{Z_\nu \backslash M_\nu N_\nu} \int_{Z_\nu \backslash M_\nu} \varphi_\nu(m_\nu n_\nu) W_{1,\nu}(m_\nu n_\nu) \\
&\qquad \cdot \overline{W}_{2,\nu}(m'_\nu n_\nu)\, \chi_\nu(m'_\nu)\, |m'_\nu|_\nu^{s-\frac{1}{2}} \\
&\qquad \cdot \chi_\nu(m_\nu)^{-1}\, |m_\nu|_\nu^{\frac{1}{2}-s}\, dm'_\nu\, dn_\nu\, dm_\nu
\end{aligned}$$

and set

$$\mathcal{K}_\infty(s,\, \chi_0,\, \chi) \,=\, \prod_{\nu|\infty} \mathcal{K}_\nu(s,\, \chi_{0,\nu},\, \chi_\nu)$$

Here $\chi_0 = \bigotimes_\nu \chi_{0,\nu}$ is the character defining φ, and $\chi = \bigotimes_\nu \chi_\nu \in \widehat{C}_0$.

Theorem. *For φ an admissible monomial vector, for suitable $\sigma > 0$, we have*

$$I(\chi_0) = \sum_{\chi \in \widehat{C}_0} \frac{1}{2\pi i} \int_{\Re(s)=\sigma} L(\chi_0 \cdot \chi^{-1}|\cdot|^{1-s},\, f_1) \cdot L(\chi|\cdot|^s,\, \bar{f}_2)\, \mathcal{K}_\infty(s,\, \chi_0,\, \chi)\, ds$$

Let S be a finite set of places including the archimedean places, all the absolutely ramified primes, and all the finite bad places for f_1 and f_2. Then, in fact, the sum is over a set $\widehat{C}_{0,S}$ of characters unramified outside S with bounded ramification at finite places.

Simplifying assumption: Both f_1 and f_2 have trivial central character.

Proof (sketch): Applying the Mellin transform and inversion to $\bar{f}_2$ via the obvious identification $Z_{\mathbb{A}} M_k \backslash M_{\mathbb{A}} \approx C$, and using the Fourier expansion

$$f_2(g) \;=\; \sum_{\xi \in Z_k \backslash M_k} W_2(\xi\, g)$$

the integral $I(\chi_0)$ is

$$\int_{Z_{\mathbb{A}} \backslash G_{\mathbb{A}}} \varphi(g)\, W_1(g) \cdot$$

$$\cdot \left(\int_{\widehat{C}} \int_{Z_{\mathbb{A}} M_k \backslash M_{\mathbb{A}}} \bar{f}_2(m' g)\, \chi(m')\, dm'\, d\chi \right) dg$$

$$= \int_{\widehat{C}} \left(\int_{Z_{\mathbb{A}} \backslash G_{\mathbb{A}}} \varphi(g)\, W_1(g) \cdot \right.$$

$$\left. \cdot \int_{Z_{\mathbb{A}} \backslash M_{\mathbb{A}}} \overline{W}_2(m'g)\, \chi(m')\, dm'\, dg \right) d\chi$$

We remark that all the characters χ which appear are unram. outside S with bounded ramification.

Assuming at this point that f_1 and f_2 generate irreducible representations locally everywhere, the Whittaker functions W_i

factor over primes

$$W_i(\Pi_\nu g_\nu) = \Pi_\nu W_{i,\nu}(g_\nu)$$

Therefore, the inner integral over $Z_{\mathbb{A}}\backslash G_{\mathbb{A}}$ and $Z_{\mathbb{A}}\backslash M_{\mathbb{A}}$ factors over primes, and

$$I(\chi_0) = \int_{\widehat{C}} \Pi_\nu \Bigg(\int_{Z_\nu \backslash G_\nu} \int_{Z_\nu \backslash M_\nu} \varphi_\nu(g_\nu)$$

$$\cdot W_{1,\nu}(g_\nu)\, \overline{W}_{2,\nu}(m'_\nu g_\nu)\, \chi_\nu(m'_\nu)\, dm'_\nu\, dg_\nu \Bigg) d\chi$$

Assume that $\nu \notin S$. Then, f_1 and f_2 are spherical at ν. It can be shown that the local integral

$$= L_\nu(\chi_{0,\nu}\cdot\chi_\nu^{-1}|\cdot|_\nu^{1/2},\ f_1)\cdot L_\nu(\chi_\nu|\cdot|_\nu^{1/2},\ \bar{f}_2)$$

i.e., the product of local factors of the standard L–functions in the theorem.

Remark. The archimedean local factors are not at all specified. The option to vary the choices is essential.

Theorem. *The Poincaré series Pé(g) has meromorphic continuation in χ_0 and in suitable archimedean data.*

The idea of the proof is that after an obvious Eisenstein series is subtracted, the

Poincaré series is in L^2 and is of sufficiently rapid decay in Siegel sets so that we can write its L^2 spectral decomposition in terms of cuspforms and continuous spectrum.

For instance, let f be a cuspform on $G_{\mathbb{A}}$ which everywhere is a spherical Hecke eigenfunction. The integral

$$\int_{Z_{\mathbb{A}} G_k \backslash G_{\mathbb{A}}} \bar{f}(g)\,\text{Pé}(g)\,dg$$

is an Euler product. At a finite ν, the corresp. local factor is $L_\nu(\chi_{0,\nu}|\cdot|_\nu^{1/2}, \bar{f})$ (up to a constant depending on the set of absolutely ramified primes in k).

Applications

For v, $w \in \mathbb{C}$, define

$$\varphi_\nu\left(\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}\right) = (x^2+1)^{-\frac{w}{2}}$$

for $\nu|\infty$ real,

$$\varphi_\nu\left(\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}\right) = (|x|^2+1)^{-w}$$

for $\nu|\infty$ complex, and choose

$$\chi_{0,\nu}(m) = |y|_\nu^v \qquad m = \begin{pmatrix} y & 0 \\ 0 & 1 \end{pmatrix} \in M_\nu$$

The corresponding monomial vector φ is admissible for $\Re(v)$ and $\Re(w)$ sufficiently large.

The integral $I = I(\chi_0) = I(v, w)$ can be written as

$$I = \sum_{\chi \in \widehat{C}_{0,S}} \frac{1}{2\pi i} \int_{\Re(s)=\sigma} L(\chi^{-1}|\cdot|^{v+1-s},\, f_1) \cdot L(\chi|\cdot|^{s},\, \bar{f}_2)\, \mathcal{K}_\infty(s,\, v,\, w,\, \chi)\, ds$$

Recall that $\mathcal{K}_\infty = \prod_{\nu|\infty} \mathcal{K}_\nu$. The function $\mathcal{K}_\nu(s, v, w, \chi_\nu)$ is analytic in a region $\mathcal{D} : \Re(s) = \sigma > \frac{1}{2} - \epsilon_0$, $\Re(v) > -\epsilon_0$ and $\Re(w) > \frac{3}{4}$, with a fixed (small) $\epsilon_0 > 0$.

Moreover,

$$\mathcal{K}_\nu(s,\, v,\, w,\, \chi_\nu) = A(v, w, \mu_{_1}, \mu_{_2}) \cdot \cdot \left(1 + \ell_\nu^2 + 4(t + t_\nu)^2\right)^{-w} \cdot \cdot \left[1 \,+\, \mathcal{O}\left(\left(\sqrt{1 + \ell_\nu^2 + 4(t + t_\nu)^2}\right)^{-1}\right)\right]$$

for $\nu|\infty$ complex, and

$$\mathcal{K}_\nu(s,\, v,\, w,\, \chi_\nu) = B(v, w, \mu_{_1}, \mu_{_2}) \cdot \cdot \left(1 + |t + t_\nu|\right)^{-w} \cdot \cdot \left[1 \,+\, \mathcal{O}\left(\left(1 + |t + t_\nu|\right)^{-\frac{1}{2}}\right)\right]$$

for $\nu|\infty$ real. The functions A and B are ratios of gamma functions.

It follows that for $\Re(w)$ sufficiently large,

$$I(0,w) = \sum_{\chi\in\widehat{C}_{0,S}} \frac{1}{2\pi} \int_{-\infty}^{\infty} L\big(\chi^{-1}|\cdot|^{\frac{1}{2}-it},\, f_1\big)$$

$$\cdot\, L\big(\chi|\cdot|^{\frac{1}{2}+it},\, \bar{f}_2\big)\, \mathcal{K}_\infty\big(\tfrac{1}{2}+it,\, 0,\, w,\, \chi\big)\, dt$$

Remark. Since $I(0,w)$ has meromorphic continuation, a mean value result can be established by standard arguments.

Remark. When $k=\mathbb{Q}$, one recovers the classical results.

Example. Assume $k=\mathbb{Q}(i)$. For $\ell\in\mathbb{Z}$, define $\Psi_\infty^{4\ell}(m) = (m/|m|)^{4\ell}$.

As $|T| \to \infty$, we have the asymptotic formula

$$\sum_{|\ell|<\frac{T}{2}} \int_0^{\sqrt{T^2-4\ell^2}} \left|L\left(\tfrac{1}{2}+it,\ f\otimes\Psi_\infty^{4\ell}\right)\right|^2 dt \sim\ C\,T^2\log T$$

where C is a non-zero (computable) constant (joint with Dorian Goldfeld).

Corollary (Sarnak 1985). *We have the following estimate:*

$$\sum_{|\ell|<T} \int_0^{T} \left|L\left(\tfrac{1}{2}+it,\ f\otimes\Psi_\infty^{4\ell}\right)\right|^2 dt \ll T^2\log T$$

Remark. By making a careful choice of the monomial vector φ, one can exhibit an error term on the order of $\mathcal{O}\left(T^{\vartheta}\right)$, with $\vartheta < 2$. in the above asymptotic formula. One can deduce that

$$\left|L\left(\tfrac{1}{2}+it,\, f\otimes\Psi_{\infty}^{4\ell}\right)\right| \ll_{\epsilon} |t|^{\frac{\vartheta}{2}+\epsilon}$$

$$\left|L\left(\tfrac{1}{2}+it,\, f\otimes\Psi_{\infty}^{4\ell}\right)\right| \ll_{\epsilon} |\ell|^{\frac{\vartheta}{2}+\epsilon}$$

In the general case, for $\chi \in \widehat{C}_0$ and $t \in \mathbb{R}$, put

$$\kappa_\chi(t, w) = \prod_{\substack{\nu|\infty \\ \nu \text{ real}}} \big(1 + |t + t_\nu|\big)^{-w} \cdot$$

$$\cdot \prod_{\substack{\nu|\infty \\ \nu \text{ complex}}} \left(\sqrt{1 + \ell_\nu^2 + 4(t + t_\nu)^2}\right)^{-w}$$

where t_ν and ℓ_ν are the parameters of the local component χ_ν of χ. Since χ is trivial on positive reals, we must have

$$\sum_{\nu|\infty} \alpha_\nu \, t_\nu = 0 \qquad (\alpha_\nu = 1, 2)$$

The asymptotic formula one obtains is of type

$$\sum_{\chi \in \widehat{C}_{0,S}} \int_{\mathfrak{I}_\chi(T)} \left| L\left(\tfrac{1}{2} + it,\, f \otimes \chi\right) \right|^2 \, dt \sim C\, T^{[k:\mathbb{Q}]} (\log T)^{r_1 + r_2}$$

for a non-zero positive constant C, where

$$\mathfrak{I}_\chi(T) = \{t : 0 \le \kappa_\chi(t,\, -1) \le T^{r_1 + r_2}\}$$