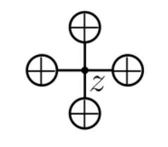
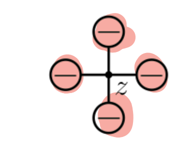
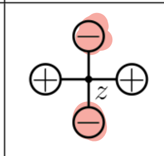
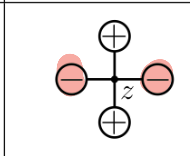
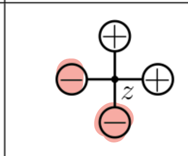
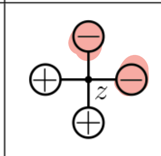
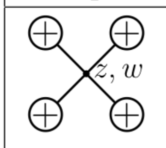
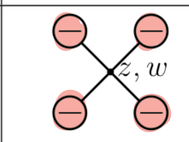
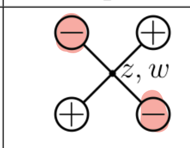
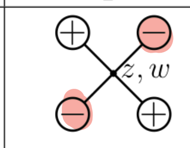
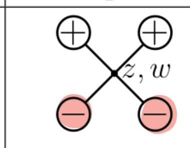
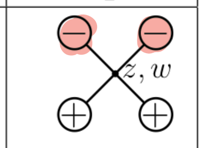
Yang-Baxter Equation.

"T-VERTEX"

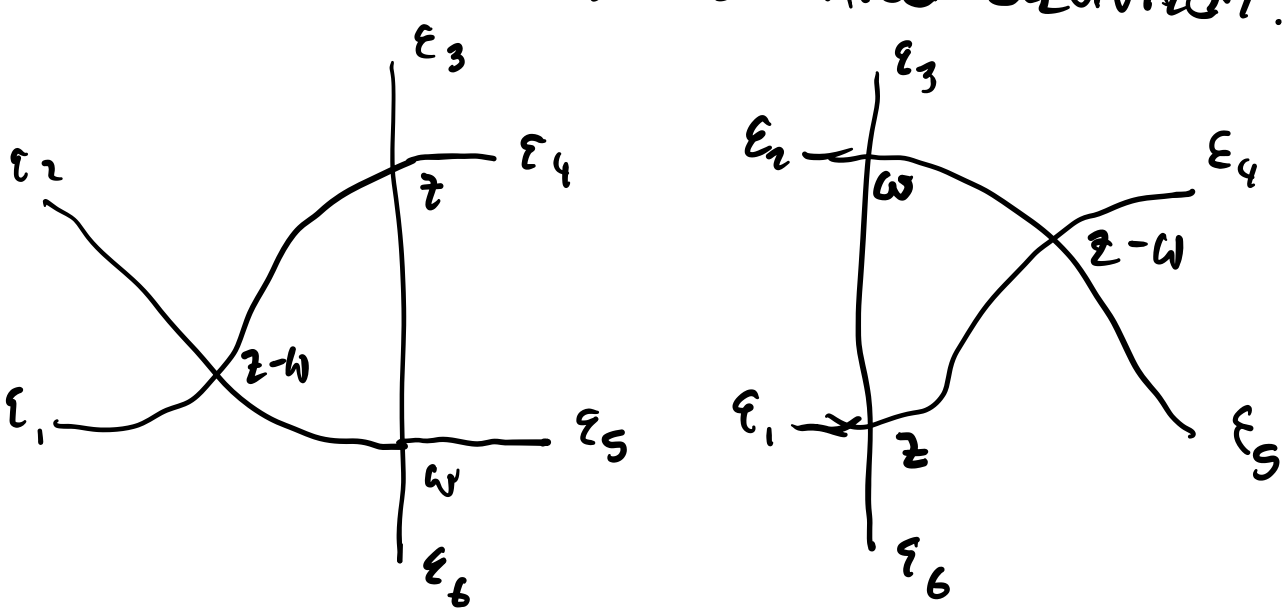
a_1	a_2	b_1	b_2	c_1	c_2
					
1	z	$-q$	z	$z(1-q)$	1

$z = z_i$ IN i-TH ROW.

a_1	a_2	b_1	b_2	c_1	c_2
					
$w-qz$	$z-qw$	$q(z-w)$	$z-w$	$(1-q)z$	$(1-q)w$

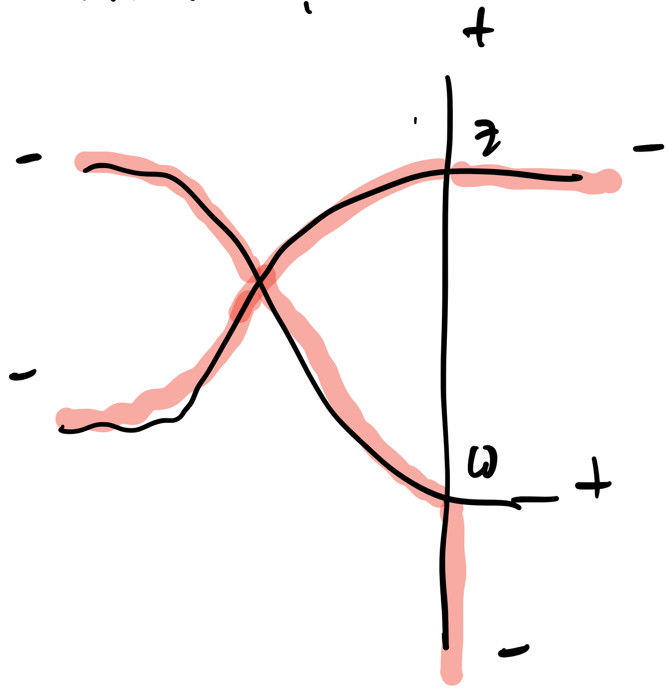
R-MATRIX.

THEOREM: FOR EVERY CHOICE OF SPINS $\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4, \varepsilon_5, \varepsilon_6 \in \{\oplus, \ominus\}$ THE FOLLOWING SYSTEMS ARE EQUIVALENT.



(SUM OVER SPINS ON INTERIOR EDGES

EXAMPLE:



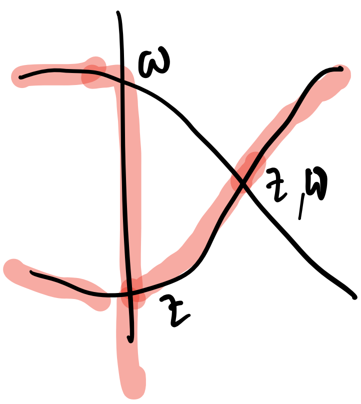
$(z-qw)$

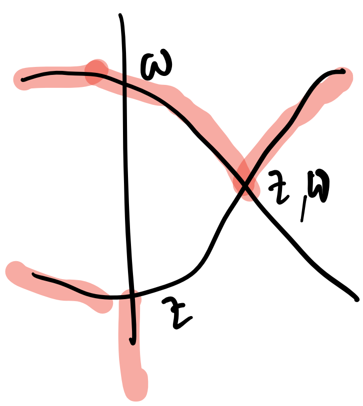
$\times\ z$

$\times\ w(1-q)$

a_1	a_2	b_1	b_2	c_1	c_2
z,w	z,w	z,w	z,w	z,w	z,w
$w-qz$	$z-qw$	$q(z-w)$	$z-w$	$(1-q)z$	$(1-q)w$

a_1	a_2	b_1	b_2	c_1	c_2
z	z	z	z	z	z
1	z	$-q$	z	$z(1-q)$	1





$(1-q)w$
z
$(z-w)$

w
$(1-q)z$
$(1-q)w$

$$(1-q)wz\left((z-\cancel{w}) + (\cancel{1}-q)w\right)$$

$$= (1-q)wz(z-qw) =$$

$(z-qw)$
$\times z$
$\times w(1-q)$

RHS LHS

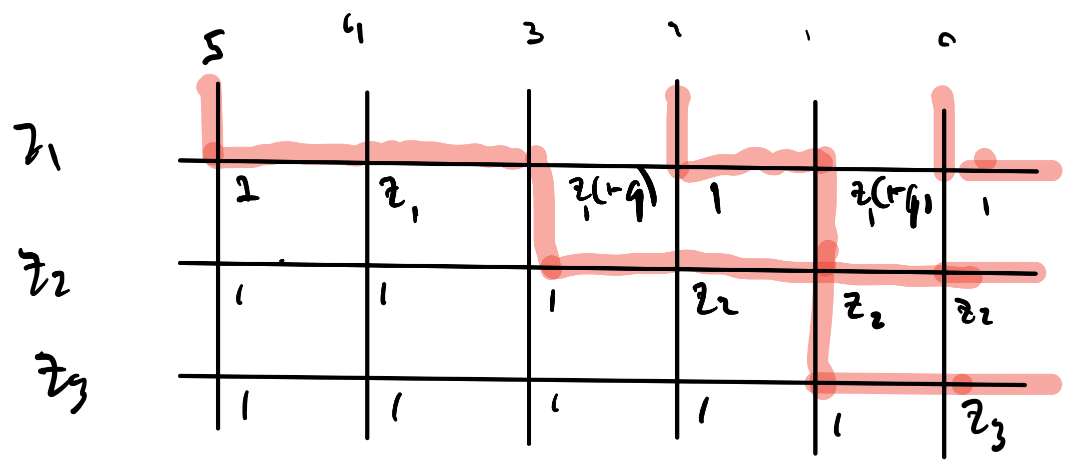
ALL 64 CASES CHECK.

λ A PARTITION $\lambda = (\lambda_1, \dots, \lambda_n)$

PUT $\ominus$ SPINS ON RIGHT BOUNDARY

$\oplus$ ON LEFT AND BOTTOM AND AT TOP

$\ominus$ IN COLUMNS $\lambda_i + n - i$



$n = 3$

$\lambda = (3, 1, 0)$

THEOREM: THERE IS A SYMMETRIC POLYNOMIAL $S_\lambda(x_1, \dots, x_n)$ SUCH THAT

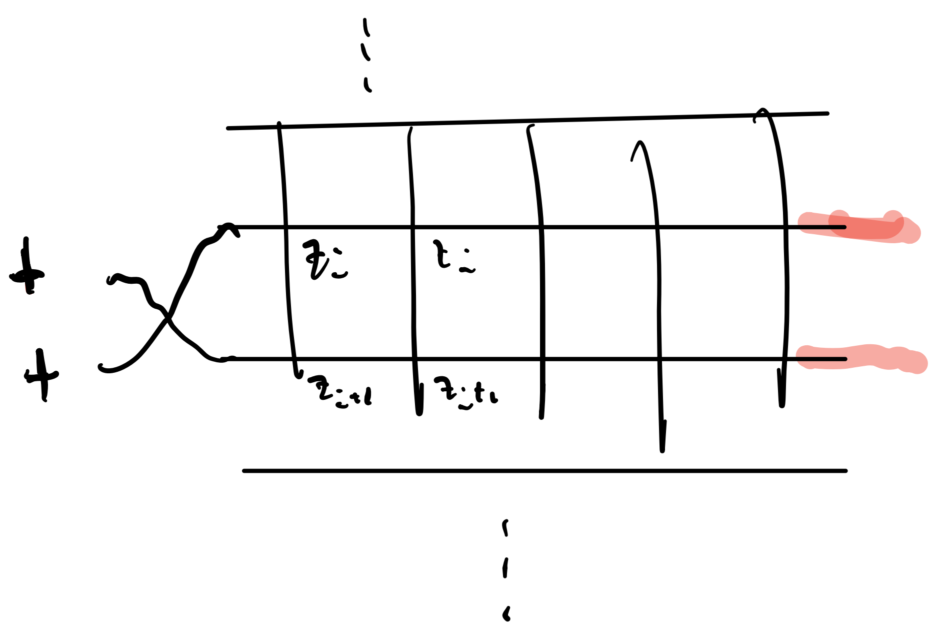
$$Z(S_\lambda) = \prod_{i<j} (z_i - q z_j) \, S_\lambda(x_1, \dots, x_n)$$

SAME

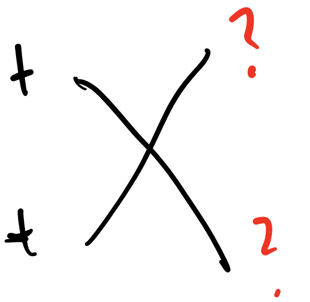
LEMMA:

$$(z_{i+1} - q z_i) \, Z(z_1, \dots, z_n; q) = (z_{i+1} - q z_i) \, Z(z; q)$$
$$= (z_i - q z_{i+1}) \, Z(s_i z; q)$$

PROOF: ATTACH R-MATRIX BETWEEN i, $i+1$ ROWS



CONSIDER THIS SYSTEM. FOR R-MATRIX



ONLY ONE POSSIBILITY

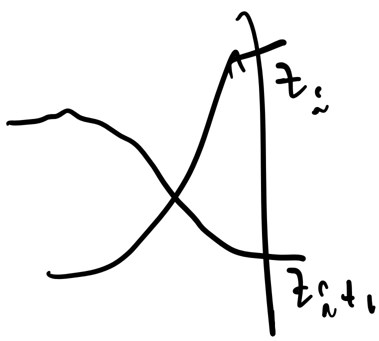
? = +

$z_{i_{n+1}} - q z_{i_n}$

THIS MODIFIED SYSTEM HAS PARTITION FN

$(z_{i_{n+1}} - q z_{i_n}) \times Z(z; q)$ = LHS IN LEMMA

TRAIN ARGUMENT:



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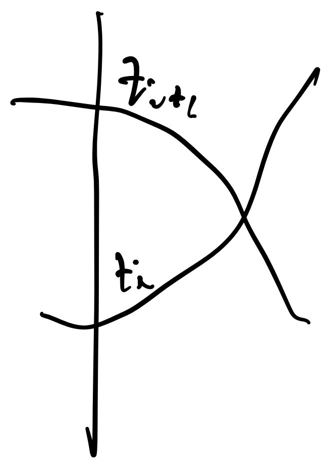
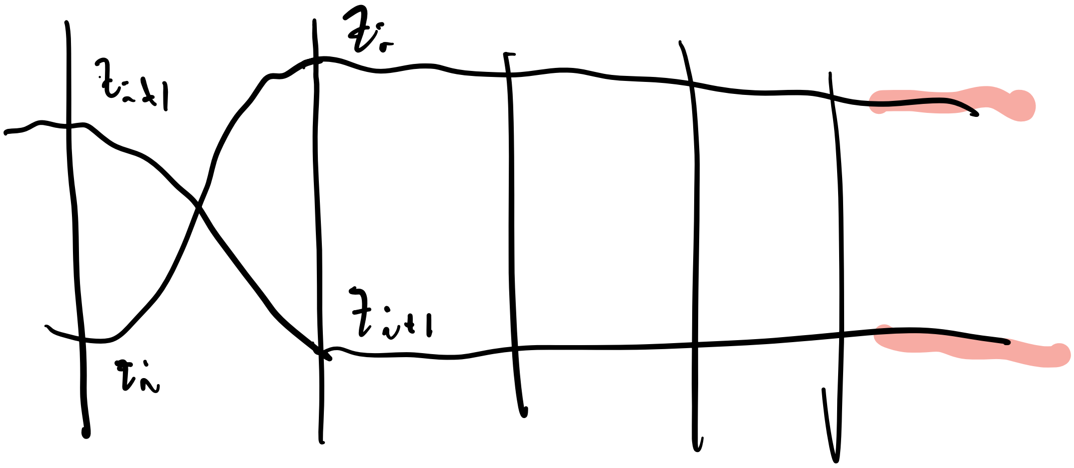
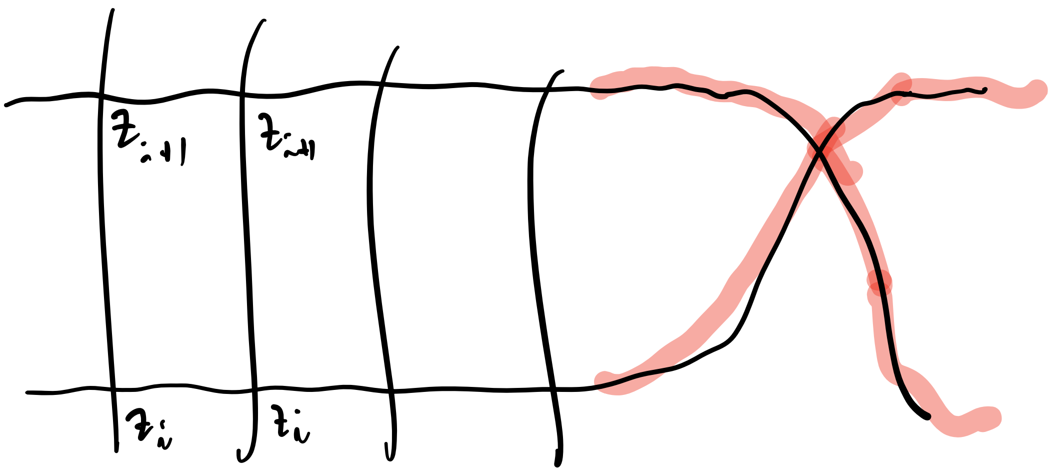




REPEAT UNTIL TRAIN ARRIVES ON RSA



$$= (z_i - q z_{i+1}) Z(\Delta_i z; q)$$

a_1	a_2	b_1	b_2	c_1	c_2
z, w	z, w	z, w	z, w	z, w	z, w
$w - qz$	$z - qw$	$q(z - w)$	$z - w$	$(1 - q)z$	$(1 - q)w$

$z_i - q z_{i+1}$

$Z(z;q)$ IS A POLY IN $z_1, \dots, z_n, q$

OF DEG

$$\frac{Z(z;q)}{\prod_{i<j} (z_i - q z_j)}$$

CLAIM THIS IS W-INVARIANT

$$W = S_n = \langle s_1, \dots, s_{n-1} \rangle$$

ENOUGH TO CHECK UNCHANGED UNDER s_i

$$= \underbrace{\frac{Z(z;q)}{z_i - q z_{i+1}}}_{\text{UNCHANGED BY LEMMA.}} \underbrace{\prod_{\substack{i<j \\ (i,j) \neq (i,i+1)}} (z_i - q z_j)^{-1}}_{\text{PERMUTED BY } s_i}$$

$$= \frac{Z(\Delta_i z; q)}{z_{i+1} - q z_i} \times \prod \text{ OTHER FACTORS.}$$

$$= \Delta_i \left(\frac{Z(z; q)}{\prod_{i<j} (z_i - q z_j)} \right)$$

SO $$Z(S_\lambda) = \prod_{i<j} (z_i - q z_{i+1}) \, S_\lambda(z; q)$$

WHERE $S_\lambda(z; q)$ IS SYMMETRIC IN z_i

WANT TO SHOW $S_\lambda(z; q)$ IS POLYNOMIAL AND IND. OF q. FROM LEMMA:

$$(z_{i+1} - q z_i) Z(z; q) = (z_i - q z_{i+1}) Z(\Delta_i z; q)$$

$\Rightarrow$ $z_i - q z_{i+1}$ DIVIDE $Z(z; q)$

IN $\mathbb{C}[z_1, \dots, z_n; q]$ UFD.

$z_i - q z_{i+1}$ $\quad$ $z_{i+1} - q z_i$ COPRIME

$$S_\lambda(z;q) = \frac{Z(\cdots)}{\prod_{i<j}(z_i - q z_j)}$$

DIVISIBLE BY $z_i - q z_{i+1}$.

AND IT IS SYMMETRIC IN

FIELD OF FRACTIONS OF $\mathbb{C}[z_1, \ldots, z_n; q]$

SO IT IS DIVISIBLE BY ALL.

THE FACTORS IN DENOM ARE COPRIME

$Z(\cdots)$ IS DIVISIBLE BY EACH SO IT IS DIVISIBLE BY THE PRODUCT. SO

$$\frac{Z(z;q)}{\prod(z_i - q z_j)}$$ IS A POLYNOMIAL.

IT IS INDEPENDENT OF q SINCE NUMERATOR, DENOM ARE POLYNOMIALS OF DEGREE $\frac{1}{2}n(n-1)$ IN q.

\# OF FACTORS IN DENOM

\# OF $\ominus$ SPINS ON VERTICAL EDGES IN ANY STATE.

$$U_q(\mathfrak{gl}(1|1))$$

$$U_q(\mathfrak{sl}_2) \qquad q = e^{2\pi i/3}.$$