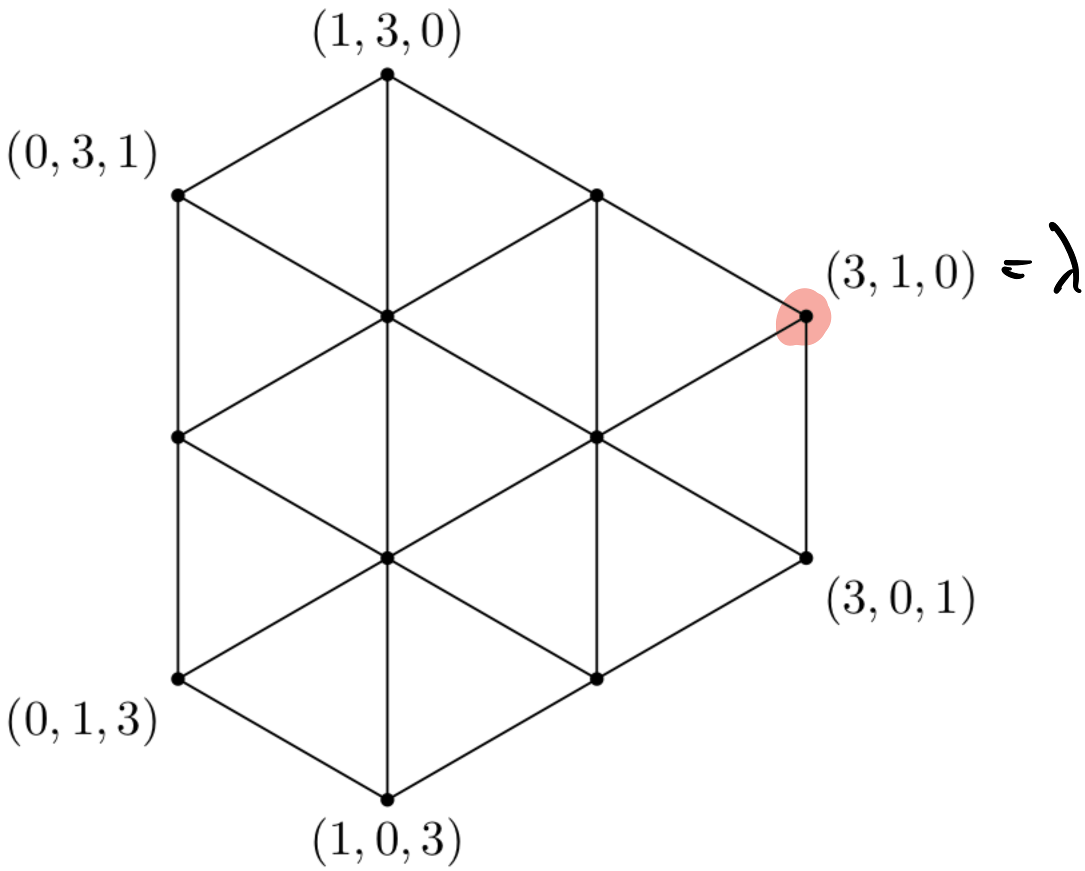
DEMAZURE CHARACTERS,
DEMAZURE CRYSTALS
LATTICE MODELS WHOSE PARTITION FUNCTIONS ARE
DEMAZURE CHARACTERS.

REVIEW:

$$\partial_i = (1 - z^{-\alpha_i})^{-1}(1 - z^{-\alpha_i}\Delta_i)$$

$\partial_i f$ SYMMETRIZES IT PARALLEL TO α_i ROOT
SO $\Delta_i \partial_i f = \partial_i f$



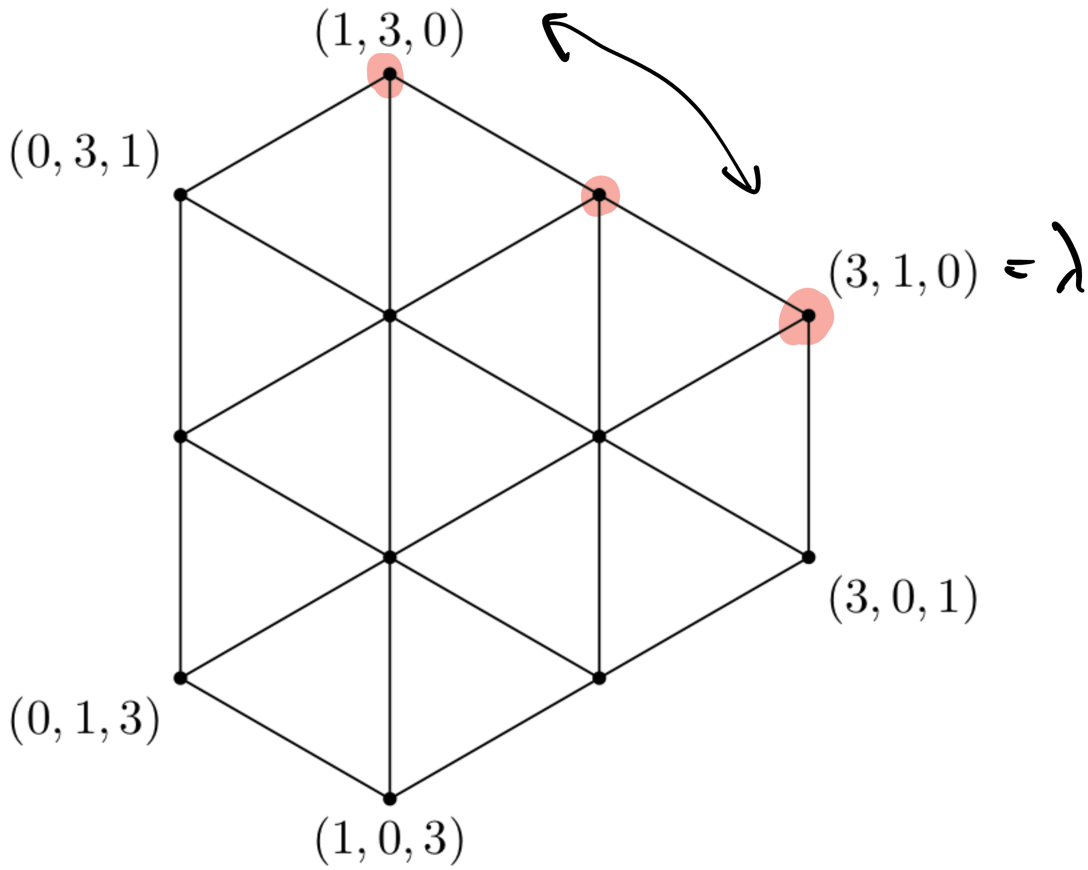
INSIDE GL(3) WEIGHT LATTICE

$z^\lambda = z_1^3 z_2$

$\alpha_1 = (1,-1,0)$

$s_1(x,y,z) = (y,x,z)$

$\partial_i z^\lambda = z_1^3 z_2 + z_1^2 z_2 + z_1 z_2^3$



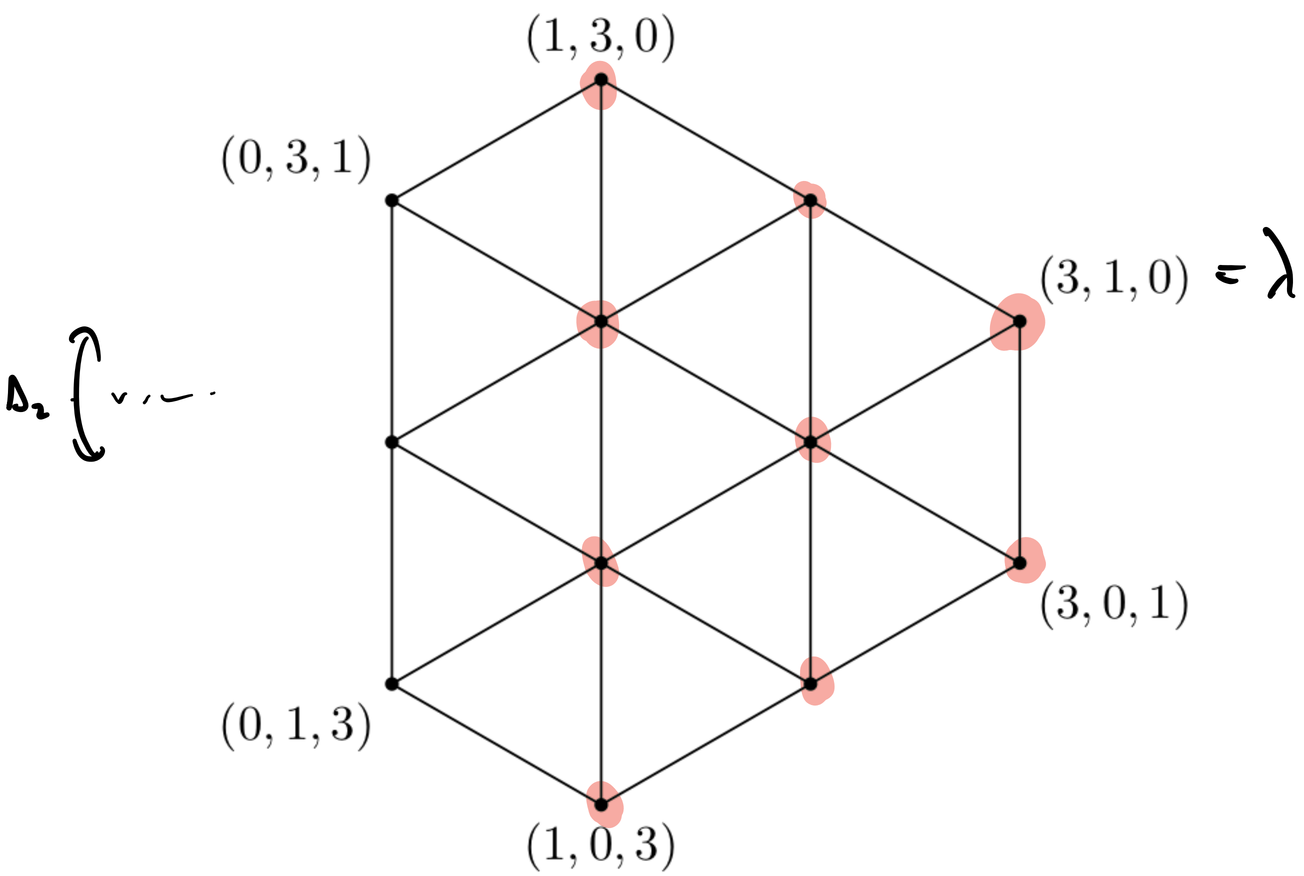
$\partial_1 z^\lambda$

∂_i SATISFY BRAID RELATIONS SO BY MATSUMOTO'S THM WE CAN DEFINE ∂_w FOR $w \in W$ $\quad w = s_{i_1} \cdots s_{i_k}$ (REDUCED)

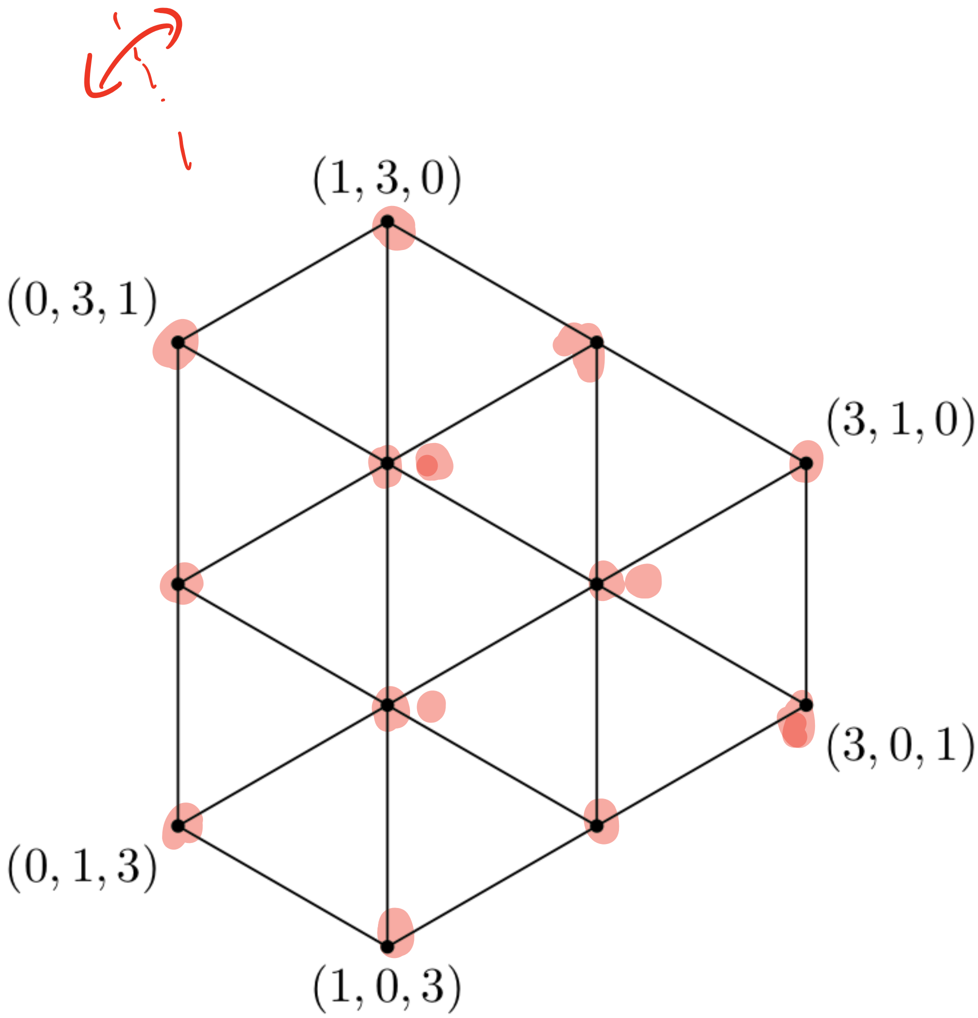
$\partial_w = \partial_{i_1} \partial_{i_2} \cdots \partial_{i_k}$

IF $w = w_0$ AND λ IS DOMINANT

$$\partial_{w_0} z^{\lambda} = \text{IRR CHAR WITH HW } \lambda$$
$$= \Delta_\lambda(z) \quad \text{SCHUR FUNCTION.}$$



$\partial_2 \partial_1 z^{\lambda}$



INNER WEIGHTS $(2,1,1)$, $(1,2,1)$ $(1,1,2)$
HAVE MULTIPLICITY 2.

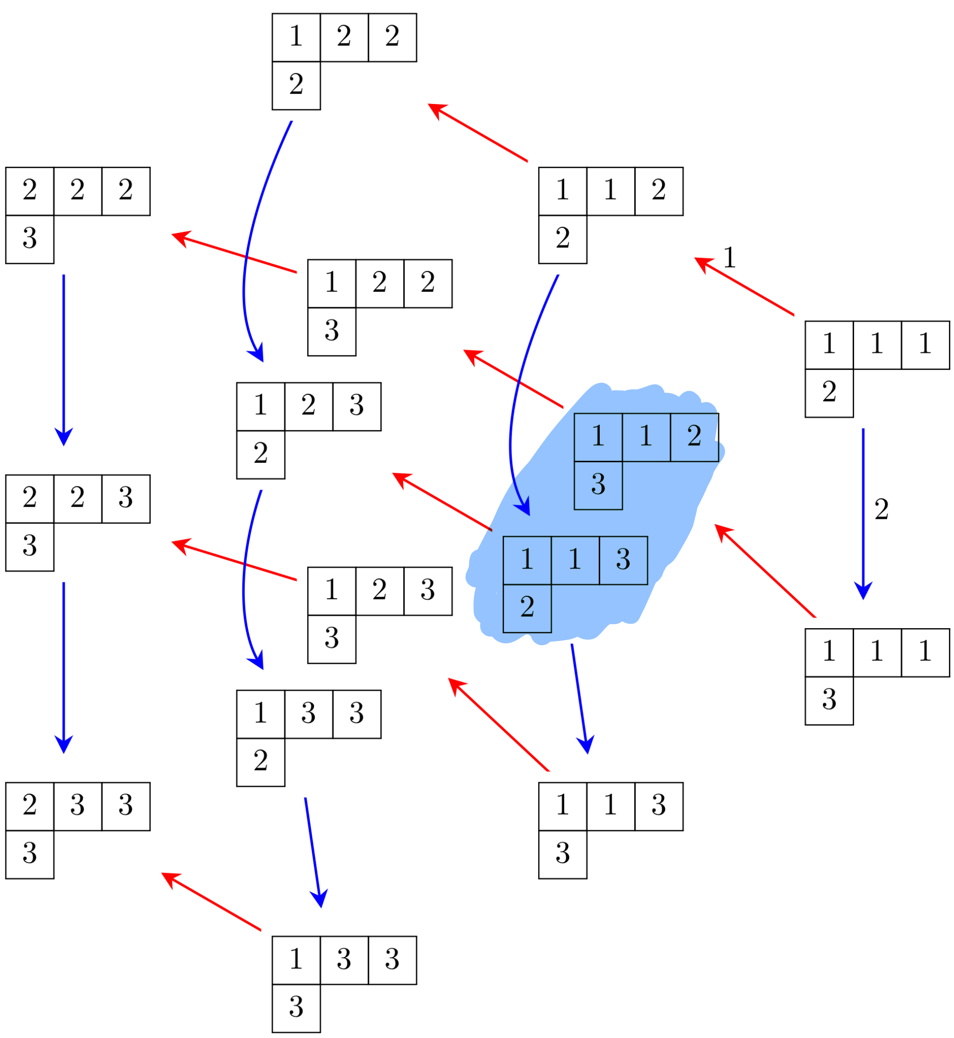
$$\partial_{w_0} z^{\lambda} = \Delta_{\lambda}(z).$$

PROOF IN MY LIE GROUPS BOOK.

COMBINATORIAL DEF OF SCHUR POLYNOMIAL
(D.E. LITTLEWOOD)

$$s_\lambda(z) = \sum_{T} z^{wt(T)}$$

$T \in$ SSYT OF SHAPE λ



THERE ARE 2 SSYT OF SHAPE $(3,1,0)$ AND WEIGHT $(2,1,1)$

THESE CONTRIBUTE

$$z_1^2 z_2 z_3 \quad \text{AND} \quad z_1^2 z_2 z_3$$

EXPLAINING WHY THIS WEIGHT HAS MULTIPLICITY 2 IN THE SCHUR POLYNOMIAL.

COMBINATORIAL DEF OF Δ_λ IS MYSTERIOUS SINCE IT DOES NOT MAKE CLEAR WHY

$\sum_T z^{wt(T)}$ IS SYMMETRICAL. KNUTH GAVE A COMBINATORIAL PROOF BY CONSTRUCTING OPERATORS ON TABLEAUX MAKING THE Δ_i-SYMMETRY CLEAR.

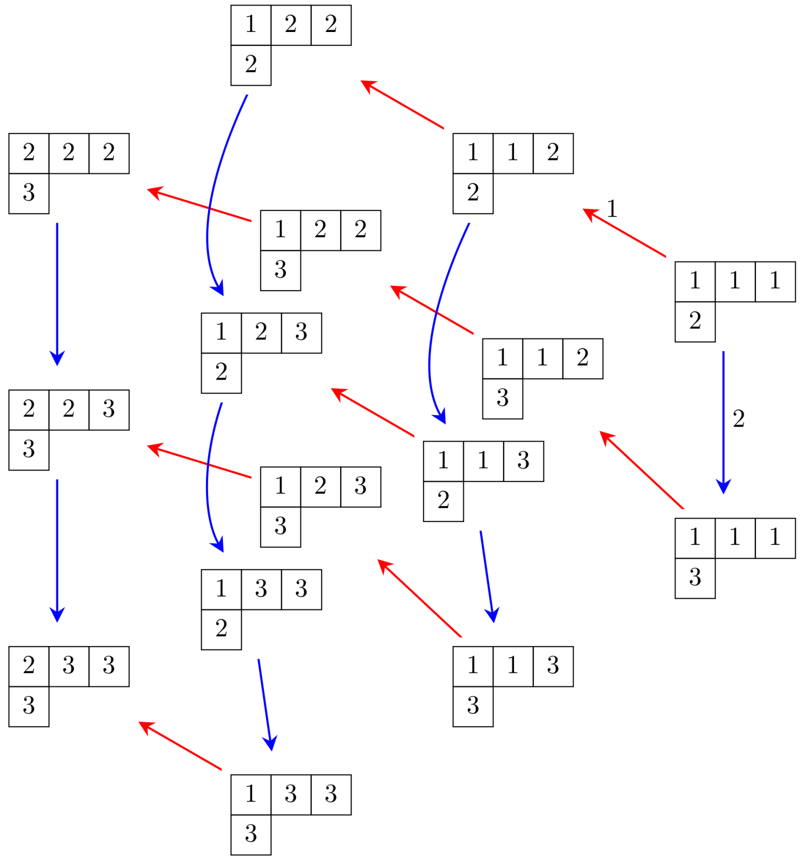
KASHIWARA AND NAKASHIMA SHOW THE SET OF SSYT HAVE A STRUCTURE OF A "CRYSTAL" WITH REMARKABLE COMBINATORIAL PROPERTIES.

A KASHIWARA CRYSTAL $\mathcal{C}$ IS A DIRECTED GRAPH. IT COMES WITH A MAP $wt : \mathcal{C} \to \Lambda$ (WEIGHT LATTICE.)

THE EDGES ARE LABELED BY ROOTS

CONSIDER AN EDGE $\quad x \xrightarrow{i} y$

IF THIS EDGE EXISTS $\quad wt(y) = wt(x) - \alpha_i$



BLUE: 2

RED: 1

$$\overset{x}{\begin{array}{|c|c|c|}\hline 1 & 1 & 2 \\ \hline 2 & \multicolumn{2}{c}{} \\ \cline{1-1}\end{array}} \xrightarrow{\quad 2 \quad} \overset{y}{\begin{array}{|c|c|c|}\hline 1 & 1 & 3 \\ \hline 2 & \multicolumn{2}{c}{} \\ \cline{1-1}\end{array}}$$

$wt = (2,2,0)$ $\qquad$ $(2,1,1) = wt(x) - \alpha_2$

$\alpha_1 = (1,-1,0)$

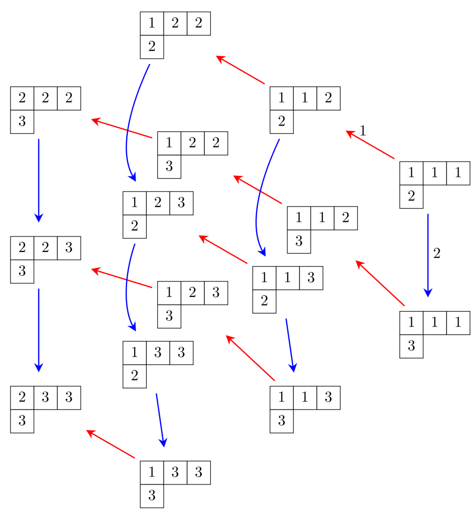
KASHIWARA OPERATORS $e_i, f_i : \mathcal{C} \to \mathcal{C} \cup \{0\}$

0 IS AN AUXILIARY SYMBOL DENOTING FAILURE TO APPLY THE OPERATOR.

$e_i(x)$ = THE UNIQUE ELT z SUCH THAT AN EDGE

$$\underset{z}{\bullet} \xrightarrow{i} \underset{x}{\bullet}$$

EXISTS. $e_i(x) = 0$ IF NO SUCH EDGE.



$$e_1\left(\begin{array}{ccc}1&1&2\\2&&\end{array}\right) = \begin{array}{|c|c|c|}\hline 1&1&1\\\hline 2&&\\\hline\end{array}$$

$f_i(x)$ = UNIQUE y ST $\overset{x}{\bullet} \xrightarrow{i} \overset{y}{\bullet}$

0 IF NO SUCH EDGE.

$$\varepsilon_i,\ \varphi_i : C \to \mathbb{Z} \quad (\text{EVEN } N)$$

$$\varepsilon_i(x) = \max\{a \mid e_i^a(x) \neq 0\},$$

= # OF TIMES e_i CAN BE APPLIED.

CRYSTALS WERE INVENTED INDEPENDENTLY BY KASHIWARA, LUSZTIG, P. LITTELMANN.

LITTELMANN AND KASHIWARA PROVED A REFINED DEMAZURE CHARACTER FORMULA.

THE OPERATORS ∂_i : FUNCTIONS ON T $\to \mathcal{O}(T)$
$\mathcal{O}(T)$

T = TORUS WHOSE RING OF REGULAR FUNCTIONS IS Λ.

$\mathcal{O}(T)$ = SPAN OF z^λ. $(\lambda \in \Lambda)$.

OR $\partial_i : \mathbb{Z}[\Lambda] \to \mathbb{Z}[\Lambda]$

CAN BE LIFTED TO $\mathbb{Z}[\mathcal{B}_\lambda] \to \mathbb{Z}[\mathcal{B}_\lambda]$

$\mathcal{B}_\lambda$ = KASHIWARA CRYSTAL OF SSYT OF SHAPE λ.

IF $T \in \mathcal{B}_\lambda$ SUPPOSE $\overset{k}{\overset{\shortparallel}{\langle wt(T), \alpha_i^\vee \rangle}} \geq 0$

$$\partial_i z^\mu = z^\mu + z^{\mu-\alpha_i} + z^{\mu-2\alpha_i} + \dots + z^{\mu-k\alpha_i}$$

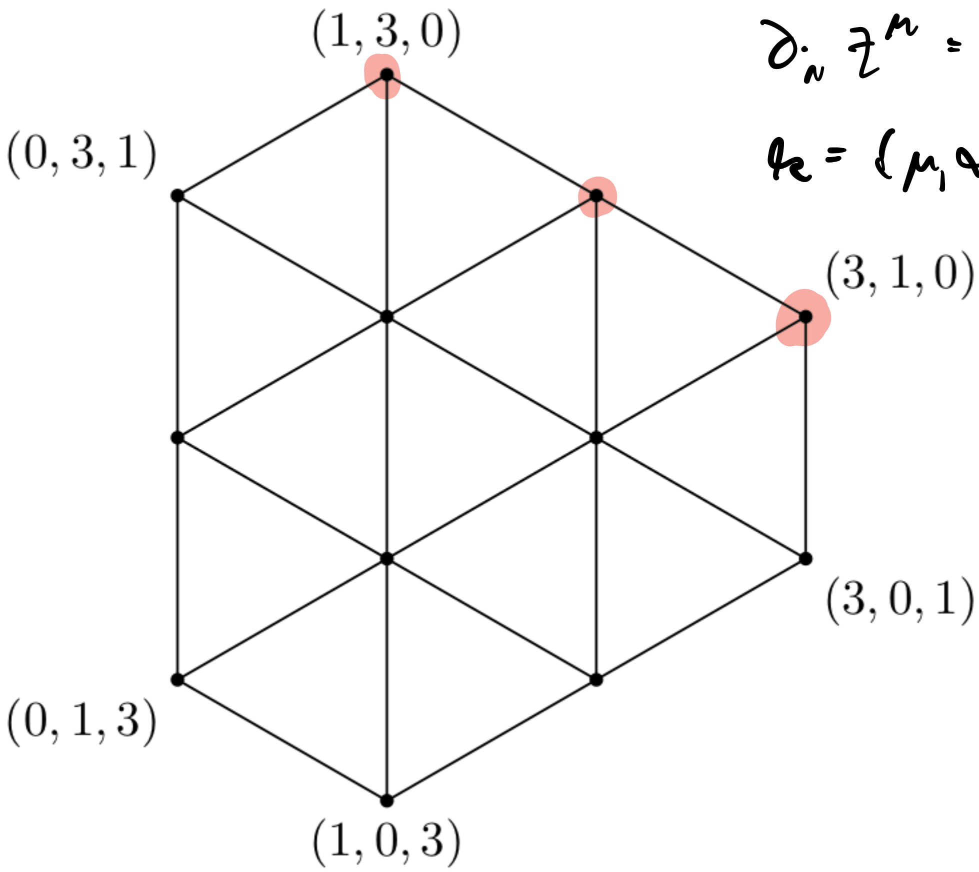
$\mu = wt(T)$

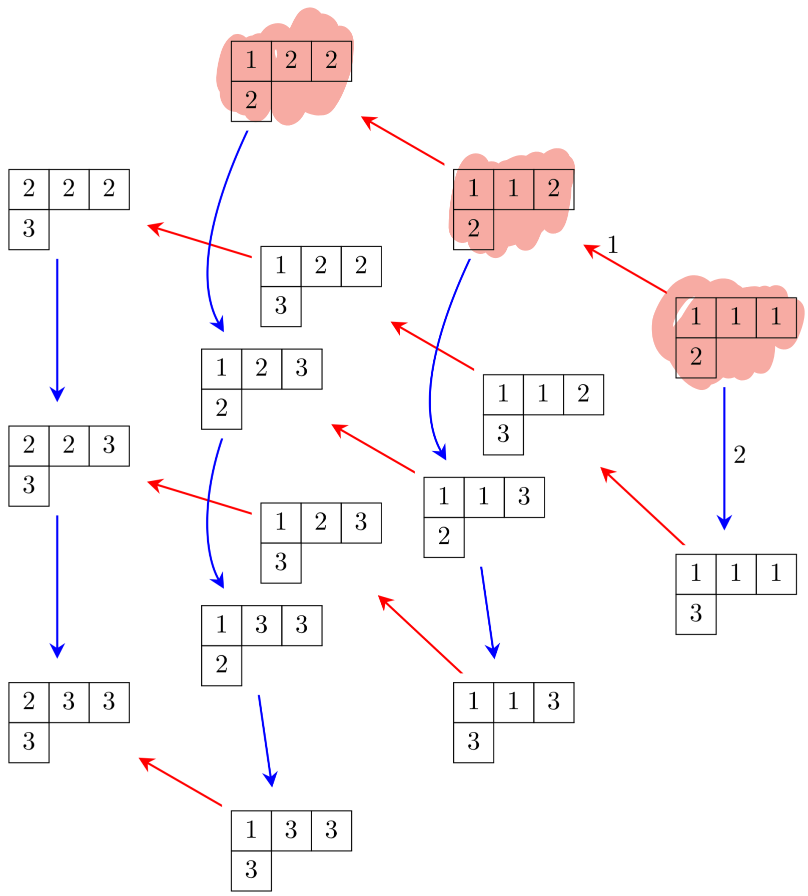
$$\partial_i z^\mu = \sum_{j=0}^{k} z^{\mu-j\alpha_i}$$

$\mu = (3,1,0)$

$\partial_i z^\mu = z^\mu + z^{\mu-\alpha_1} + z^{\mu-2\alpha_1}$

$k = \langle \mu, \alpha_i^\vee \rangle = 2$





DEFINITION: (LITTELMANN)

$$\partial_i T = \sum_{j=0}^{k} f_i^{\,j}(T) \quad \text{IN} \quad \mathbb{Z}[B_\lambda]$$

$$\partial_i \begin{array}{|c|c|c|}\hline 1 & 1 & 1 \\ \hline 2 \\ \cline{1-1}\end{array} = \begin{array}{|c|c|c|}\hline 1 & 1 & 1 \\ \hline 2 \\ \cline{1-1}\end{array} + \begin{array}{|c|c|c|}\hline 1 & 1 & 2 \\ \hline 2 \\ \cline{1-1}\end{array} + \begin{array}{|c|c|c|}\hline 1 & 2 & 2 \\ \hline 2 \\ \cline{1-1}\end{array}$$

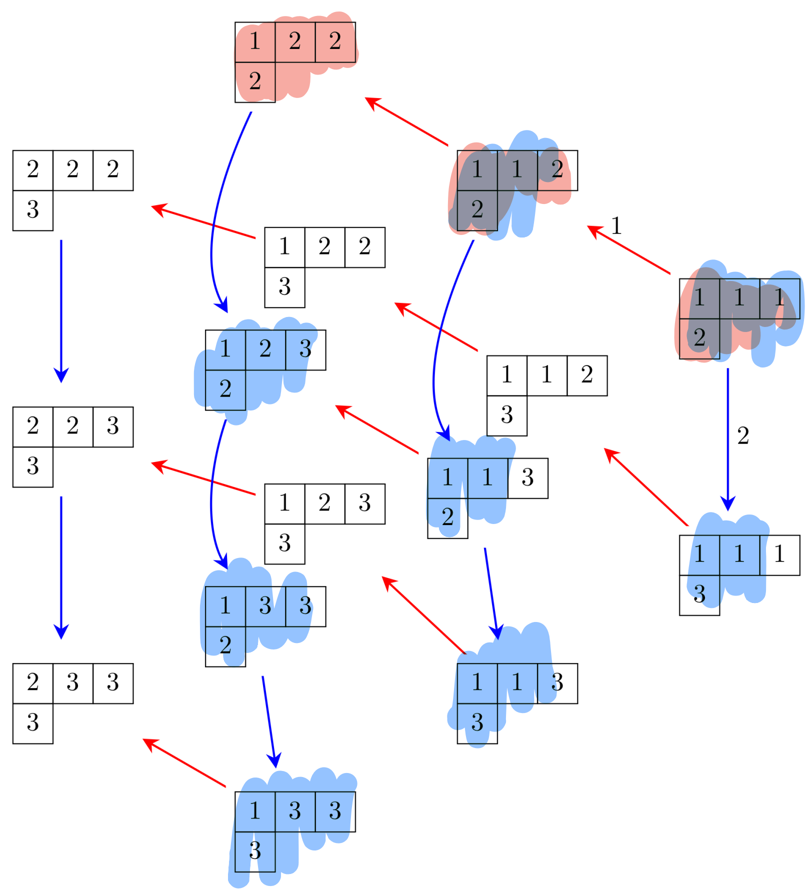
THEOREM (LITTELMANN) APPLIED TO THE HIGHEST WEIGHT ELT THESE OPERATORS BRAID SO

$$\partial_w T_\lambda = \partial_{i_{n_1}} \cdots \partial_{i_{n_r}} T_\lambda$$

HIGHEST WEIGHT ELT.

$$w = \Delta_{i_{n_1}} \cdots \Delta_{i_{n_r}}$$

IS INDEPENDENT OF THE REDUCED EXPRESSION.

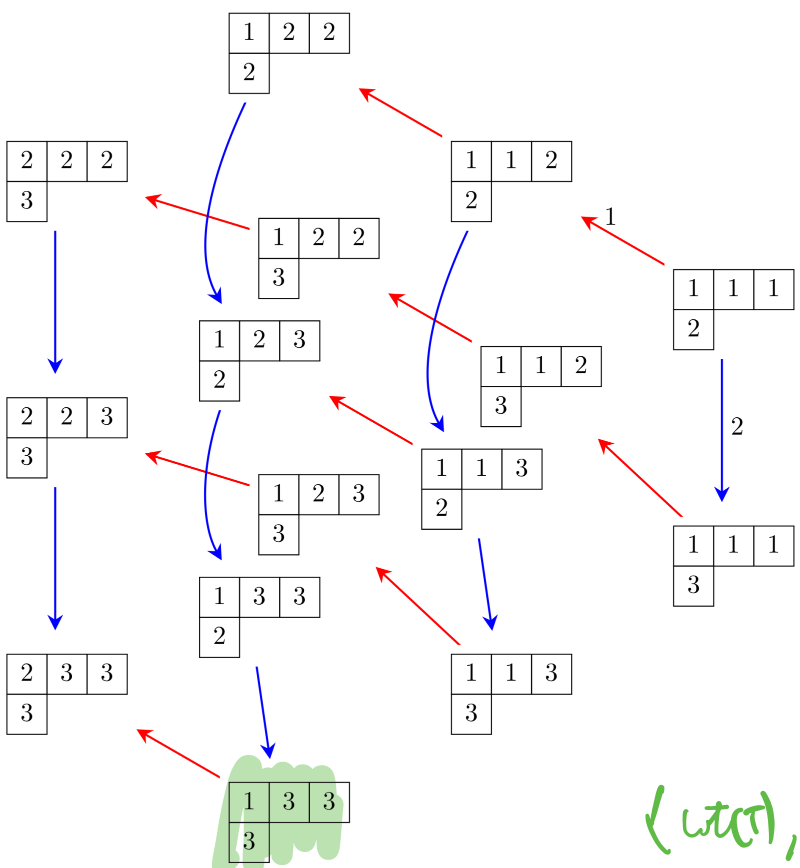


IF $\langle wt(T), \alpha_i^\vee \rangle < 0$

$\partial_i T = -e_i(T) - e_i^2(T) \cdots$

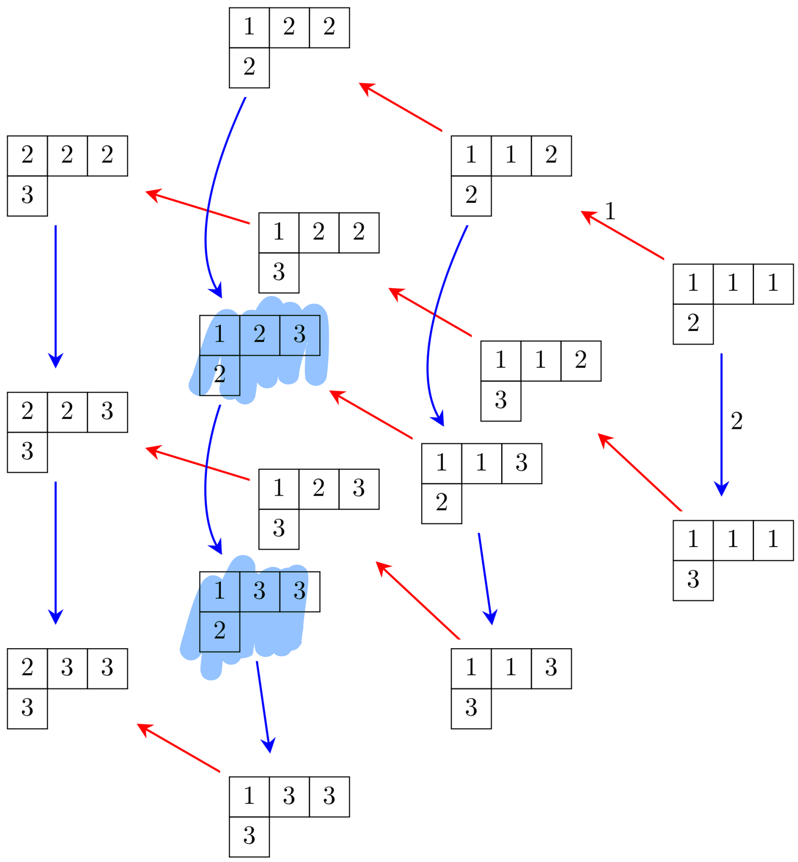
\# OF APPLICATIONS IS

$-\langle wt(T), \alpha_i^\vee \rangle - 1$. (I THINK



$\langle wt(T), \alpha_2^\vee \rangle$

$\langle (1,0,3), (0,1,-1) \rangle = -2$.



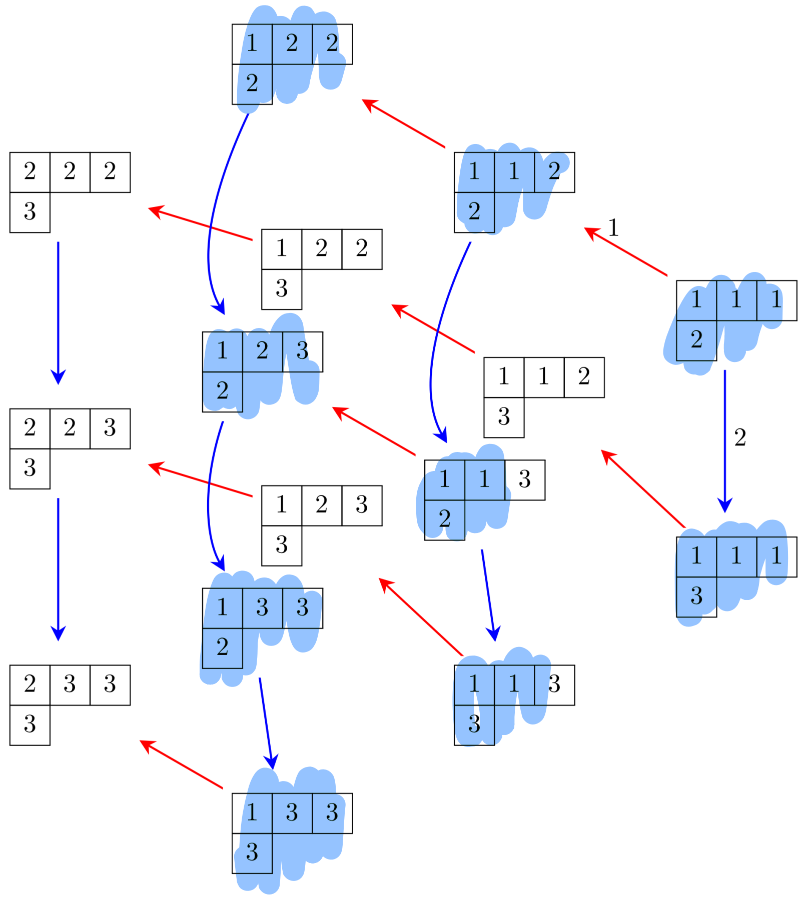
$$\partial_2 \begin{bmatrix} 1 & 3 & 3 \\ 3 \end{bmatrix} = - \begin{bmatrix} 1 & 3 & 3 \\ 2 \end{bmatrix} - \begin{bmatrix} 1 & 2 & 3 \\ 2 \end{bmatrix}$$

$$e_2(\cdots)$$

THEOREM: THERE IS A SUBSET OF $\mathcal{B}_\lambda$ $\mathcal{B}_\lambda(w)$ SUCH THAT IN $\mathbb{Z}[\Lambda]$

$$\partial_w T_{HW.} = \sum_{v \in \mathcal{B}_\lambda(w)} v$$

THIS IS CALLED A DEMAZURE CRYSTAL.



$w = s_2 s_1$

THIS IS THE DEMAZURE CRYSTAL $\mathcal{B}(s_2 s_1)$.

$\mathcal{B}(w_0) = \mathcal{B}$.