Demazure Operators and Demazure Crystals

LET G BE A REDUCTIVE COMPLEX ANALYTIC LIE GROUP

T = MAX'L TORUS

$\Lambda = X^*(T)$ = GROUP OF RATIONAL CHARACTERS

WEIGHT LATTICE

$\Phi \subset \Lambda$ ROOT SYSTEM

EXAMPLE: $G = GL(n, \mathbb{C})$

$$T = \begin{pmatrix} * & & \\ & \ddots & \\ & & * \end{pmatrix} \qquad z = \begin{pmatrix} z_1 & & \\ & \ddots & \\ & & z_n \end{pmatrix}$$

$\Lambda \cong \mathbb{Z}^n \ni \lambda$

THIS IS INTERPRETED AS A CHAR. OF T

$$z \mapsto z^\lambda = z_1^{\lambda_1} \cdots z_n^{\lambda_n},$$

$e_i \in \mathbb{Z}^n$ $(i = 1, \ldots, n)$ STANDARD BASIS VECTS

$$\alpha = e_i - e_j \qquad i \neq j \qquad 1 \leq i, j \leq n$$

POSITIVE ROOTS: $e_i - e_j$ $\quad i < j$

SIMPLE ROOTS $\quad \alpha_i = e_i - e_{i+1} \quad 1 \leq i \leq n-1.$

SIMPLE ROOT $\leftrightarrow$ $s_i \in \mathrm{AUT}(\Lambda)$ OR $\mathrm{AUT}(T)$

$$s_i = (i, i+1)$$

AS AN AUTOMORPHISM OF Λ

$$s_i(\lambda) = \lambda - \langle \alpha_i^\vee, \lambda \rangle \alpha_i$$

$\alpha_i^\vee \in \mathrm{HOM}(\Lambda, \mathbb{Z})$ "SIMPLE COROOTS"

$\alpha_i^\vee$ CAN BE IDENTIFIED WITH α FOR $GL(n)$ OR ANY SIMPLY-LACED GROUP

$\langle \alpha_i^\vee, \lambda \rangle$ = DOT PRODUCT.

$$s_i(\lambda_1, \dots, \lambda_n) = (\lambda_1, \dots, \lambda_{i+1}, \lambda_i, \dots)$$

$$= \lambda - (\lambda_i - \lambda_{i+1}) \cdot \alpha_i$$

$$\|$$

$$\langle \alpha_i^\vee, \lambda \rangle = (\alpha_i, \lambda)$$

$\uparrow$

USUAL DOT PROD ON $\mathbb{Z}^n$.

AS ON WEDNESDAY I CAN WORK IN A RING $\mathcal{R}$ CONTAINING $W = S_n$ AND $\mathcal{O}(T) = \mathbb{C}[z_1, z_1^{-1}, \dots, z_n, z_n^{-1}]$

RING OF FUNCTIONS ON T

$$w f w^{-1} = {}^{w}f$$

$({}^{w}f)(z) = f(w^{-1}z)$

$(w f w^{-1})(z) = (w f)(w^{-1}z)$

$w(f(w^{-1}z)) = {}^{w}f$

$\Delta_1 \qquad f(z) = z_2/z_3$

$\Delta_1 f \Delta_1^{-1}(z) = \Delta_1(f(z_2, z_1, z_3))$

$= \Delta_1(z_1/z_3) = z_2/z_3.$

${}^{\Delta_1}f(z)$

AS AN OPERATOR ON $\mathcal{O}(T)$

f = MULTIPLICATION BY ITSELF.

$$\partial_i = (1 - z^{-\alpha_i})^{-1}(1 - z^{-\alpha_i} \cdot \Delta_i)$$

$$\partial_i f(z) = \frac{f(z) - z^{-\alpha_i} \cdot f(\Delta_i z)}{1 - z^{-\alpha_i}}$$

"ISOBARIC DEMAZURE OPERATORS".

LET US COMPUTE $\partial_i z^{\lambda}$

$$\partial_i z^\lambda = \frac{z^\lambda - z^{-\alpha_i} z^{\lambda - \langle \alpha_i^\vee, \lambda \rangle \alpha_i}}{1 - z^{-\alpha_i}}$$

$$z^\lambda \frac{1 - z^{-\alpha_i(\langle \alpha_i^\vee, \lambda \rangle + 1)}}{1 - z^{-\alpha_i}}$$

THIS IS A FINITE GEOMETRIC SERIES.

IF $\langle \alpha_i^\vee, \lambda \rangle = k \geq 0$ THIS EQUALS

$$z^\lambda \cdot \left(1 + z^{-\alpha_i} + z^{-2\alpha_i} + \dots + z^{-\langle \lambda, \alpha_i^\vee \rangle \alpha_i}\right)$$

SYMMETRIC I.E. Δ_i-INVARIANT.

$$z^\lambda + z^{\lambda - \alpha_i} + \dots + z^{\Delta_i \lambda}$$

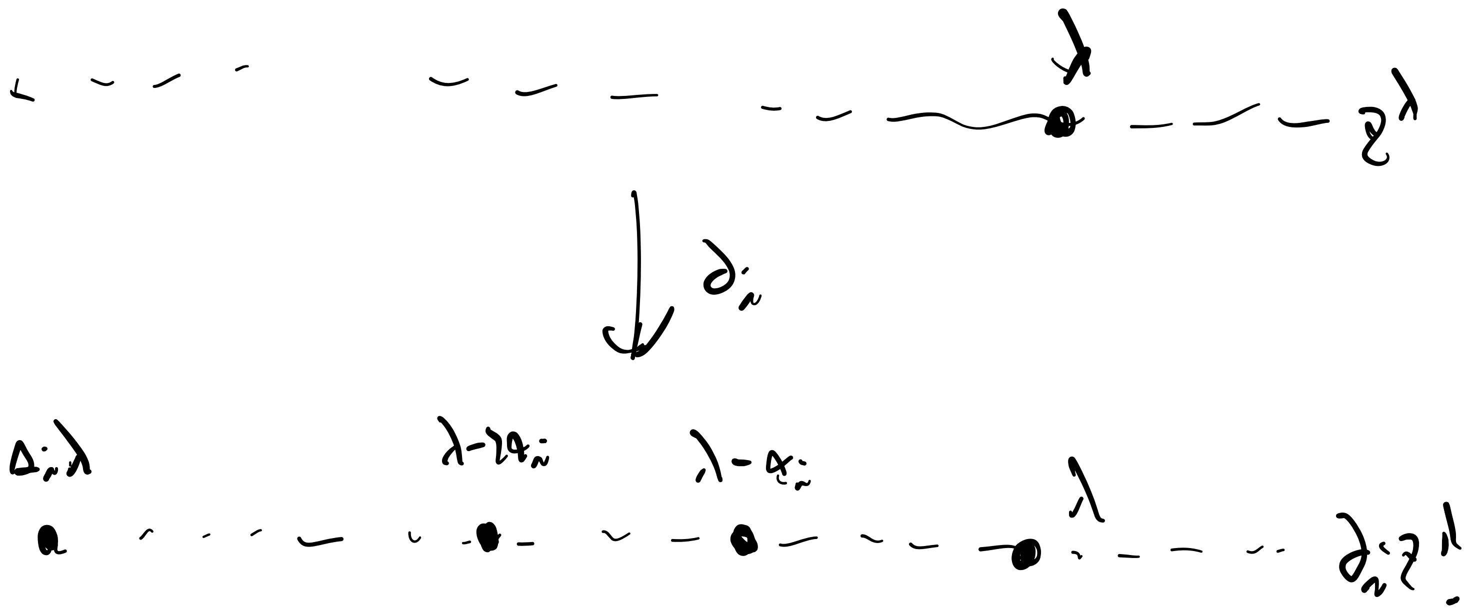
IF $\langle \alpha_i^\vee, \lambda \rangle = -1$, $\partial_i z^\lambda = 0$

IF $\langle \alpha_i^\vee, \lambda \rangle < -1$ THIS IS

$$-\left(z^{\lambda + \alpha_i} + z^{\lambda + 2\alpha_i} + \dots + z^{\Delta_i \lambda - \alpha_i}\right)$$

STILL Δ_i INVARIANT.

FOR $GL(2)$:



WEIGHTS λ, $\lambda_i - \alpha_i$, ...

∂_i SYMMETRIZES. WE CAN SEE THIS DIRECTLY:

$$\Delta_i \partial_i = \partial_i$$

PROOF IN R

$$\Delta_i \partial_i = \Delta_i (1 - z^{-\alpha_i})^{-1}(1 - z^{-\alpha_i}\Delta_i)$$

$$(1 - z^{+\alpha_i})^{-1}(\Delta_i - z^{\alpha_i} 1_W)$$

$$\Delta_i z^{-\alpha_i} \Delta_i = z^{\alpha_i}$$

MULTIPLY NUM & DENOM BY $-z^{-\alpha_i}$

$$= \left(-z^{-\alpha_i}+1\right)^{-1}\left(z^{-\alpha_i}\Delta_i - 1\right) = \partial_i$$

SO $\Delta_i(\partial_i(f)) = \partial_i(f)$

SO $\partial_i f$ IS Δ_i-SYMMETRIC.

THEOREM: $\partial_i^2 = \partial_i$ AND THEY SATISFY BRAID RELATIONS.

(PROVED SIMILAR FACT FOR $D_i = (x_i - x_{i+1})^{-1}(1 - \Delta_i)$ ON WEDNESDAY. FOR THIS I WON'T PROVE)

THEREFORE WE MAY APPLY MATSUMOTO'S THEOREM AND DEFINE

$$\partial_w = \partial_{i_1} \cdots \partial_{i_k}$$

$$\Delta_{i_1} \cdots \Delta_{i_k} = w \quad \text{(REDUCED)}.$$

FOR REFERENCE

$$\partial_i = (1 - z^{-\alpha_i})^{-1}(1 - z^{-\alpha_i} \cdot \Delta_i)$$

$$\partial_i f(z) = \frac{f(z) - z^{-\alpha_i} \cdot f(\Delta_i z)}{1 - z^{-\alpha_i}}$$

THEOREM: LET λ BE DOMINANT.

$\langle \lambda, \alpha_i^\vee \rangle \geq 0$ ALL i

THEN $\partial_{w_0} z^\lambda$ = CHARACTER OF IRR REP'N OF HIGHEST WEIGHT λ.

∂_{w_0} = LONGEST DEMAZURE OPERATOR.

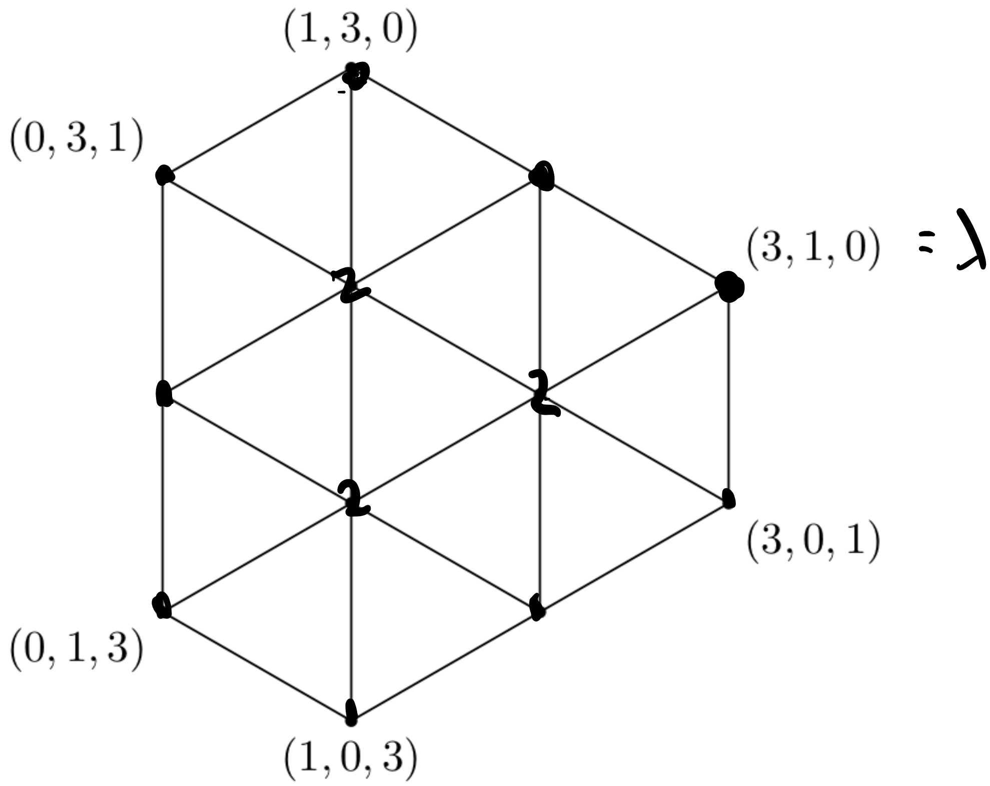
FOR $GL(n)$ $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$

FOR EXAMPLE A PARTITION IS A DOMINANT WEIGHT

I'LL ILLUSTRATE THIS FOR $\lambda = (3,1,0)$

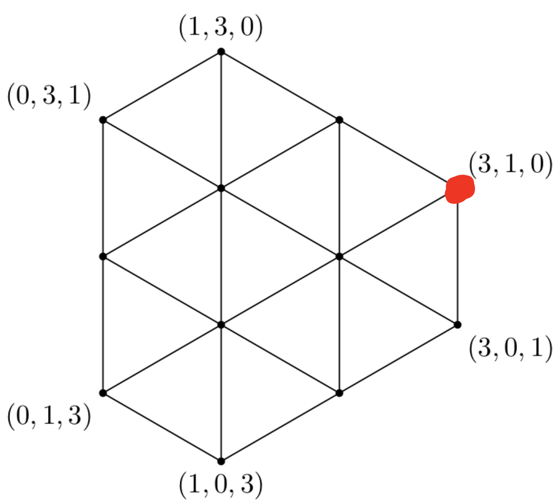
$G = GL(3)$.

$$z^\lambda = z_1^3 \cdot z_2$$

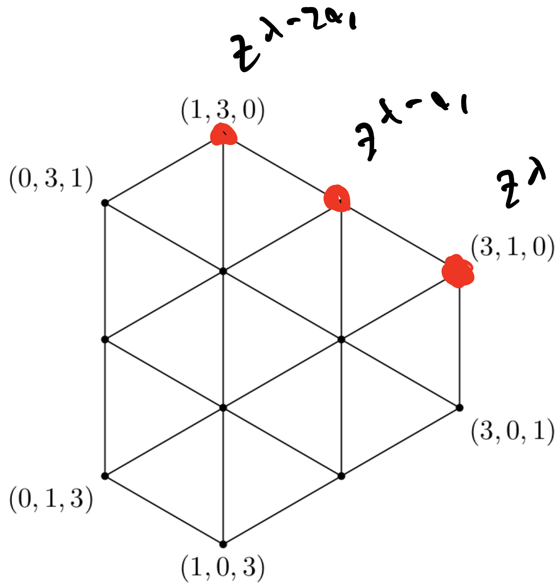


CHARACTER OF IRREP'N $\Pi_\lambda^{GL(3)}$

$$\Delta_\lambda(z) = z_1^3 z_2 + z_1^2 z_2^2 + z_1 z_2^3$$
$$+ z_1^3 z_3 + 2 z_1^2 z_2 z_3 + 2 z_1 z_2^2 z_3 + \dots$$

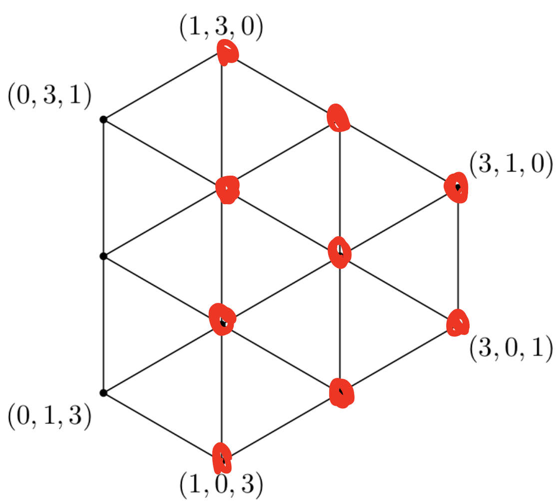


z^λ



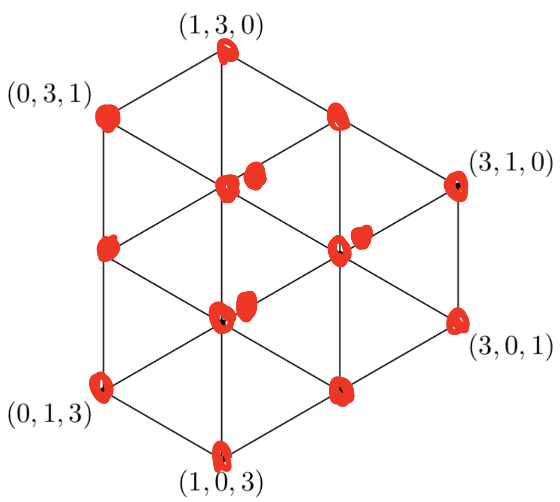
$\partial_1 z^\lambda$

$\langle \alpha_1^\vee, \lambda \rangle = 2$



$\partial_2 \partial_1 z^\lambda$

"DEMAZURE CHARACTER".

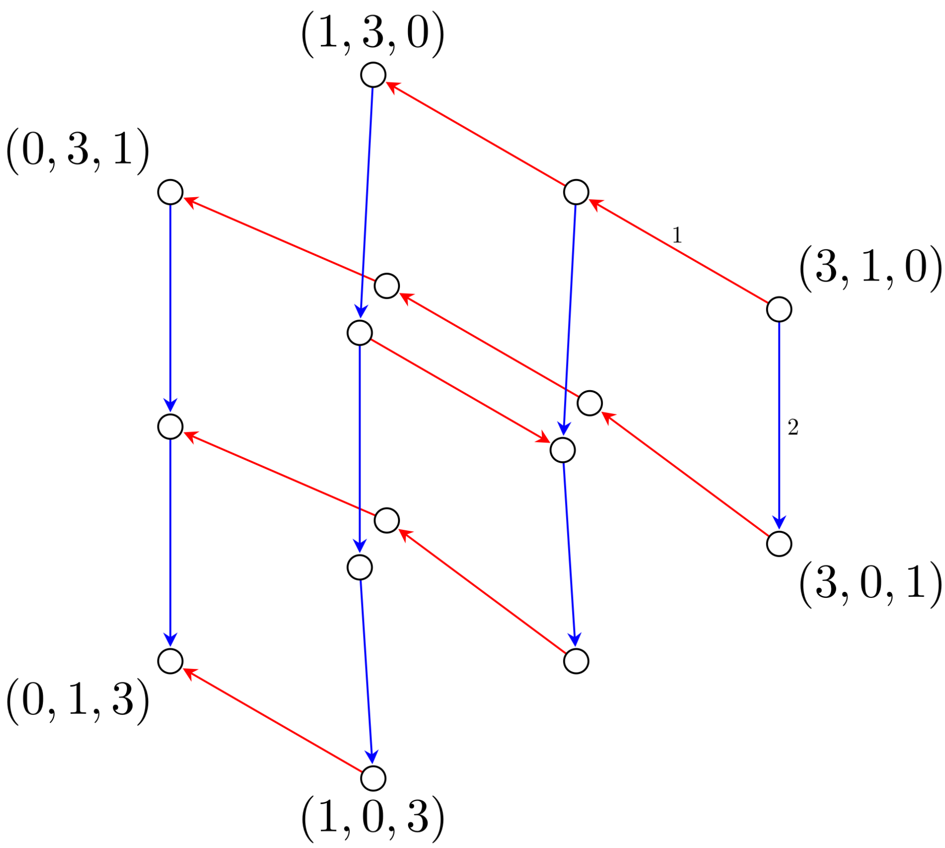


$\partial_1 \partial_2 \partial_1 z^\lambda$

$= \partial_w z^\lambda$

$= D_\lambda(z^\lambda)$.

$\partial_2 \partial_1 \partial_2 z^\lambda$ = SAME RESULT.



THE $\partial_w z^\lambda$ (λ DOMINANT)
ARE CALLED DEMAZURE CHARACTERS
OR KEY POLYNOMIALS

THERE IS A REFINEMENT
(LITTELMANN, KASHIWARA) OF $\partial_{w_0} z^\lambda = \Delta_\lambda$
INVOLVING CRYSTALS.

$$\Delta_i \partial_{w_0} z^\lambda = \partial_{w_0} z^\lambda$$

CHOOSE $w_0 = \Delta_{i_1} \cdots \Delta_{i_k}$

WITH $i_1 = i$.

$$\partial_{w_0} = \prod (1 - z^{-\alpha_i})^{-1} \sum_w (-1)^{\ell(w)} z^{\rho - w(\rho)}$$

APPEARS IN WCF

$$D_i = z_{i+1}^{-1} (\partial_i - 1)$$