

Chapter 10

Crystals for Stanley Symmetric Functions

As we have seen in (3.3), the Schur polynomial $s_\lambda(x_1, \dots, x_\ell)$ can be viewed as the character of the $GL(\ell)$ crystal \mathcal{B}_λ . Since the Schur polynomials form a basis for the ring of symmetric polynomials, an interesting question is to expand a given symmetric polynomial $f(x)$ in terms of Schur polynomials. Suppose that the Schur expansion is integral and *positive*, meaning that in the expansion

$$f(x) = \sum_{\lambda} a_{\lambda} s_{\lambda}(x)$$

in terms of Schur polynomials, all the coefficients a_{λ} are nonnegative integers. Then one can try to give a crystal theoretic interpretation of this expansion. More precisely, one can try to find a crystal structure on the combinatorial objects \mathcal{C} defining the symmetric polynomial $f(x)$. Determining the highest weight elements in this crystal \mathcal{C} , then yields a combinatorial interpretation for the coefficients a_{λ} :

$$a_{\lambda} = |\{c \in \mathcal{C} \mid e_i c = 0 \text{ for all } 1 \leq i < \ell, \text{ wt}(c) = \lambda\}|.$$

In this chapter, we carry this out for the Stanley symmetric functions F_w following [Morse and Schilling (2016)]. Stanley symmetric functions F_w [Stanley (1984)] are indexed by permutations $w \in S_n$. We begin in Section 10.1 by defining the Stanley symmetric functions and the combinatorial objects (decreasing factorization of w) that govern them. In Section 10.2, we introduce the crystal structure on decreasing factorizations and then use the crystal in Section 10.3 to prove properties of the Stanley symmetric functions. In particular, it turns out that the crystal on decreasing factorizations is related to the tableaux model via the Edelman-Greene insertion [Edelman and Greene (1987)] of Chapter 7.3.

10.1 Stanley symmetric functions

Stanley symmetric functions are indexed by permutations $w \in S_n$ of the symmetric group. The symmetric group S_n is a Coxeter group (as introduced in Chapter 2) generated by the simple transpositions s_i for $1 \leq i < n$, where each s_i interchanges i and $i + 1$. The word $i_1 i_2 \dots i_m$ of letters $i_j \in \{1, 2, \dots, n - 1\}$ is called a *reduced word* for w if $w = s_{i_1} s_{i_2} \dots s_{i_m}$ and there is no shorter word with this property.

(Note that unlike in other chapters, for convenience of notation we do not write reduced words as tuples.) The length $\ell(w)$ of w is equal to m if the word is reduced. We denote by w_0 the longest element in S_n .

An element $v \in S_n$ is called *decreasing* if there is a reduced word $i_1 i_2 \cdots i_m$ for v such that $i_1 > i_2 > \cdots > i_m$. The identity in S_n is considered to be decreasing. The *content* $\text{cont}(v)$ is the set of letters appearing in its reduced word(s). Note that a decreasing element v is completely determined by its content. Given $w \in S_n$, a *decreasing factorization* of w is a factorization $w^k \cdots w^1$ such that $w = w^k \cdots w^1$ with $\ell(w) = \ell(w^1) + \cdots + \ell(w^k)$ and each factor w^i is decreasing. We denote the set of all decreasing factorizations of w by \mathcal{W}_w . Then for any $w \in S_n$, the *Stanley symmetric function* $F_w(x)$ is defined as

$$F_w(x) = \sum_{w^k \cdots w^1 \in \mathcal{W}_w} x_1^{\ell(w^1)} \cdots x_k^{\ell(w^k)}. \tag{10.1}$$

One of Stanley’s motivations to study these functions was to understand the reduced words for a given w . Let us denote the set of all reduced words for w by $\text{Red}(w)$. For example, since every single letter i by itself is decreasing, the coefficient of the square free term $x_1 x_2 \cdots x_{\ell(w)}$ is precisely the number of reduced words $|\text{Red}(w)|$.

For the next example, it is useful to define the *monomial symmetric functions*

$$m_\lambda(x) = \sum_{\alpha} x^\alpha$$

where the sum runs over all distinct permutations $\alpha = (\alpha_1, \alpha_2, \dots)$ of the entries of $\lambda = (\lambda_1, \lambda_2, \dots)$.

Note that the definition of Stanley symmetric functions, in principle, involves infinitely many variables x_1, x_2, \dots . However, if one restricts to precisely ℓ decreasing factors (some of which might be trivial), then one obtains Stanley symmetric polynomials in ℓ variables x_1, x_2, \dots, x_ℓ . Let us denote the set of decreasing factorizations of $w \in S_n$ into ℓ decreasing factors by \mathcal{W}_w^ℓ .

Example 10.1. We show how to compute the Stanley symmetric polynomial for the long element $w_0 = s_1 s_2 s_1 = s_2 s_1 s_2 \in S_3$. Restricting ourselves to three variables x_1, x_2, x_3 or equivalently to $\mathcal{W}_{w_0}^3$, we find the following set of decreasing factorizations with three factors for w_0 :

$$\begin{aligned} (1)(2)(1), & \quad ()(1)(21), & \quad (1)()(21), & \quad (1)(21)(), \\ (2)(1)(2), & \quad ()(21)(2), & \quad (21)()(2), & \quad (21)(2)(). \end{aligned}$$

Therefore,

$$\begin{aligned} F_{s_1 s_2 s_1}(x_1, x_2, x_3) &= 2x_1 x_2 x_3 + x_1^2 x_2 + x_1^2 x_3 + x_2^2 x_3 + x_1 x_2^2 + x_1 x_3^2 + x_2 x_3^2 \\ &= 2m_{(1,1,1)}(x_1, x_2, x_3) + m_{(2,1)}(x_1, x_2, x_3). \end{aligned}$$

Note that the square free term is contained in $m_{(1,1,1)}(x)$ which indeed has coefficient 2, the number of reduced words of $w_0 \in S_3$.

It turns out that the Stanley symmetric functions are indeed symmetric functions.

Theorem 10.2 ([Stanley (1984)]). *The Stanley symmetric functions F_w for $w \in S_n$ satisfy the following properties:*

- (1) $F_w(x)$ is a symmetric function in $x = (x_1, x_2, \dots)$.
- (2) Let $a_{w,\lambda} \in \mathbb{Z}$ be the coefficient of the Schur function s_λ in F_w . Then there exist partitions $\lambda(w)$ and $\mu(w)$, so that $a_{w,\lambda(w)} = a_{w,\mu(w)} = 1$ and

$$F_w(x) = \sum_{\lambda(w) \leq \lambda \leq \mu(w)} a_{w,\lambda} s_\lambda(x).$$

[Edelman and Greene (1987)] and separately [Lascoux and Schützenberger (1985)] showed that the Schur expansion coefficients $a_{w,\lambda}$ are nonnegative.

Theorem 10.3 ([Edelman and Greene (1987); Lascoux and Schützenberger (1985)]). *We have $a_{w,\lambda} \in \mathbb{Z}_{\geq 0}$.*

Other interpretations of the coefficients $a_{w,\lambda}$ can be found in [Haiman (1992); Reiner and Shimozono (1998)].

In the next section we provide another proof of the results in Theorems 10.2 and 10.3, based on [Morse and Schilling (2016)]. In particular, we interpret $a_{w,\lambda}$ as the number of highest weight elements of weight λ in a crystal graph.

10.2 Crystal on decreasing factorizations

We are now ready to define a crystal structure $\mathcal{B}(w)$ of type $A_{\ell-1}$ on \mathcal{W}_w^ℓ for every $w \in S_n$. As a set $\mathcal{B}(w)$ is the set \mathcal{W}_w^ℓ , that is, the set of all decreasing factorizations of w into at most ℓ factors. The weight function wt of $w^\ell \cdots w^1 \in \mathcal{B}(w)$ is defined to be $(\ell(w^1), \ell(w^2), \dots, \ell(w^\ell))$. The Kashiwara raising and lowering operators e_i and f_i only act on the factors $w^{i+1}w^i$. The action is defined by first bracketing certain letters and then moving an unbracketed letter from one factor to the other.

Let us begin by describing the bracketing procedure in analogy to the bracketing procedure in Section 2.4. Start with the largest letter b in $\text{cont}(w^{i+1})$ and pair it with the smallest $a > b$ in $\text{cont}(w^i)$. If there is no such a in $\text{cont}(w^i)$, then b is unpaired. The pairing proceeds in decreasing order on elements of $\text{cont}(w^{i+1})$, and with each iteration, previously paired letters of $\text{cont}(w^i)$ are ignored. Define

$$L_i(w^\ell \cdots w^1) = \{b \in \text{cont}(w^{i+1}) \mid b \text{ is unpaired in the } w^{i+1}w^i\text{-pairing}\}$$

and

$$R_i(w^\ell \cdots w^1) = \{b \in \text{cont}(w^i) \mid b \text{ is unpaired in the } w^{i+1}w^i\text{-pairing}\}.$$

We may now define the crystal operations, beginning with e_i . If $L_i(w^\ell \cdots w^1) = \emptyset$, then we define $e_i(w^\ell \cdots w^1) = 0$. Otherwise, $e_i(w^\ell \cdots w^1)$ is defined by replacing the factors w^{i+1} and w^i by \tilde{w}^{i+1} and \tilde{w}^i such that

$$\text{cont}(\tilde{w}^{i+1}) = \text{cont}(w^{i+1}) \setminus \{b\} \quad \text{and} \quad \text{cont}(\tilde{w}^i) = \text{cont}(w^i) \cup \{b - t\}$$

for $b = \min(L_i(w^\ell \cdots w^1))$ and $t = \min\{j \geq 0 \mid b - j - 1 \notin \text{cont}(w^{i+1})\}$. The following Lemma contains the facts needed to see that replacing w^{i+1} and w^i in this way produces another reduced factorization of w .

Lemma 10.4. *The content of w^{i+1} contains $b, b-1, \dots, b-t$ but not $b-t-1$. The content of w^i contains $b, b-1, \dots, b-t+1$ but not $b+1$ and not $b-t$. We have*

$$w^{i+1}w^i = \tilde{w}^{i+1}\tilde{w}^i. \quad (10.2)$$

Proof. The definition of t implies that $\text{cont}(w^{i+1})$ contains $b, b-1, \dots, b-t$ but not $b-t-1$. To see that $\text{cont}(w^i)$ does not contain $b+1$, we note that if it did, then $b+1$ in $\text{cont}(w^i)$ could not be paired with any element y of $\text{cont}(w^{i+1})$ before b , since y would be strictly greater than b and each element of $\text{cont}(w^i)$ can only be paired with an element of $\text{cont}(w^{i+1})$ that is smaller than it. Thus when we come to pairing $b \in \text{cont}(w^{i+1})$ there will be an unpaired $y > b$ in $\text{cont}(w^i)$ and b will get paired. But this is a contraction since by assumption b is unpaired.

Now by definition, b is the rightmost unpaired element of $\text{cont}(w^{i+1})$, so that $b-1, \dots, b-t$ are all paired. Let us ask what elements they may be paired with. The $b-1$ in $\text{cont}(w^{i+1})$ must be paired with an element z of $\text{cont}(w^i)$ that is greater than it. Thus $z > b-1$. But z cannot be greater than b , because if such a z is unpaired when $b-1$ is paired, it must be unpaired when b is paired, in which case the b would get paired with z . Therefore $z = b$ and so $\text{cont}(w^i)$ contains b , which is paired with $b-1$ in $\text{cont}(w^{i+1})$. Next we consider $b-2$ in $\text{cont}(w^{i+1})$, and similar reasoning shows that it has to be paired with $b-1$. Continuing in this way, we see that $\text{cont}(w^i)$ must contain $b, \dots, b-t+1$ and these are paired with $b-1, \dots, b-t$ in $\text{cont}(w^{i+1})$. We will show that $\text{cont}(w^i)$ does not contain $b-t$ at the end of the proof.

Now we may write

$$w^{i+1} = us_b s_{b-1} \cdots s_{b-t} u' \quad \text{and} \quad w^i = u'' s_b s_{b-1} \cdots s_{b-t+1} u''',$$

where u and u'' are the product of reflections s_k with $k > b$, u' is the product of reflections s_k with $k < b-t$, and u''' is the product of reflections s_k with $k < b-t+1$. We know that u' does not involve s_{b-t-1} and u'' does not involve s_{b+1} , so u' and u'' commute with s_b, \dots, s_{b-t} .

We make note of the identity

$$(s_b s_{b-1} \cdots s_{b-t})(s_b s_{b-1} \cdots s_{b-t+1}) = (s_{b-1} \cdots s_{b-t})(s_b s_{b-1} \cdots s_{b-t})$$

which we leave the reader to check. This, together with the commutations noted above, means that

$$w^{i+1}w^i = (us_{b-1} \cdots s_{b-t}u)(u''s_b \cdots s_{b-t}u'''). \quad (10.3)$$

Now we can show that $\text{cont}(w^i)$ does not contain $b-t$. If it did, then s_{b-t} would be the first factor in the decreasing factorization of u''' . So the above factorization would contain two adjacent s_{b-t} , which we could cancel, obtaining $\ell(w^{i+1}w^i) < \ell(w^{i+1}) + \ell(w^i)$, contradicting the definition of a decreasing factorization.

Finally, we note that the two factors in (10.3) are \tilde{w}^{i+1} and \tilde{w}^i , obtaining (10.2). \square

Note that since $\ell(\tilde{w}^i) = \ell(w^i) + 1$ and $\ell(\tilde{w}^{i+1}) = \ell(w^{i+1}) - 1$, we have $\text{wt}(e_i(x)) = \text{wt}(x) + 1$ for $x \in \mathcal{W}_w^\ell$.

Similarly, $f_i(w^\ell \cdots w^1)$ is defined by replacing the factors $w^{i+1}w^i$ by $\tilde{w}^{i+1}\tilde{w}^i$ such that

$$\text{cont}(\tilde{w}^{i+1}) = \text{cont}(w^{i+1}) \cup \{a + s\} \quad \text{and} \quad \text{cont}(\tilde{w}^i) = \text{cont}(w^i) \setminus \{a\}$$

for $a = \max(R_i(w^\ell \cdots w^1))$ and $s = \min\{j \geq 0 \mid a + j + 1 \notin \text{cont}(w^i)\}$. If $R_i(w^\ell \cdots w^1) = \emptyset$, then $f_i(w^\ell \cdots w^1) = 0$. We leave the reader to formulate and prove the analog of Lemma 10.4 for the f_i and to deduce that $\mathcal{B}(w)$ is a crystal; see Exercise 10.1.

Example 10.5. Let $(s_3s_2)(s_3s_1)(s_2) \in \mathcal{W}_w^3$ for $w = s_3s_2s_3s_1s_2 \in S_4$. To apply e_2 , we need to first bracket the letters in $\text{cont}(w^3) = 32$ with those in $\text{cont}(w^2) = 31$. The letter 3 in $\text{cont}(w^3)$ is unbracketed since there is no bigger letter in $\text{cont}(w^2)$, but the letter 2 in $\text{cont}(w^3)$ is bracketed with 3 in $\text{cont}(w^2)$. Hence $b = \min(L_2(w^3w^2w^1)) = 3$ and $t = \min\{j \geq 0 \mid b - j - 1 \notin \text{cont}(w^3)\} = 1$. Therefore, $e_2((s_3s_2)(s_3s_1)(s_2)) = (s_2)(s_3s_2s_1)(s_2)$. Similarly, $f_2((s_3s_2)(s_3s_1)(s_2)) = (s_3s_2s_1)(s_3)(s_2)$.

Theorem 10.6 ([Morse and Schilling (2016)]). $\mathcal{B}(w)$ is a Stembridge crystal of type $A_{\ell-1}$.

A proof of this theorem is given in [Morse and Schilling (2016), Appendix]. See also Exercise 10.2.

Example 10.7. The crystal $\mathcal{B}(w_0)$ of type A_2 for $w_0 \in S_3$ is provided in Figure 10.1.

10.3 Applications

As we have seen in Theorem 5.20, normal (and in particular Stembridge) crystals are closed under taking tensor products and connected components in a crystal graph correspond to irreducible components. In addition, the irreducible components are in bijection with the highest weight elements and dominant weights. Therefore, the irreducible components of our crystal $\mathcal{B}(w)$ of type $A_{\ell-1}$ are isomorphic to \mathcal{B}_λ for some partition λ . Recall from (3.3) that the Schur function $s_\lambda(x_1, \dots, x_\ell)$ is the character of the crystal \mathcal{B}_λ of type $A_{\ell-1}$

$$s_\lambda(x_1, \dots, x_\ell) = \sum_{b \in \mathcal{B}_\lambda} x^{\text{wt}(b)}.$$

Denote by $\mathcal{W}_{w,\lambda}^\ell$ all elements in \mathcal{W}_w^ℓ of weight λ .

Choosing ℓ sufficiently large, the above arguments immediately yield the following result regarding the Schur expansion of the Stanley symmetric functions.

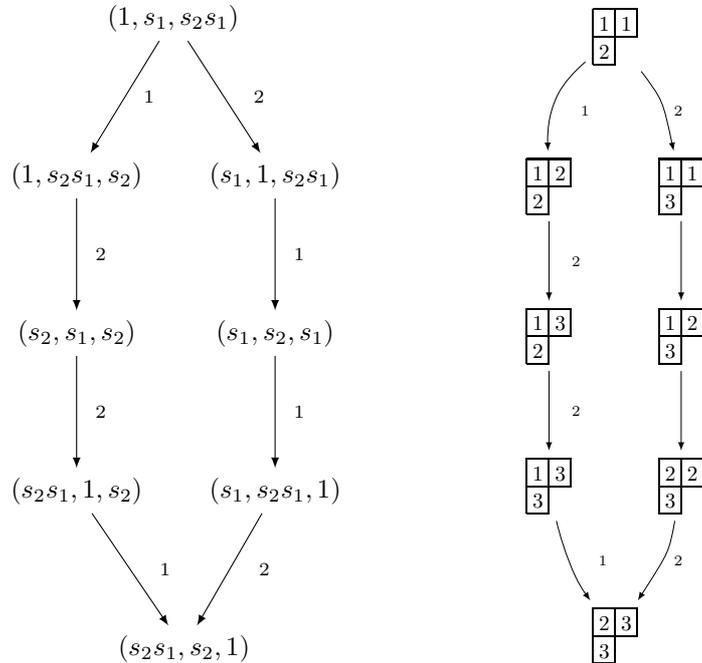


Fig. 10.1 Crystal $\mathcal{B}(w_0)$ of type A_2 for $w_0 = s_1s_2s_1 \in S_3$ on the left and the highest weight crystal $\mathcal{B}_{(2,1)}$ of type A_2 in terms of Young tableaux on the right.

Corollary 10.8 ([Morse and Schilling (2016)]). *For any $w \in S_n$, the coefficient $a_{w,\lambda}$ in*

$$F_w(x) = \sum_{\lambda} a_{w,\lambda} s_{\lambda}(x) \tag{10.4}$$

enumerates the highest weight factorizations in $\mathcal{W}_{w,\lambda}^{\ell}$, where $\ell > \ell(w)$. That is,

$$a_{w,\lambda} = |\{w^{\ell} \cdots w^1 \in \mathcal{W}_{w,\lambda}^{\ell} \mid e_i(w^{\ell} \cdots w^1) = 0 \text{ for all } 1 \leq i < \ell\}|.$$

[Edelman and Greene (1987)] (see also [Fomin and Greene (1998), Theorem 1.2]) characterized the coefficients $a_{w,\lambda}$ as the number of semistandard tableaux of shape λ' (the conjugate of λ) whose column-reading word is a reduced word of w .

We can understand this result in terms of a variant of the Edelman–Greene (EG) insertion discussed in Chapter 7.3. In fact, it can be extended to the full crystal (not just highest weight elements) as follows. Given a decreasing factorization $v^{\ell} \cdots v^1 \in \mathcal{W}_w^{\ell}$, consider $\bar{v}^1 \cdots \bar{v}^{\ell}$ by reversing all factors. In particular, each decreasing factor v^i turns into an increasing factor \bar{v}^i . Now successively insert the factors \bar{v}^i for $i = 1, 2, \dots, \ell$ using the EG insertion of Chapter 7.3. Recall that, in this insertion, a letter a is inserted into a row by finding the smallest letter $b > a$. If $b = a + 1$ and a is also contained in the row, then $a + 1$ is inserted into the next row down. Otherwise,

b is replaced by a and then b is inserted into the next row down. In both cases, we consider b to be bumped. For each inserted factor \bar{v}^i , the cells in the new shape are recorded by letters i . This yields a correspondence $\varphi_{\text{EG}}: v^\ell \cdots v^1 \mapsto (P, Q)$, where P is the EG insertion tableau and Q is the EG recording tableau, where each letter in the factor \bar{v}^i is recorded with the letter i in Q .

Example 10.9. Take $v^3v^2v^1 = (1)(2)(32)$ a factorization of the permutation $s_1s_2s_3s_2 \in S_4$. Then $\bar{v}^1\bar{v}^2\bar{v}^3 = (23)(2)(1)$ with insertions for $i = 1, 2, 3$:

$$\left(\begin{array}{|c|c|} \hline 2 & 3 \\ \hline \end{array}, \begin{array}{|c|c|} \hline 1 & 1 \\ \hline \end{array} \right) \quad \left(\begin{array}{|c|c|} \hline 2 & 3 \\ \hline 3 & \end{array}, \begin{array}{|c|c|} \hline 1 & 1 \\ \hline 2 & \end{array} \right) \quad \left(\begin{array}{|c|c|} \hline 1 & 3 \\ \hline 2 & \end{array}, \begin{array}{|c|c|} \hline 1 & 1 \\ \hline 2 & \end{array} \right) = (P, Q).$$

If $v^3v^2v^1 = (32)(31)(2)$, then the insertion sequence yields:

$$\left(\begin{array}{|c|} \hline 2 \\ \hline \end{array}, \begin{array}{|c|} \hline 1 \\ \hline \end{array} \right) \quad \left(\begin{array}{|c|c|} \hline 1 & 3 \\ \hline 2 & \end{array}, \begin{array}{|c|c|} \hline 1 & 2 \\ \hline 2 & \end{array} \right) \quad \left(\begin{array}{|c|c|c|} \hline 1 & 2 & 3 \\ \hline 2 & 3 & \end{array}, \begin{array}{|c|c|c|} \hline 1 & 2 & 3 \\ \hline 2 & 3 & \end{array} \right) = (P, Q).$$

The map φ_{EG} turns out to be a crystal morphism, when restricted to the recording tableaux as the next theorem shows. We denote the projection of φ_{EG} onto the recording tableau by $\varphi_{\text{EG}}^{\text{rec}}$, that is, $\varphi_{\text{EG}}^{\text{rec}}(v^\ell \cdots v^1) = Q$ if $\varphi_{\text{EG}}(v^\ell \cdots v^1) = (P, Q)$.

Theorem 10.10 ([Morse and Schilling (2016)], Theorem 4.11). *For any permutation $w \in S_n$, the crystal isomorphism*

$$\mathcal{B}(w) \cong \bigoplus_{\lambda} \mathcal{B}_{\lambda}^{\oplus a_{w,\lambda}}$$

is explicitly given by $\varphi_{\text{EG}}^{\text{rec}}$. In particular,

$$\varphi_{\text{EG}}^{\text{rec}} \circ e_i = e_i \circ \varphi_{\text{EG}}^{\text{rec}} \quad \text{and} \quad \varphi_{\text{EG}}^{\text{rec}} \circ f_i = f_i \circ \varphi_{\text{EG}}^{\text{rec}}.$$

Example 10.11. Take $v^3v^2v^1 = (32)(31)(2)$ from Example 10.9. Then $e_2(v^3v^2v^1) = (2)(321)(2)$ and

$$\varphi_{\text{EG}}^{\text{rec}}((2)(321)(2)) = \begin{array}{|c|c|c|} \hline 1 & 2 & 2 \\ \hline 2 & 3 & \end{array},$$

which is the same as $e_2 \circ \varphi_{\text{EG}}^{\text{rec}}((32)(31)(2))$.

Proof of Theorem 10.10. EG-insertion enjoys many of the same properties as RSK-insertion. For example, given that cell c_x is added to a tableau when x is EG-inserted, and cell c_y is added when y is then EG-inserted into the result, c_y lies strictly east of c_x when $y > x$, and c_y lies strictly lower than c_x when $y < x$.

Fix $w \in S_n$. We first note that φ_{EG} is a bijection between $\mathcal{W}_{w,\alpha}^\ell$ and the set of pairs of same-shaped tableaux (P, Q) where the column-reading word of the transpose of P is a reduced expression for w and Q is semistandard of weight α . That is, given $v^\ell \cdots v^1 \in \mathcal{W}_{w,\alpha}^\ell$, let $(P, Q) = \varphi_{\text{EG}}(v^\ell \cdots v^1)$ and recall that $P = P^\ell$ where P^ℓ is defined by inserting the (distinct) letters of $\text{cont}(v^\ell)$ from smallest to largest into $P^{\ell-1}$. By the above remarks about the properties of the EG-insertion, $Q^\ell/Q^{\ell-1}$ is a horizontal $\ell(v^\ell)$ -strip and we iteratively find Q to be semistandard of

weight α . The column reading word of the transpose of P is a reduced expression for w by Proposition 7.17. It is not difficult to see that the process is invertible by reverse EG-bumping letters from P^i that lie in the positions determined by cells of $\text{shape}(Q^i)/\text{shape}(Q^{i-1})$ taken from right to left.

Denote the letters in $\text{cont}(v^{i+1})$ by $y_{\alpha_{i+1}} \cdots y_1$ and the letters in $\text{cont}(v^i)$ by $x_{\alpha_i} \cdots x_1$ in decreasing order. Let $a = x_j$ be the leftmost unbracketed letter in the pairing in Section 10.2. Inserting the letters x_1, \dots, x_{α_i} under the EG-insertion yields α_i insertion paths that move strictly to the right in the tableaux P^i by the above remark on the properties of the EG insertion. Since $a = x_j$ is the leftmost unbracketed letter in $\text{cont}(v^i)$, there exists an index $1 \leq m \leq \alpha_{i+1}$ such that $x_j \leq y_m < y_{m+1} < \cdots < y_{\alpha_{i+1}}$ and $y_1 < y_2 < \cdots < y_{m-1} < x_{j-1}$. In addition, all letters y_1, \dots, y_{m-1} are bracketed under the crystal bracketing which means that the insertion paths of these letters are weakly to the left of the insertion path of x_{j-1} and no letter can bump x_j . Also, the letters $x_{j+1}, \dots, x_{\alpha_i}$ are bracketed under the crystal bracketing so that of the letters i in Q^i corresponding to the insertion paths of x_j, \dots, x_{α_i} precisely one is not bracketed with an $i + 1$ in Q^{i+1} .

Under the application of f_i the letter $a = x_j$ moves from $\text{cont}(v^i)$ to the letter $a + s$ in $\text{cont}(v^{i+1})$. As a result, the insertion paths of $x_{j+1}, \dots, x_{\alpha_i}$ either stay (partially) in their old track or move left (partially) to the insertion of the previously inserted letter. Similarly, the insertion paths of the corresponding y_h move (partially) left. The new letter $a + s$ in v^{i+1} after the application of f_i , then causes the previously unpaired letter i in Q^{i+1} in the insertion to become an $i + 1$, possibly by shifting the insertion paths of the subsequent y_h to the right. This proves the claim for f_i .

The proof for e_i is similar. □

Restricting Theorem 10.10 to highest weight elements yields the following corollary.

Corollary 10.12. *For any permutation $w \in S_n$ and partition λ , there is a bijection between the highest weight factorizations,*

$$\{v^\ell \cdots v^1 \in \mathcal{W}_{w,\lambda}^\ell \mid e_i(v^\ell \cdots v^1) = 0 \text{ for all } 1 \leq i < \ell\},$$

and the semistandard tableaux of shape λ' whose column-reading word is a reduced word of w . Explicitly, the bijection is given by the conjugate of the insertion tableau $\varphi_{\text{EG}}^{\text{ins}}(v^\ell \cdots v^1) = P$ of the highest weight element $v^\ell \cdots v^1$.

Example 10.13. *Take $v^3 v^2 v^1 = (1)(2)(32)$ of Example 10.9. The element $(1)(2)(32)$ is highest weight of weight $(2, 1, 1)$ and the column-reading word of the conjugate of P*

$$P' = \begin{array}{|c|c|c|} \hline 1 & 2 & 3 \\ \hline 3 & & \\ \hline \end{array}$$

is 3123, which is indeed a reduced word for $s_1 s_2 s_3 s_2$. This demonstrates the bijective correspondence of Corollary 10.12.

Another immediate outcome of the crystal $\mathcal{B}(w)$ on decreasing factorizations is Stanley's famous result [Stanley (1984)] that the number of reduced expressions for the longest element $w_0 \in S_n$ is equal to the number of standard tableaux of staircase shape $\rho = (n-1, n-2, \dots, 1)$. See Theorem 7.18. Namely, in $\mathcal{B}(w_0)$ there is only one highest weight element given by the factorization $(s_1)(s_2s_1)(s_3s_2s_1) \cdots (s_{n-1}s_{n-2} \cdots s_1)$. Hence $\mathcal{B}(w_0)$ is isomorphic to the highest weight crystal \mathcal{B}_ρ . The reduced words of w_0 are precisely given by the factorizations of weight $(1, 1, \dots, 1)$. In \mathcal{B}_ρ they are the standard tableaux of shape ρ . The bijection between the reduced words of w_0 and standard tableaux of shape ρ induced by the crystal isomorphism is precisely φ_{EG}^Q (which, due to the initial reversal of the factorization, gives the conjugate of the standard tableau from the straight EG insertion). An example of this crystal isomorphism for $\mathcal{B}(s_1s_2s_1)$ in S_3 is given in Figure 10.1.

By Theorem 10.10, the crystal $\mathcal{B}(w)$ relates to the crystals on the recording tableaux under the EG correspondence. It was proved in [Edelman and Greene (1987)] that two reduced words EG insert to the same P tableau if and only if they are Coxeter–Knuth equivalent. Two reduced words are Coxeter–Knuth equivalent if one can be obtained from the other by a sequence of Coxeter–Knuth relations on three consecutive letters

$$(a+1)a(a+1) \sim a(a+1)a, \quad bac \sim bca, \quad cab \sim acb, \quad (10.5)$$

where the last two relations only hold when $a < b < c$. The Coxeter–Knuth graph $\mathcal{CK}(w)$ for $w \in S_n$ is a graph on the reduced words for w where two words are connected if they differ by a relation in (10.5).

There is an interesting relation between the crystal $\mathcal{B}(w)$ and its decomposition into irreducible components and the connected components of the Coxeter–Knuth graph.

Proposition 10.14. *Let $w \in S_n$. The connected components of $\mathcal{CK}(w)$ are in one-to-one correspondence with the connected components of $\mathcal{B}(w)$.*

Proof. Every reduced word of w can be viewed as an element of $\mathcal{B}(w)$ by placing each letter in its own factor (assuming that ℓ is bigger than $\ell(w)$). Suppose $\mathbf{i}, \mathbf{j} \in \text{Red}(w)$ differ by a single relation (10.5) with \mathbf{i} having 3 consecutive letters of the left hand side and \mathbf{j} the corresponding letters of the right hand side. Viewing \mathbf{i} and \mathbf{j} as elements of $\mathcal{B}(w)$, it is not hard to check that $f_i f_{i+1} e_i e_{i+1}(\mathbf{i}) = \mathbf{j}$ which proves that two elements in the same component in $\mathcal{CK}(w)$ are also in the same crystal component.

Conversely, suppose $b, b' \in \mathcal{B}(w)$ with $e_i(b) = b'$, so that b and b' lie in the same component in $\mathcal{B}(w)$. We can view b and b' as reduced words of w by disregarding the grouping into factors. By [Morse and Schilling (2016), Lemma 3.8], b' is obtained from b by a sequence of braid and commutation moves. By a close inspection of the proof of [Morse and Schilling (2016), Lemma 3.8] only the Coxeter–Knuth relations (10.5) are used. A similar argument holds for f_i . This implies that if

$b, b' \in \mathcal{B}(w)$ are in the same component, then the corresponding reduced words are in the same component in $\mathcal{CK}(w)$. \square

Given that the Edelman–Greene correspondence maps a factorization to a pair of tableaux and Theorem 10.10 relates the crystal on decreasing factorizations to the crystal on the recording tableaux Q , a natural question to ask is whether there is a “dual” crystal on the P -tableaux. By [Hamaker and Young (2014)], two reduced words have the same recording tableau under the EG insertion if and only if they are connected by Little bumps [Little (2005)]. See Exercise 10.3.

In [Morse and Schilling (2016)], the crystal is defined more generally on certain affine permutations into cyclically decreasing factors. Lam [Lam (2006)] defined analogues of the Stanley symmetric functions in terms of cyclically decreasing elements. In [Morse and Schilling (2016)], the crystal on these affine permutations is used to study k -Schur structure coefficients and further applications to flag Gromov–Witten invariances, fusion coefficients, and positroid varieties.

Exercises

Exercise 10.1. Formulate and prove the analog of Lemma 10.4 for the crystal operators f_i . Also prove, for $x, y \in \mathcal{W}_w^\ell$ that $e_i(x) = y$ if and only if $f_i(y) = x$. Deduce that \mathcal{W}_w^ℓ is a crystal.

Exercise 10.2. Prove Theorem 10.6 in several steps.

- (1) Prove that $\mathcal{B}(w)$ satisfies the Stembridge Axioms S0–S3 of Section 4.2.
- (2) Deduce Stembridge Axioms S0'–S3' for $\mathcal{B}(w)$ by appealing to the order reversing map $\star: i \mapsto n - i$, which is extended to words by $a_1 \cdots a_h \mapsto a_h^\star \cdots a_1^\star$ and decreasing factorizations $\star: w^\ell \cdots w^1 \mapsto w^{1^\star} \cdots w^{\ell^\star}$, where w^{i^\star} is the element in S_n corresponding to $\text{cont}(w_i)^\star$.

Hint: See [Morse and Schilling (2016), Appendix].

Exercise 10.3. (Open) Describe a “dual” crystal structure on the P -tableaux under the Edelman–Greene correspondence.

Exercise 10.4. (Open) Generalize the results of this chapter to root systems beyond type A .