TOKUYAMA MODELS

INVOLVE A PARAMETER q

$q=0$ INTERESTING

$q=1$ INTERESTING

REF: "BOOK" CH. 4.

AGGARWAL

373 (T, TH 10:30)
381T

APPEARED IN AGGARWAL'S THURSDAY LECTURE AS FERMIONIC MODELS FOR SCHUR POLYNOMIALS.

HISTORY: FREE-FERMIONIC 6-VERTEX MODEL WAS STUDIED BY FELDERHOF, KOREPIN (1970 c.c.)
PARTICULAR MODELS IN FREE-FERMIONIC CASE HAVE SCHUR POLYNOMIALS OR MORE GENERALLY

(*) $$\underbrace{\prod_{i<j} (z_i - q z_j)}_{} \, \Delta_\lambda(z_1, \dots, z_n)$$

DEFORMED WEYL DENOMINATOR

AS PARTITION FUNCTIONS.

THIS IS A COMBINATORIAL FACT EQUIVALENT TO A FORMULA OF TOKUYAMA EXPRESSING

(*) AS A SUM OF GELFAND TSETLIN PATTERNS. 1983

IT WAS TRANSLATED INTO A STATEMENT

ABOUT LATTICE MODELS BY HAMEL AND KING.

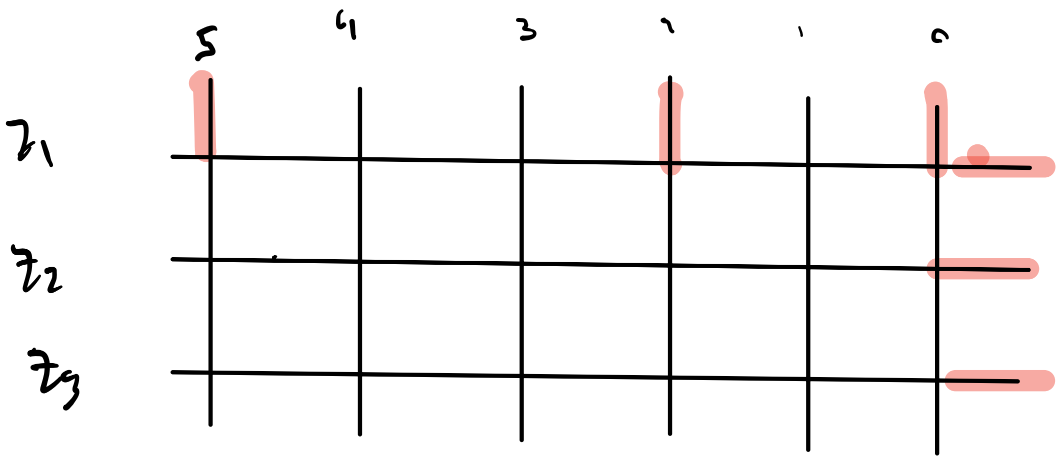
BRUBAKER, BUMP FRIEDBERG INDEPENDENTLY
PAUL ZINN-JUSTIN INTRODUCE YANG-BAXTER
APPROACH TO TOKUYAMA'S FORMULA.

RELEVANT YBE IS IN EARLIER WORK OF
KOREPIN.

WHAT ARE THE MODELS?

GRID WITH n ROWS AND N COLUMNS

$N \geq \lambda_1 + n - 1$



ROWS ARE ASSOCIATED WITH PARAMETERS
$z_1, \dots, z_n$

IN A STATE OF THE MODEL EVERY EDGE IS ASSIGNED A "SPIN" $\oplus$ $\ominus$

SPINS ON BOUNDARY EDGES ARE FIXED

$\lambda = (3,1,0)$ $\qquad$ $\rho = (2,1,0)$ "WEYL VECTOR"

$\lambda + \rho = (5,2,0)$ $\qquad$ $\rho = (n-1, n-2, \dots, 0)$

$s_i \rho = \rho - \alpha_i$

PUT $\ominus$ SPINS ON RIGHT BOUNDARY

TOP BOUNDARY AT LOCATIONS

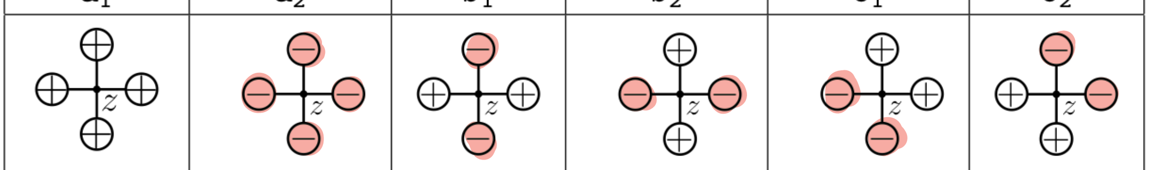
$\lambda_i + n - i$.

$\lambda = (\lambda_1, \dots, \lambda_n)$

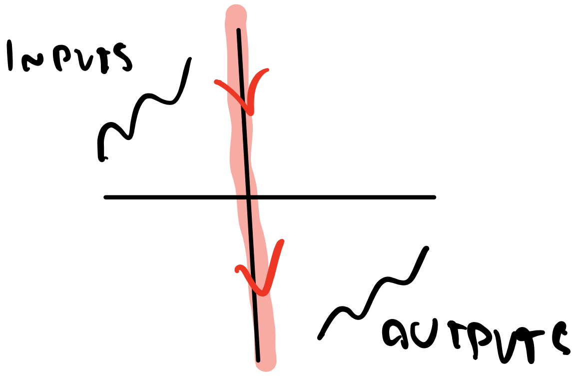
A PARTITION

IN A STATE OF THE MODEL ONLY THE FOLLOWING CONFIGURATIONS ARE ALLOWED

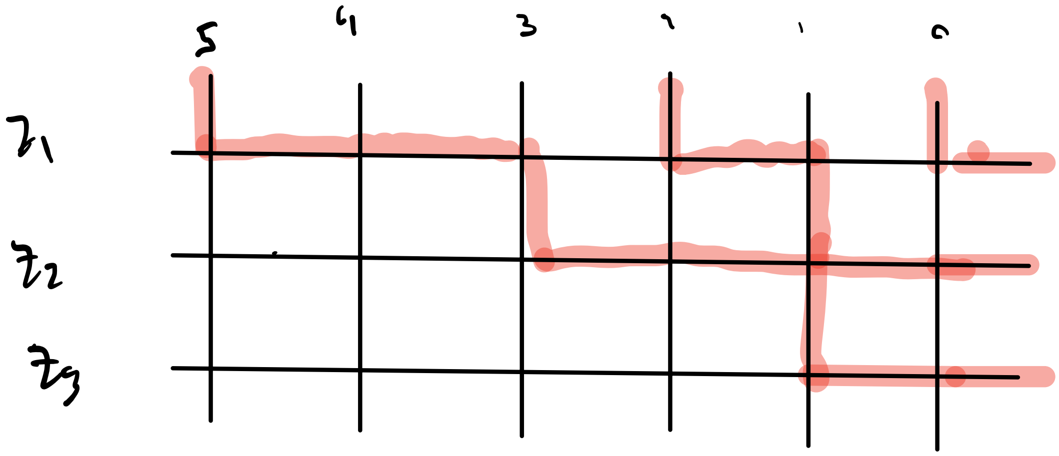
(4.5)



a_1	a_2	b_1	b_2	c_1	c_2
1	z	$-q$	z	$z(1-q)$	1



b_1



TYPICAL STATE. Δ

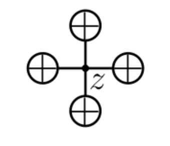
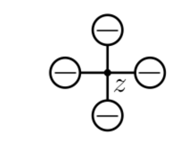
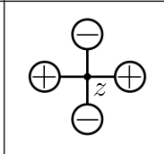
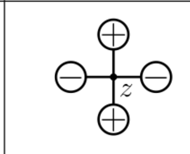
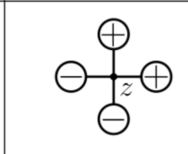
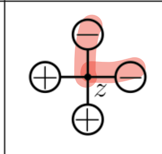
Δ IS ASSIGNED A BOLTZMANN WEIGHT

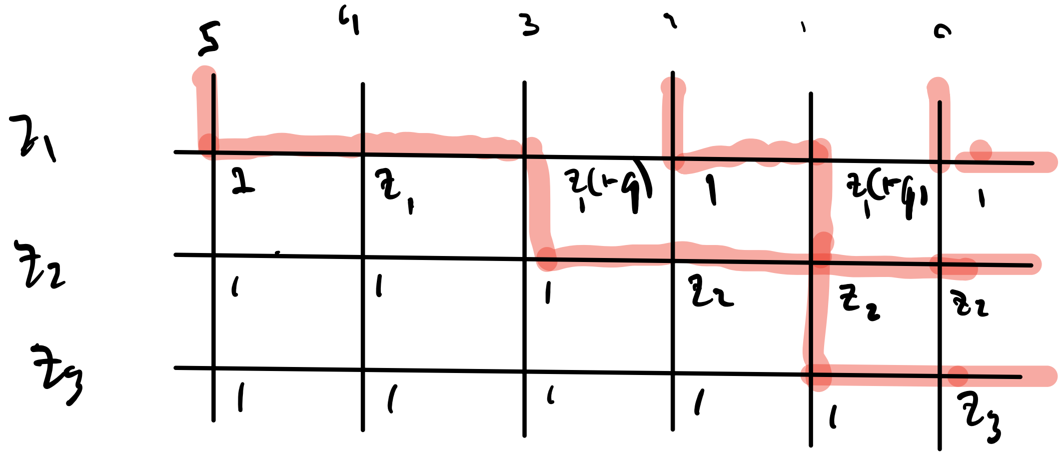
$$\underset{\text{MODEL}}{Z(S)} = \sum_{\substack{\text{STATES} \\ \Delta}} \underset{\text{B.W.}}{\beta(\Delta)}$$

PARTITION FUNCTION

IN STATISTICAL MECHANICS

$$\frac{\beta(\Delta)}{Z(S)} = \text{PROBABILITY OF } \Delta .$$

a_1	a_2	b_1	b_2	c_1	c_2
					
1	z	$-q$	z	$z(1-q)$	1



$$\beta(\Delta) = \prod_{\substack{v \\ \text{VERTICES}}} \beta_v(\Delta)$$

AT A VERTEX v IN n-TH ROW TAKE $z = z_n$. FOR THIS STATE

$$\beta_v = \underset{\substack{\uparrow \\ \text{FIRST ROW}}}{z_1^3(1-q)^2}\ \underset{\substack{\uparrow \\ \text{2-ND}}}{z_2^3} \cdot \underset{\substack{\uparrow \\ \text{THIRD}}}{z_3}.$$

THEOREM: $Z(S) = \prod_{i<j} (z_i - q z_j) \Delta_\lambda(z)$

SCHUR POLYNOMIALS ARE CHARS OF IRREPS OF $GL(n, \mathbb{C})$

$$\chi_\lambda^{GL(n)}(g) = \Delta_\lambda(z_1, \dots, z_n)$$

$z_1, \dots, z_n$ EIGENVALUES OF g.

THERE ARE TWO FORMULAS FOR Δ_λ.

$$\Delta_\lambda(z_1, \dots, z_n) = \frac{\det(z_i^{n+i-\lambda_i})}{\det(z_i^{n-i})}$$

DENOMINATOR

$$\begin{vmatrix} z_1^2 & z_2^2 & z_3^2 \\ z_1 & z_2 & z_3 \\ 1 & 1 & 1 \end{vmatrix} = \prod_{i<j} z_i - z_j$$

$n = 3$

VANDERMONDE DETERMINANT.

NUMERATOR IS AN ALTERNATING POLYNOMIAL.

$$\begin{vmatrix} z_1^{\lambda_1+2} & z_2^{\lambda_1+2} & z_3^{\lambda_1+2} \\ z_1^{\lambda_2+1} & z_2^{\lambda_2+1} & z_3^{\lambda_2+1} \\ z_1^{\lambda_3} & z_2^{\lambda_3} & z_3^{\lambda_3} \end{vmatrix}$$

$$\mathrm{NUM}(\Delta_i z) = -\mathrm{NUM}(z)$$

$\Rightarrow$ $z_i - z_{i+1}$ DIVIDES NUMERATOR.

SO ALL FACTORS $z_i - z_j$ DIVIDE NUMERATOR SO

$$\frac{\det(z_i^{n+i-\lambda_i})}{\det(z_i^{n-i})}$$

IS A SYMMETRIC POLYNOMIAL.

JACOBI'S DEFINITION OF SCHUR POLYNOMIAL

THE OTHER DEFINITION:

LET T BE A SEMISTANDARD YOUNG TABLEAU OF SHAPE λ.

$$YD(\lambda) = \begin{array}{|c|c|c|} \hline 1 & 1 & 2 \\ \hline 3 & \multicolumn{2}{c}{} \\ \cline{1-1} \end{array}$$

$\lambda = (3, 1, 0)$

$YD(\lambda) =$ λ_i BOXES IN ROW i.

SSYT IS A FILLING $YD(\lambda)$ WITH $1, 2, \ldots, n$ ROWS WEAKLY INCREASING COLUMNS STRICTLY INCREASING.

$$wt(T) = (\mu_1, \mu_2, \ldots)$$

$\mu_i = \#$ OF i'S IN T.

$$wt\left(\begin{array}{|c|c|c|} \hline 1 & 1 & 2 \\ \hline 3 & \multicolumn{2}{c}{} \\ \cline{1-1} \end{array}\right) = (2, 1, 1) \qquad z^{wt(T)} = z_1^2 z_2 z_3$$

COMBINATORIAL D.E.
DEFINITION: (LITTLEWOOD)

$$D_\lambda(z_1, \dots, z_n) = \sum_T z^{wt(T)}$$

QUESTIONS: WHY IS THIS SYMMETRIC?
WHY IS THIS EQUIV. TO JACOBI'S DEF?

BOTH QUESTIONS ANSWERED BY TOKUYAMA MODEL.

AFTER PROVING

$$\frac{Z(S_{\lambda,q})}{\prod_{i<j}(z_i - q z_j)}$$ IS A SYMMETRIC POLYNOMIAL

INDEPENDENT OF q.

SPECIALIZE: $q = 1 \rightsquigarrow$ JACOBI DEFINITION

$q = 0$ COMBINATORIAL DEFINITION