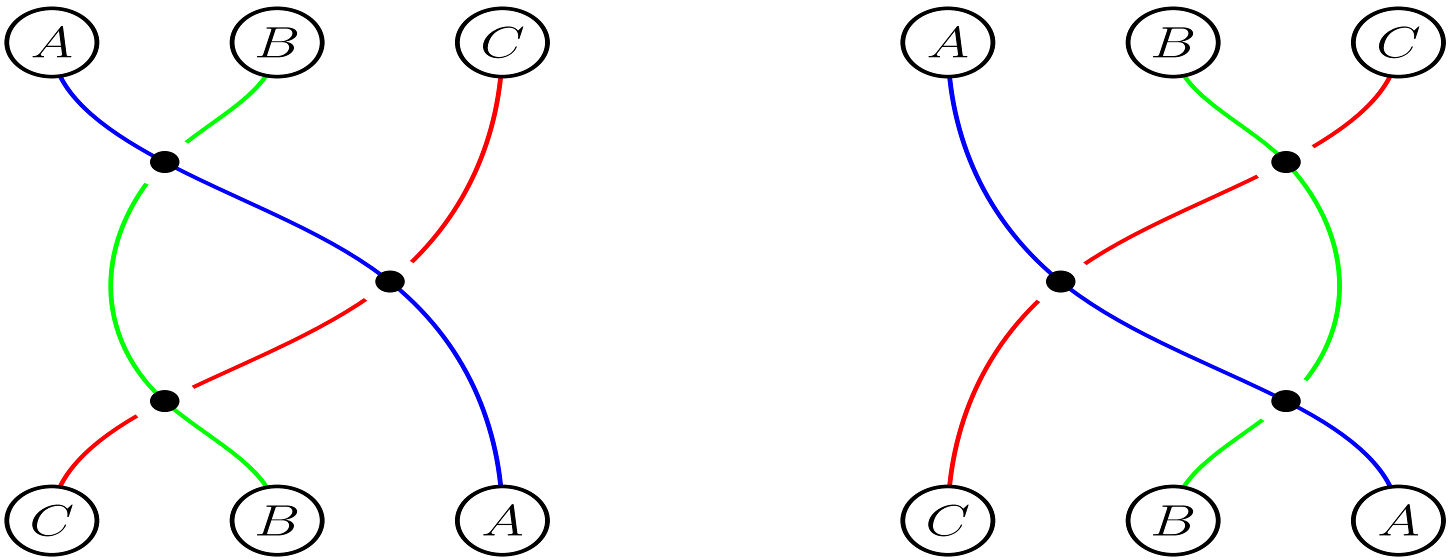
From Whittaker Functions to Quantum Groups

by Daniel Bump



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Between Physics and Number Theory

We will describe a new connection between existing topics in Mathematical Physics and Number Theory.

- In Physics, **Solvable Lattice Models** are an important example in Statistical Mechanics.
- The key to their secrets is the **Yang-Baxter** equation.
- In Number Theory **Whittaker functions** appear in the Fourier coefficients of automorphic forms. Particularly **Metaplectic Whittaker functions** are somewhat mysterious.
- The key to their properties are **intertwining integrals** which relate different **principal series representations** of covers of $\mathrm{GL}(r, \mathbb{Q}_p)$.

We will connect these topics, reviewing two papers:

- **Metaplectic Ice** (2012) by Brubaker, Bump, Friedberg, Chinta, Gunnells
- **A Yang-Baxter equation for Metaplectic Ice** (2016) by Buciumas, Brubaker, Bump

Overview

If H is a quantum group (quasitriangular Hopf algebra) and if U, V are modules, then there is an isomorphism $U \otimes V \longrightarrow V \otimes U$ making the modules into a **braided category**. This isomorphism is described by an **R-matrix**. A particular quantum group is $H = U_q(\hat{\mathfrak{gl}}(n))$. For $z \in \mathbb{C}$ it has an n-dimensional module $V(z)$.

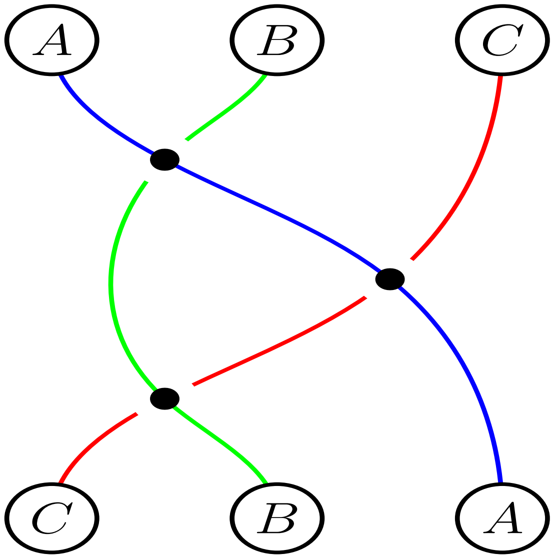
If F is a nonarchimedean local field containing the group μ_{2n} of n-th roots of unity, there is a "metaplectic" n-fold cover of $\mathrm{GL}(r, F)$. It is a central extension:

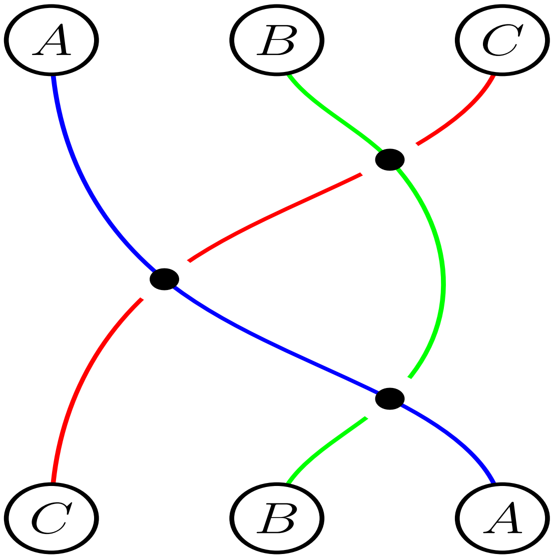
$$0 \longrightarrow \mu_{2n} \longrightarrow \widetilde{\mathrm{GL}}(r, F) \longrightarrow \mathrm{GL}(r, F) \longrightarrow 0.$$

The spherical representations are parametrized by a "Langlands parameter" $\boldsymbol{z} = (z_1, \cdots, z_n)$. There is a vector space $\mathcal{W}_{\boldsymbol{z}}$ of **Whittaker functions** and standard **intertwining maps** $\mathcal{W}_{\boldsymbol{z}} \longrightarrow \mathcal{W}_{w(\boldsymbol{z})}$ ($w \in W$, the **Weyl group**).

We will explain that $\mathcal{W}_{\boldsymbol{z}} \cong V(z_1) \otimes \cdots \otimes V(z_n)$ and the intertwining maps are described by R-matrices. The supersymmetric quantum group $U_q(\hat{\mathfrak{gl}}(n|1))$ also plays a role.

Part I: Quantum Groups





Statistical Mechanics

In Statistical Mechanics, each state $\mathfrak{s}_i$ of a system $\mathfrak{S}$ with energy E_i has probability proportional to e^{-kE_i}. The exact probability is $Z^{-1}e^{-kE_i}$ where

$$Z = \sum_{\mathfrak{s}_i \in \mathfrak{S}} e^{-kE_i} \qquad\qquad (k\text{=Boltzmann's constant})$$

is the **partition function.** In this setting it is real valued, but:

- In 1952, Lee and Yang showed that sometimes the partition function may be extended to an analytic function of a complex parameter. Its zeros lie on the unit circle. This is like a **Riemann hypothesis**.
- We will represent p-adic Whittaker functions as complex analytic partition functions. The **Langlands parameters** are in $\mathbb{C}^\times$.

Solvable Lattice Models

In some cases the partition function may be solved exactly. These examples are important for studying phase transitions. Unfortunately all known examples are two-dimensional.

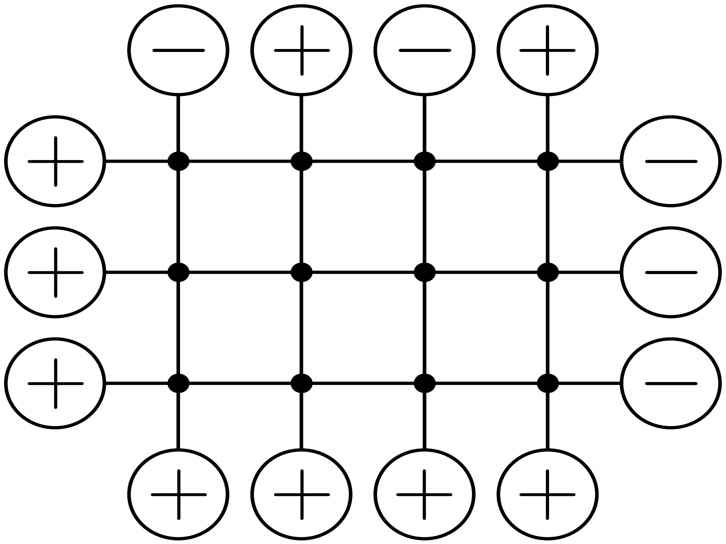
Solvable Lattice Models

In these examples, the state of a model is described by specifying the values of parameters associated with the vertices or edges of a planar graph.

- The **Ising Model** was solved by Onsager in 1944.
- The **six-vertex model** was solved by Lieb, Sutherland and Baxter in the 1960's.
- The **six-vertex model** is a key example in the history of **Quantum Groups.**
- A key principle underlying these examples is the **Yang-Baxter equation**. (This is not one equation but a class of equations.)
- The algebraic context for understanding Yang-Baxter equations is quantum groups.
- Once the theory of quantum groups is in place, many examples of Yang-Baxter equations present themselves. For us the important ones are associated with the quantum groups $U_q(\mathfrak{gl}(n))$ and $U_q(\mathfrak{gl}(n|1))$.

Two Dimensional Ice

Begin with a grid, usually (but not always) rectangular:



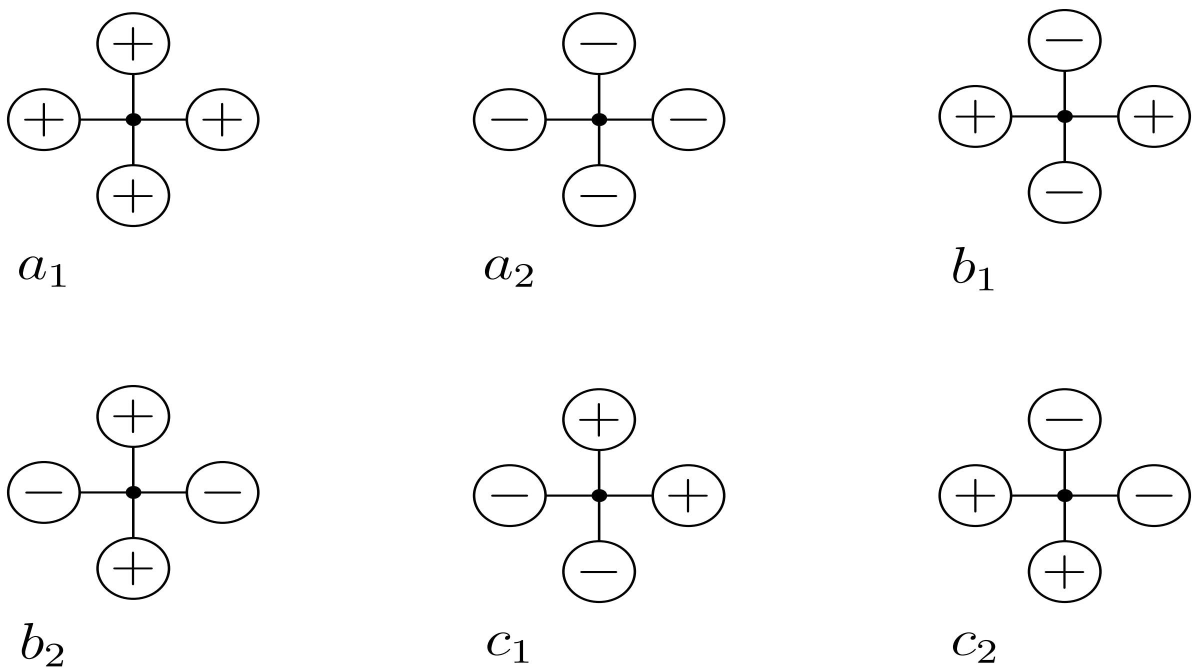
Each exterior edge is assigned a fixed **spin** $+$ or $-$. The inner edges are also assigned spins but these will vary.

A state of the system is an assignment of spins to the interior edges

The allowed configuration of spins around a vertex has six allowed possibilities

Six Vertex Model

A **state** of the model is an assignment of spins to the inner edges. (The outer edges have preassigned spins. Every vertex is assigned a set of **Boltzmann weights**. These depend on the spins of the four adjacent edges. For the six-vertex model there are only six nonzero Boltzman weights:



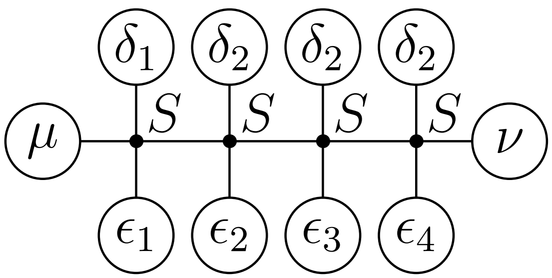
The **Boltzmann weight** of the state is the product of the weights at the vertices. The **partition function** is the sum over the states of the system.

Baxter

In the **field-free case** let S be Boltzmann weights for some vertex with $a = a_1 = a_2$ and $b = b_1 = b_2$ and $c = c_1 = c_2$. Following Lieb and Baxter let

$$\Delta_S = \frac{a^2 + b^2 - c^2}{2ab}.$$

Given one row of "ice" with Boltzmann weights S at each vertex:



$\left(\begin{array}{c} \text{Here } \delta_i, \varepsilon_i \text{ are fixed.} \\ \text{Toroidal domain: } \mu \text{ is considered an} \\ \text{interior edge after gluing the left and} \\ \text{right edges. So we sum over } \mu \text{ and} \\ \text{the other interior edges.} \end{array} \right)$

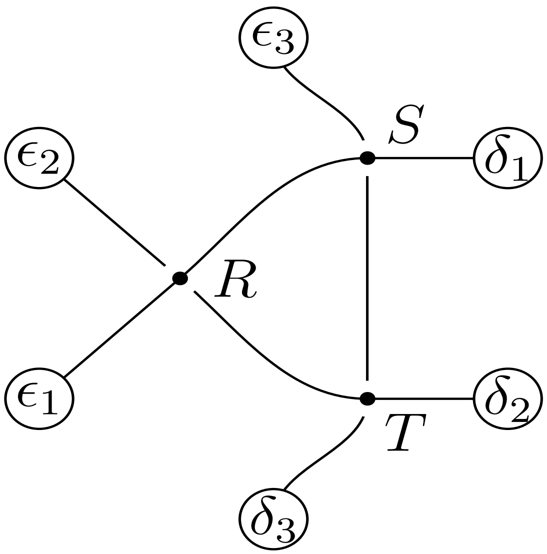
Let $\delta = (\delta_1, \delta_2, \cdots)$ and $\varepsilon = (\varepsilon_1, \varepsilon_2, \cdots)$ be the states of the top and bottom rows. The partition function then is a **row transfer matrix** $\Theta_S(\delta, \varepsilon)$. The partition function with several rows is a product transfer matrices.

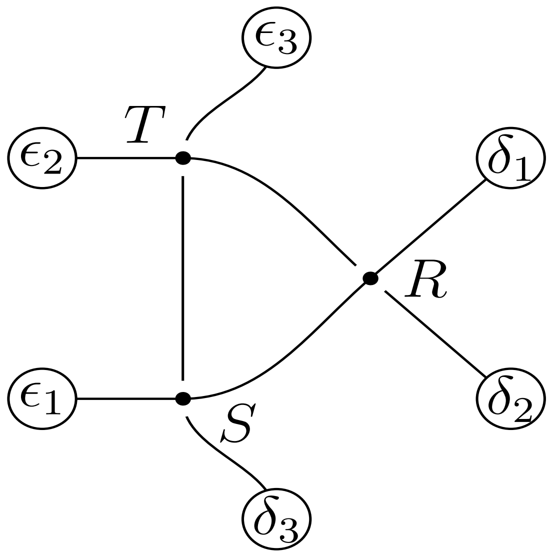
Theorem 1. (Baxter) *If* $\Delta_S = \Delta_T$ *then* Θ_S *and* Θ_T *commute.*

(This is closely related to solvability of the model.)

The Yang-Baxter equation

Theorem 2. (Baxter) *Let S and T be vertices with field-free Boltzmann weights. If $\Delta_S = \Delta_T$ then there exists a third R with $\Delta_R = \Delta_S = \Delta_T$ such that the two systems have the same partition function for any spins $\delta_1, \delta_2, \delta_3, \varepsilon_1, \varepsilon_2, \varepsilon_3$.*

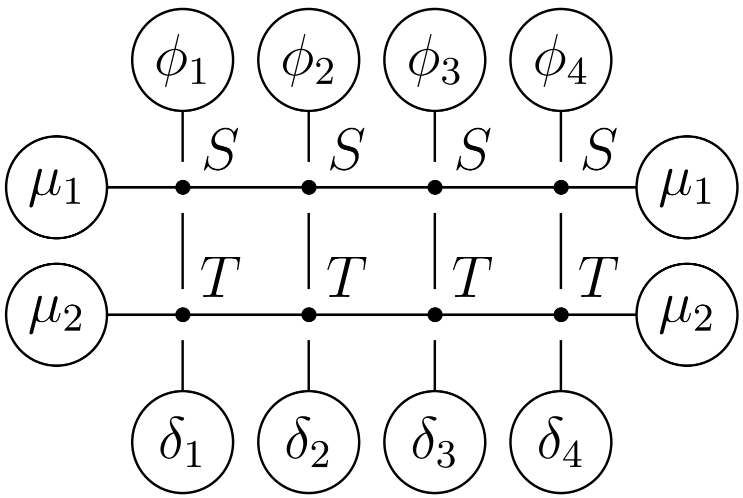




Remember: $\Delta_R = \dfrac{a^2 + b^2 - c^2}{2ab}$.

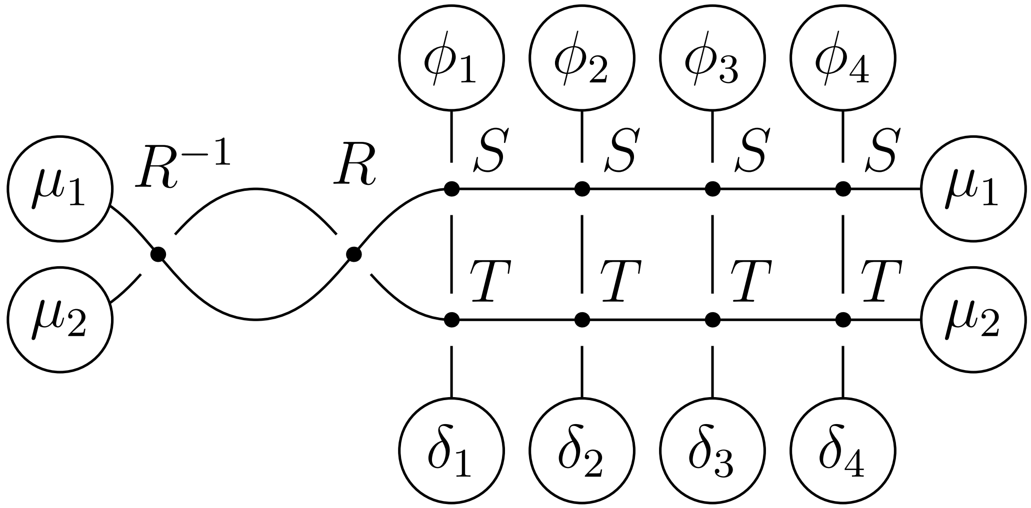
Commutativity of Transfer Matrices

This is used to prove the commutativity of the transfer matrices as follows. Consider the following system, whose partition function is the product of transfer matrices
$\Theta_S\Theta_T(\phi,\delta) = \sum_\varepsilon \Theta_S(\phi,\varepsilon)\Theta_T(\varepsilon,\delta)$:

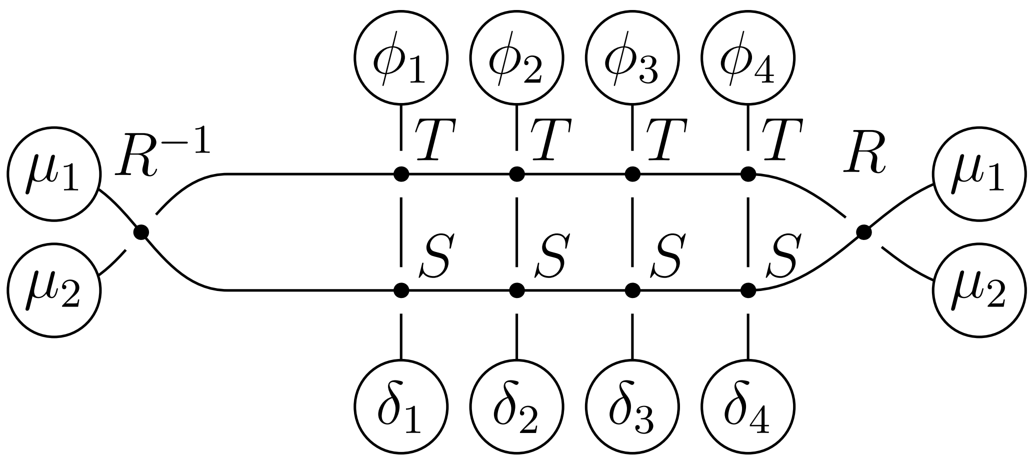


Toroidal Boundary Conditions: μ_1, μ_2 are interior edges and so we sum over μ_1, μ_2.

Insert R and another vertex R^{-1} that undoes its effect:



Now use YBE repeatedly:

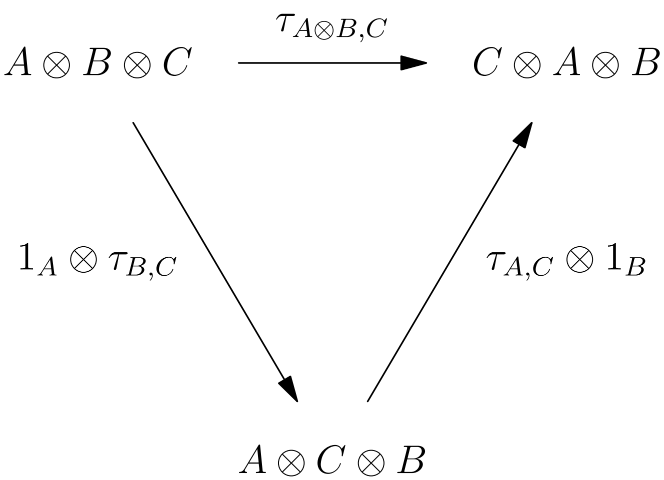


Due to toroidal BC now R, R^{-1} are adjacent again and they cancel. The transfer matrices have been shown to commute.

Braided Categories

A **monoidal category** (**Maclane**) has associative operation $\otimes$.
In **symmetric monoidal category** (**Maclane**) we assume:

Coherence: any two ways of going from one **permutation** of $A_1 \otimes A_2 \otimes \cdots$ to another give the same result.

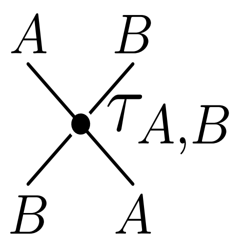


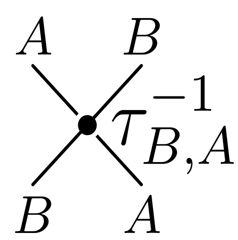
Important generalization!

Maclane **assumed** that $\tau_{A,B}\colon A \longrightarrow B$ and $\tau_{B,A}\colon B \longrightarrow A$ are **inverses**. But ...

Joyal and Street proposed eliminating this assumption. This leads to the important notion of a **braided monoidal category.**

In a **symmetric** monoidal category $\tau_{A,B}\colon A \longrightarrow B$ and $\tau^{-1}_{B,A}\colon A \longrightarrow B$ are the same. In a **braided** monoidal category they may be **different**.

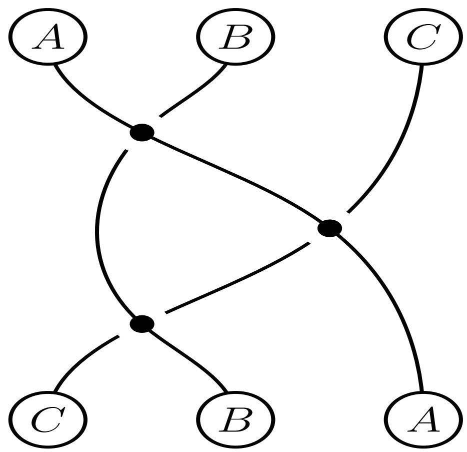


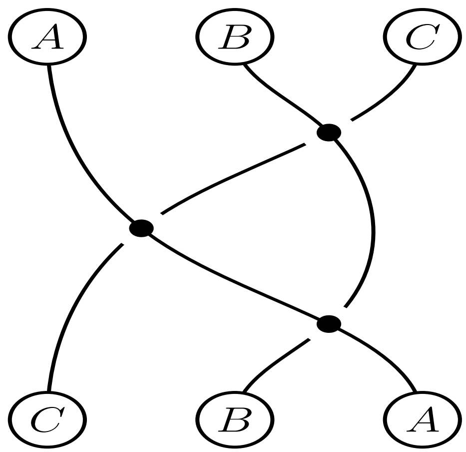


Represent them as braids.

The top row is $A \otimes B$
The bottom row is $B \otimes A$

Coherence: Any two ways of going from $A_1 \otimes A_2 \otimes \cdots$ to itself gives the same identity **provided** the two ways are the same in the Artin braid group.





These diagrams describe two morphisms $A \otimes B \otimes C \longrightarrow C \otimes B \otimes A$.
The morphisms are the same since the braids are the same.

Quantum Groups

Hopf algebras are convenient substitutes – essentially generalizations – of the notion of a group. The category of modules or comodules of a Hopf algebra is a **monoidal category.**

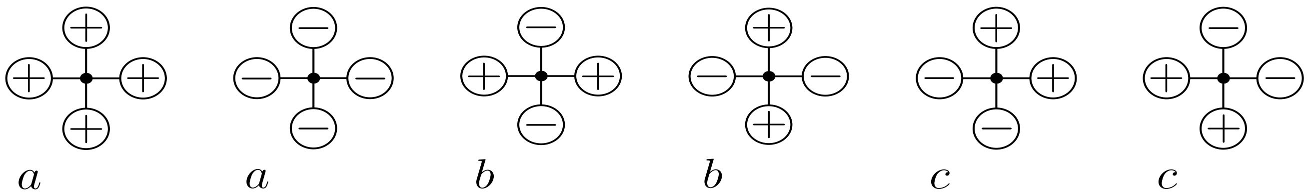
The group G has morphisms $\mu\colon G \times G \longrightarrow G$ and $\Delta\colon G \longrightarrow G \times G$, namely the multiplication and diagonal map. These become the multiplication and the comultiplication in the Hopf algebra.

The modules over a group form a **symmetric monoidal** category. There are two types of Hopf algebras with an analogous property.

- In a cocommutative Hopf algebra, the **modules** form a **symmetric monoidal category.**
- In a **commutative** Hopf algebra, the **comodules** form a **symmetric monoidal category**.

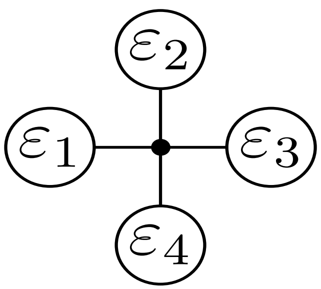
A **quantum group** is a Hopf algebra whose modules (or comodules) form a **braided monoidal category.** The axioms needed for this define a **quasi-triangular Hopf algebra** (Drinfeld). These are quantum group.

The R-matrix



Field-free Boltzmann weights

From this point of view, the Boltzmann weights at a field-free vertex R go into a matrix, an endomorphism of $V_1 \otimes V_2$, where V_1 and V_2 are two-dimensional vector spaces, each with a basis labeled v_+ and v_-.



$R^{\varepsilon_3 \otimes \varepsilon_4}_{\varepsilon_1 \otimes \varepsilon_2}$

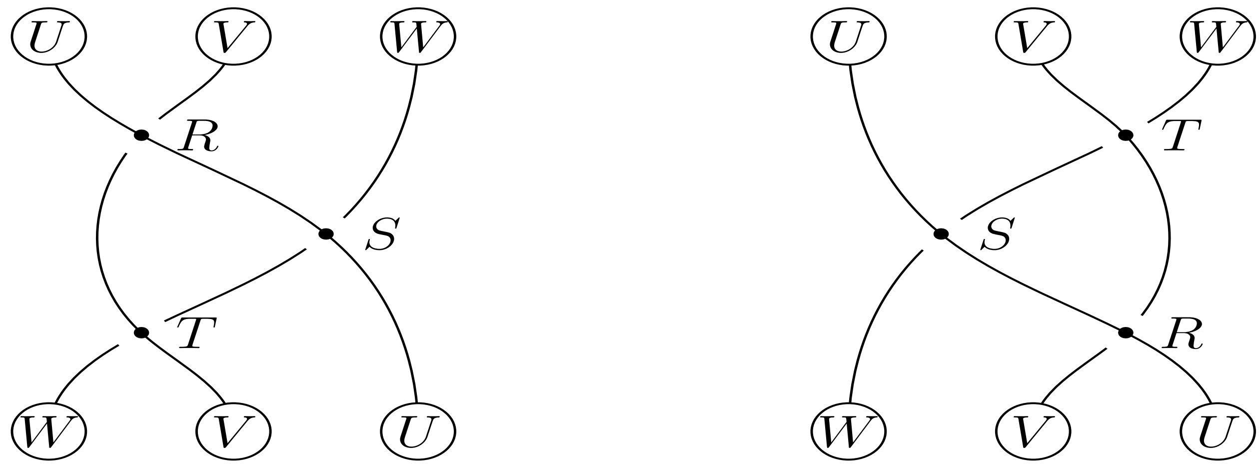
We interpret the vertex as a **linear transformation**

$$R(v_{\varepsilon_1} \otimes v_{\varepsilon_2}) = \sum R^{\varepsilon_3 \otimes \varepsilon_4}_{\varepsilon_1 \otimes \varepsilon_2}(v_{\varepsilon_3} \otimes v_{\varepsilon_4})$$

If $\tau \colon V_1 \otimes V_2 \longrightarrow V_2 \otimes V_1$ is $\tau(x \otimes y) = y \otimes x$
Then $\tau R \colon V_1 \otimes V_2 \longrightarrow V_2 \otimes V_1$ is a **candidate**
for a **morphism** in a **braided monoidal category**

Braid Picture

Given R,S,T with $\Delta_R = \Delta_S = \Delta_T$, rotate **Yang-Baxter equation** picture:



So associating with R, S, T two dimensional vector spaces and interpreting τR: $U \otimes V \longrightarrow V \otimes U$ as a morphism in a suitable category, the **Yang-Baxter equation** means the category is braided.

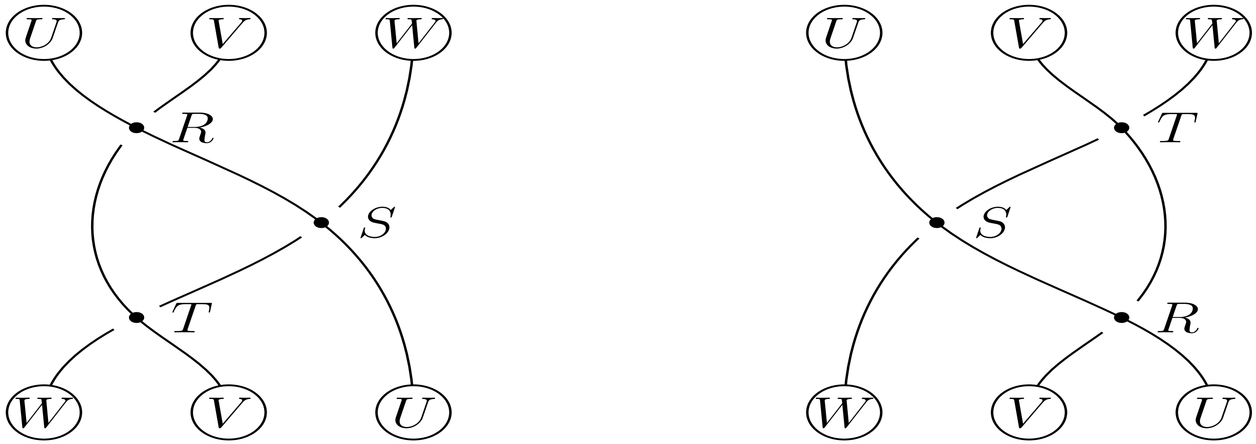
Tannakian Program:

⟨text|**Yang − Baxter equation**⟩	⇒	⟨text|**Braided category**⟩	⇒

Quantum Group

In the Baxter case, the relevant quantum group is $U_q(\hat{\mathfrak{sl}}_2)$.

Parametrized Yang-Baxter equations



If we unravel the definitions, the Yang-Baxter equation means that

$$R_{12}S_{13}T_{23} = T_{23}S_{13}R_{12}$$

where R_{12} is the endomorphism $R \otimes I_W$ acting on the first and third components of $U \otimes V \otimes W$ and trivially on W, etc. A **parametrized YBE** gives an R-matrix $R(\gamma)$ for every element γ of a group Γ such that

$$R_{12}(\gamma)R_{13}(\gamma\delta)R_{23}(\delta) = R_{23}(\delta)R_{13}(\gamma\delta)R_{12}(\gamma).$$

The Baxter YBE discussed gives a parametrized equation with $\Gamma = \mathbb{C}^\times$.

Affine quantum groups

Where do parametrized Yang Baxter equations come from?

Given a Lie group G with Lie algebra $\mathfrak{g}$ we can try to build a deformation of the universal enveloping algebra $U(\mathfrak{g})$. This is $U_q(\mathfrak{g})$, invented by Drinfeld and Jimbo after the example $\mathfrak{g} = \mathfrak{sl}_2$ was found by Kulish, Reshetikhin and Sklyanin.

The quantum group $U_q(\mathfrak{g})$ will have modules corresponding to the irreducible representations of G.

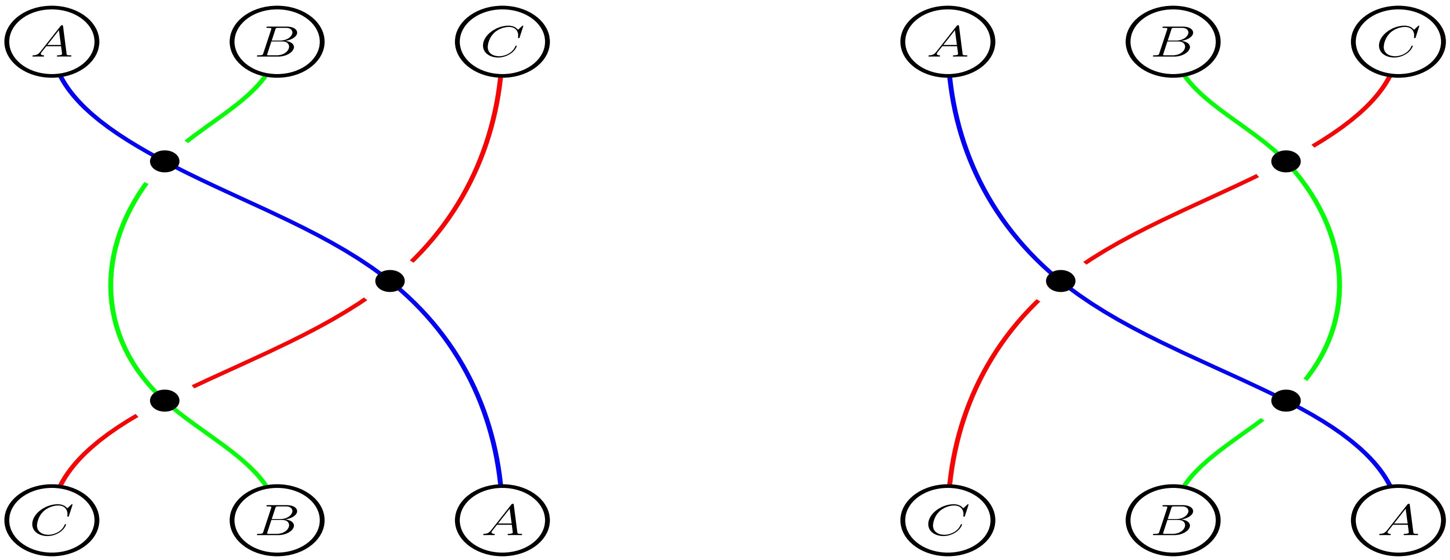
The **untwisted affine Lie algebra** $\hat{\mathfrak{g}}$ was constructed first by physicists, then appeared as a special case of the Kac-Moody Lie algebras. It is a central extension

$$0 \longrightarrow \mathbb{C} \cdot c \longrightarrow \hat{\mathfrak{g}} \longrightarrow \mathfrak{g} \otimes \mathbb{C}[t, t^{-1}] \longrightarrow 0.$$

Given a G-module V, $U_q(\hat{\mathfrak{g}})$ will have **one copy of V for every $z \in \mathbb{C}^\times$**. The corresponding Yang-Baxter equation will be a parametrized one.

For the Baxter example, $\mathfrak{g} = \mathfrak{sl}_2$ and z corresponds to (a, b, c). The parameter q is chosen so $\Delta = \frac{1}{2}(q + q^{-1})$.

Part II: Tokuyama's formula and the Casselman-Shalika formula



Whittaker models

Let F be a nonarchimedean local field, G a reductive algebraic group like $\mathrm{GL}(r)$. Langlands associated another algebraic group $\hat{G}$ (the connected L-group) such that **very roughly** (some) representations of $G(F)$ are related to the conjugacy classes of $\hat{G}(\mathbb{C})$. Let T and $\hat{T}$ be maximal tori in G and $\hat{G}$.

- If $\boldsymbol{z} \in \hat{T}(\mathbb{C})$ there is a representation $\pi_{\boldsymbol{z}}$ of $G(F)$ called a spherical principal series. It is infinite-dimensional.
- If $\boldsymbol{z}$ and $\boldsymbol{z}'$ are conjugate then there exists a Weyl group element such that $w(\boldsymbol{z}) = \boldsymbol{z}'$. Correspondingly there is an **intertwining operator** $\mathcal{A}_w\colon \pi_{\boldsymbol{z}} \to \pi_{w(\boldsymbol{z})}$ which is usually an isomorphism.

The representation $\pi_{\boldsymbol{z}}$ may (usually) be realized on a space $\mathcal{W}_{\boldsymbol{z}}$ of functions on the group called the **Whittaker model**. The representation π is infinite-dimensional. But let K be a (special) maximal compact subgroup. Then the space of K-fixed vectors is (usually) one-dimensional. Correspondingly there is a unique K-fixed vector in the Whittaker model.

The Weight Lattice

Let G and $\hat{G}$ be a reductive group and its dual, with dual maximal tori T and $\hat{T}$. Then there is a lattice Λ, the **weight lattice** which may be identified with either the group $X^*(\hat{T})$ of characters of $\hat{T}$, or the group $T(F)/T(\mathfrak{o})$ where $\mathfrak{o}$ is the ring of integers in the local field F. There is a cone Λ^+ in Λ consisting of **dominant weights**.

- If $\lambda \in \Lambda$ let $\boldsymbol{z} \longmapsto \boldsymbol{z}^\lambda$ ($\boldsymbol{z} \in \hat{T}$) be λ regarded as a character of $\hat{T}$.
- Let $\varpi^\lambda \in T(F)$ correspond.

Example. If $G = \mathrm{GL}(r)$ then $\hat{G} = \mathrm{GL}(r)$ also. T and $\hat{T}$ are the diagonal tori. The weight lattice $\Lambda \cong \mathbb{Z}^r$ and if $\lambda = (\lambda_1, \cdots, \lambda_r)$ then

$$\boldsymbol{z} = \begin{pmatrix} z_1 & & \\ & \ddots & \\ & & z_r \end{pmatrix}, \qquad \varpi^\lambda = \begin{pmatrix} \varpi^{\lambda_1} & & \\ & \ddots & \\ & & \varpi^{\lambda_r} \end{pmatrix}, \qquad \begin{array}{l} \boldsymbol{z}^\lambda = z_1^{\lambda_1} \cdots z_r^{\lambda_r}, \\ \\ \varpi \in \mathfrak{o} \text{ a prime element.} \end{array}$$

The Weyl Character Formula

Let $\hat{G}(\mathbb{C})$ be a reductive Lie group. Irreducible characters χ_λ are parametrized by **dominant weights** λ. The **Weyl character formula** is

$$\prod_{\alpha\in\Phi}(1-\boldsymbol{z}^\alpha)\chi_\lambda(\boldsymbol{z})=\sum_{w\in W}(-1)^{l(w)}\boldsymbol{z}^{w(\lambda+\rho)+\rho}\qquad\qquad \rho=\frac{1}{2}\sum_{\alpha\in\Phi^+}\alpha$$

The Casselman-Shalika Formula

Let F be a nonarchimedean local field and $\mathfrak{o}$ its ring of integers. Let $\hat{G}$ be the Langlands dual of the split reductive group G. Let T and $\hat{T}$ be dual maximal tori of $G,\hat{G}$. The weight lattice Λ of $\hat{G}$ is in bijection with $T(F)/T(\mathfrak{o})$. Let $\lambda\in\Lambda$ be a dominant weight an t_λ be a representative in $T(F)$. The **Casselman-Shalika formula** shows that for the **spherical Whittaker function W**

$$W(t_\lambda)=\begin{cases}(*)\prod_{\alpha\in\Phi}(1-q^{-1}\boldsymbol{z}^\alpha)\chi_\lambda(\boldsymbol{z}) & \text{if } \lambda \text{ is dominant,}\\ 0 & \text{otherwise.}\end{cases}$$

Here q is the residue field size. The unimportant constant $(*)$ is a power of q. **The expression is a deformation of the Weyl character formula.**

The Weyl character formula

The **Weyl character formula** can be thought of as a formula for

$$\prod_{\alpha \in \Phi^+} (1 - \boldsymbol{z}^\alpha) \chi_\lambda(\boldsymbol{z}).$$

λ = dominant weight of some Lie group $\hat{G}$

Φ^+ = positive roots

χ_λ = character of an irreducible representation with highest weight λ

Tokuyama's deformation of the WCF

The Weyl character formula has a deformation (**Tokuyama, 1988**). This was stated for $\mathrm{GL}(r)$ but other Cartan types may be done similarly. It is a formula for:

$$\prod_{\alpha \in \Phi^+} (1 - v\boldsymbol{z}^\alpha) \chi_\lambda(\boldsymbol{z}).$$

Tokuyama expressed this as a sum over strict Gelfand-Tsetlin patterns with shape $\lambda + \rho$.

Tokuyama's Formula: Crystal Version

There are two ways of expressing Tokuyama's formula.

$$\prod_{\alpha\in\Phi^+}(1-v\boldsymbol{z}^\alpha)\chi_\lambda(\boldsymbol{z}).$$

- As a sum over the Kashiwara **crystal based** $\mathcal{B}_{\lambda+\rho}$.
- As a partition function.

We will not discuss the first way, except to mention that it was a milestone towards the discovery of the connection between Whittaker functions and quantum groups.

Crystal bases are combinatorial analogs of representations of Lie groups. In Kashiwara's development, they come from the theory of quantum groups.

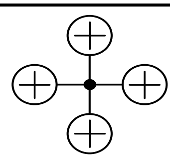
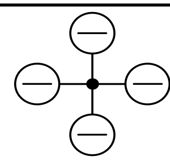
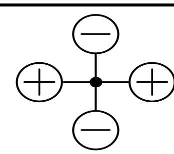
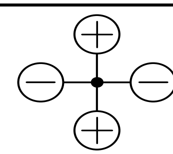
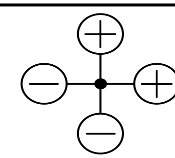
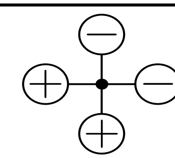
Tokuyama's Formula: Six Vertex Model

There is another way of writing Tokuyama's formula that was first found by **Hamel and King**.

In this view, Tokuyama's formula represents

$$Z(\mathfrak{S}) = \prod_{\alpha \in \Phi^+} (1 - v\boldsymbol{z}^{\alpha}) \chi_\lambda(\boldsymbol{z})$$

as the **partition function** of a six-vertex model system $Z(\mathfrak{S})$. We may use the following Boltzmann weights. We will label this vertex $T(z_i)$

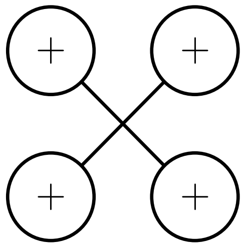
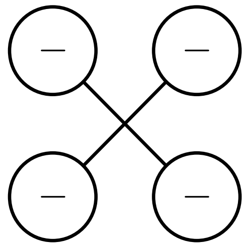
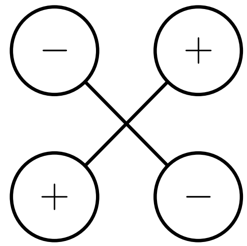
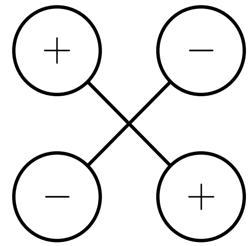
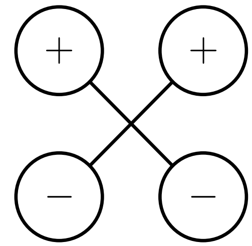
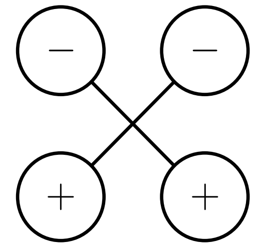
$T(z_i)$	 a_1	 a_2	 b_1	 b_2	 c_1	 c_2
	1	z_i	$-v$	z_i	$(1-v)z_i$	1

Here if $z_1, \cdots, z_r$ are the eigenvalues of $\boldsymbol{z} \in \mathrm{GL}(r, \mathbb{C})$, we take an ice model with r rows, and the above weights at each vertex in the r-th row. (The boundary conditions depend on λ.)

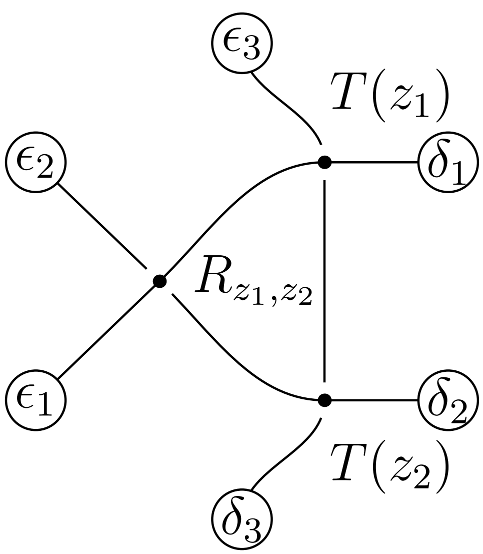
These weights are **free-fermionic** meaning $a_1 a_2 + b_1 b_2 = c_1 c_2$.

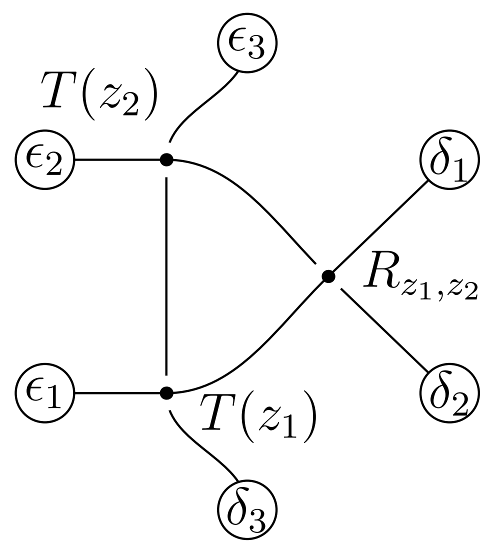
The R-matrix

The Yang-Baxter equation was introduced into this setting by **Brubaker, Bump and Friedberg**. Here is the R-matrix (also free-fermionic):

R_{z_1,z_2}						
	$z_2 - vz_1$	$z_1 - vz_2$	$v(z_1 - z_2)$	$z_1 - z_2$	$(1-v)z_1$	$(1-v)z_2$

- Yang-Baxter equation: the following partition functions are equal.
- It can be used to prove Tokuyama's theorem.

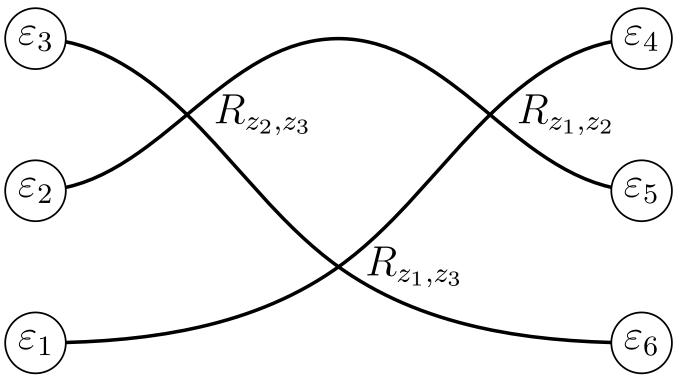


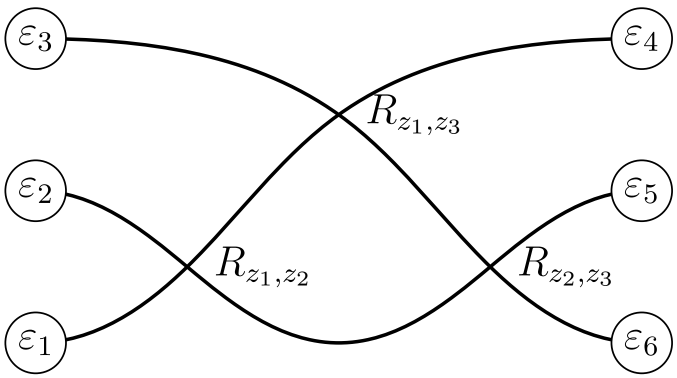


Another Yang-Baxter equation

We can make a Yang-Baxter equation involving only the R_{z_1,z_2}.

- The following partition functions are equal.





- This is a **parametrized YBE with parameter group $\mathbb{C}^\times$**.
- So we expect the corresponding quantum group to be affine.
- It turns out to be $U_v(\hat{\mathfrak{gl}}(1|1))$. Here $\hat{\mathfrak{gl}}(1|1)$ is a **Lie superalgebra.**

Digression: the Free-Fermionic Quantum Group

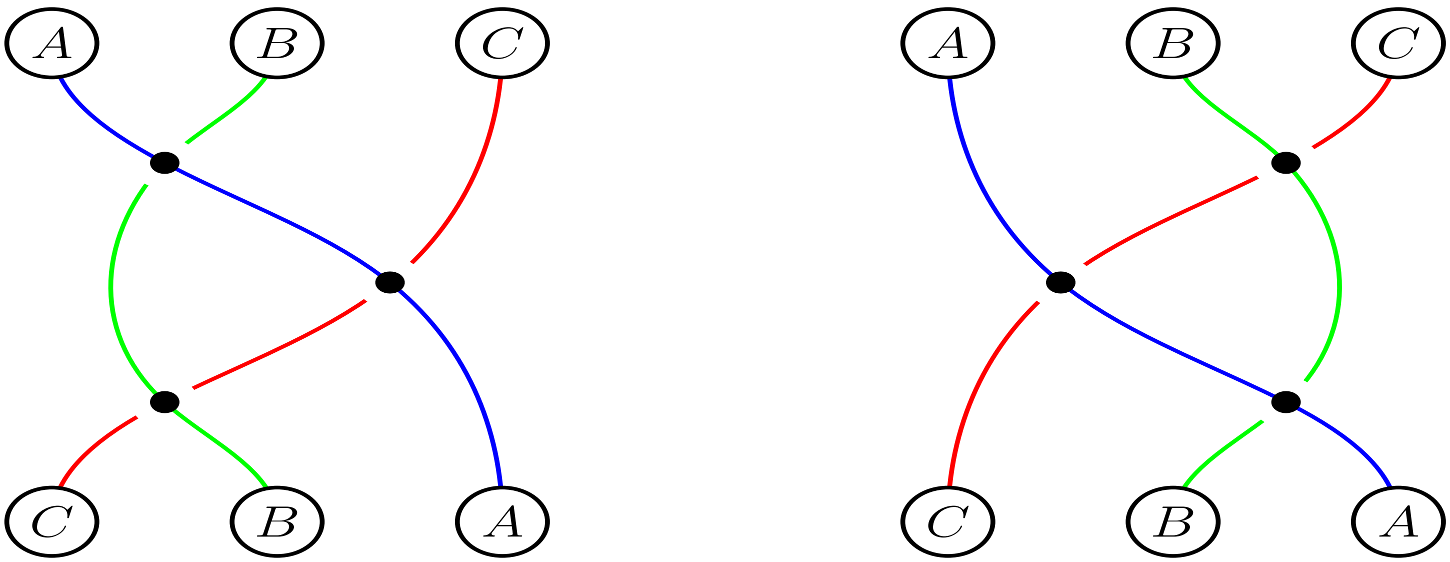
It was shown by **Korepin** and by **Brubaker-Bump-Friedberg** that **all** free-fermionic Boltzmann weights ($a_1 a_2 + b_1 b_2 = c_1 c_2$) can be assembled into a parametrized YBE **with nonabelian parameter group** $\mathrm{GL}(2) \times \mathrm{GL}(1)$.

Thus the weights R_{z_1, z_2} and $T(z_1)$, $T(z_2)$ all fit into a single YBE.

This **does not generalize** to the metaplectic case that we consider next.

The corresponding quantum group appears to be new. It has been studied by **Buciumas**.

Part II: Metaplectic Ice



The Metaplectic Group

If F contains the group μ_{2n} of n-th roots of unity, there is a "metaplectic" n-fold cover of $\mathrm{GL}(r, F)$. It is a central extension:

$$0 \longrightarrow \mu_{2n} \longrightarrow \widetilde{\mathrm{GL}}(r, F) \longrightarrow \mathrm{GL}(r, F) \longrightarrow 0.$$

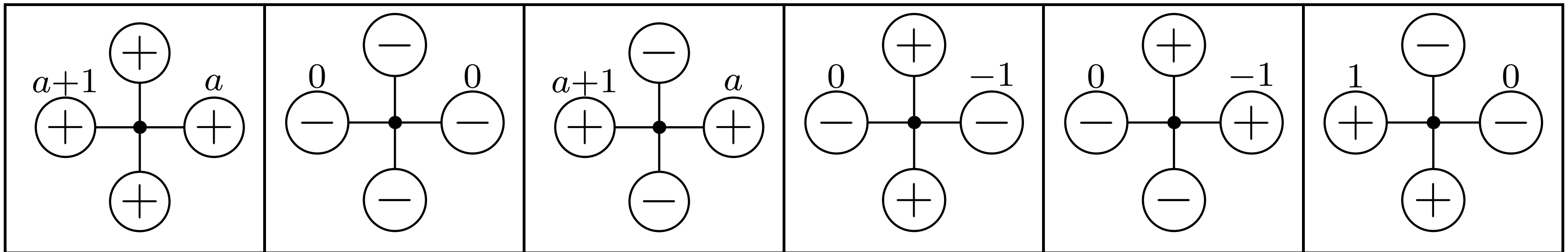
This fact may be generalized to **general reductive groups**. Whittaker models are **no longer unique**.

Metaplectic Ice: the Question

- In 2012, **Brubaker, Bump, Chinta, Friedberg and Gunnells** showed that Whittaker functions on $\widetilde{\mathrm{GL}}(r, F)$ could be represented as partition functions of generalized six-vertex models.

- They were not able to find the R-matrix for reasons that are now finally understood.

The partition function

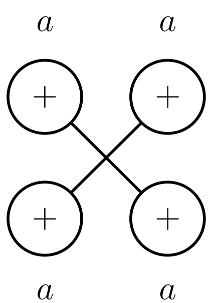
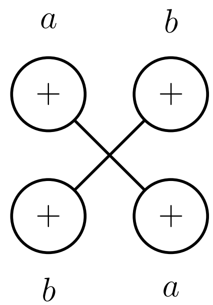
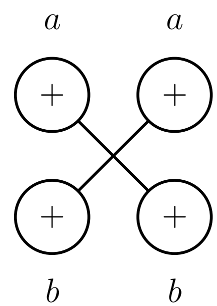
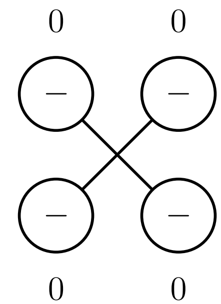
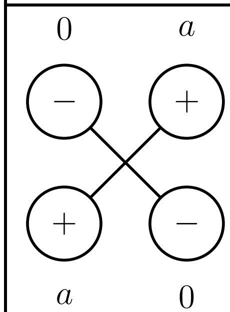
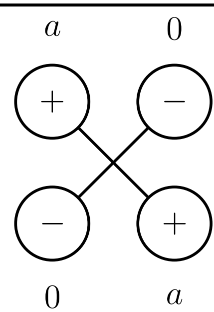
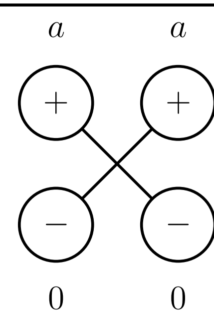
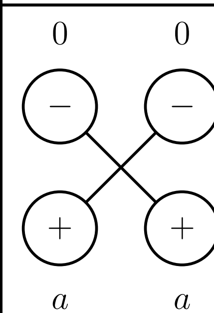


1	z_i	$g(a)$	z_i	$(1-v)z_i$	1

- We decorate each horizontal edges spin with an integer modulo n.
- The $-$ spins **must be** decorated with 0.
- $v = q^{-1}$ where q is the cardinality of the residue field.
- Here $g(a)$ is a Gauss sum.
- $g(a)g(-a) = v$ if $a \not\equiv 0 \bmod n$, while $g(0) = -v$.
- As before, λ is built into the boundary conditions.
- The partition function equals a metaplectic Whittaker function evaluated at ϖ^λ.
- The decorated edges span $(\mathbb{Z}/2\mathbb{Z})$-graded **super** vector space with n even-graded basis vectors and 1 odd-graded vector.

Metaplectic Ice: the Answer

In very recent work of **Brubaker, Buciumas and Bump:**

			
$z_2^n - v z_1^n$	$g(a-b)(z_1^n - z_2^n)$	$(1-v)z_1^c z_2^{n-c}$	$z_1^n - v z_2^n$
			
$v(z_1^n - z_2^n)$	$z_1^n - z_2^n$	$(1-v)z_1^a z_2^{n-a}$	$(1-v)z_1^{n-a} z_2^a$

- $c \equiv a - b$. The representatives a and c are chosen between 0 and n.
- This R-matrix gives the Yang-Baxter equation for metaplectic ice.

Supersymmetric partition function

- The quantum group is $U_v(\hat{\mathfrak{gl}}(n|1))$ with $v = q^{-1}$
- A procedure introduces Gauss sums into the R-matrix.
- The $U_v(\hat{\mathfrak{gl}}(n|m))$ R-matrix was discovered by **Perk** and **Schultz**. It was not related to $U_v(\hat{\mathfrak{gl}}(n|m))$ until later. It appears in the theory of superconductivity.

Intertwining integrals

The Whittaker model is no longer one-dimensional. Let $\mathcal{W}_{\boldsymbol{z}}$ be the finite-dimensional space of spherical Whittaker functions. We recall that the standard intertwining integrals $\mathcal{A}_w: \mathcal{W}_{\boldsymbol{z}} \longrightarrow \mathcal{W}_{w(\boldsymbol{z})}$ for Weyl group elements w.

Theorem 3. (**Brubaker, Buciumas, Bump**) *The quantum group $U_v(\hat{\mathfrak{gl}}(n))$ has one n-dimensional module $V(z)$ for each $z \in \mathbb{C}^\times$. We have an isomorphism $\mathcal{W}_{\boldsymbol{z}} \cong V(z_1) \otimes \cdots \otimes V(z_n)$ that realizes the standard intertwining integrals as $U_v(\hat{\mathfrak{gl}}(n))$-module homomorphisms.*

Lie superalgebra is not involved in this statement. It is very plausible that an identical statement is true for general Cartan types.