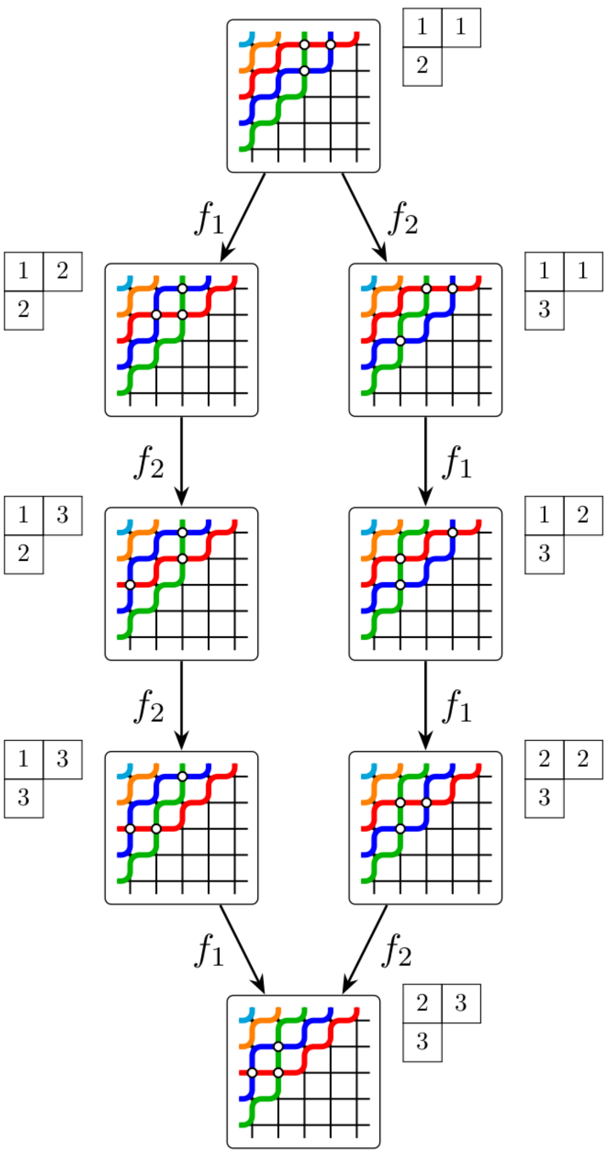
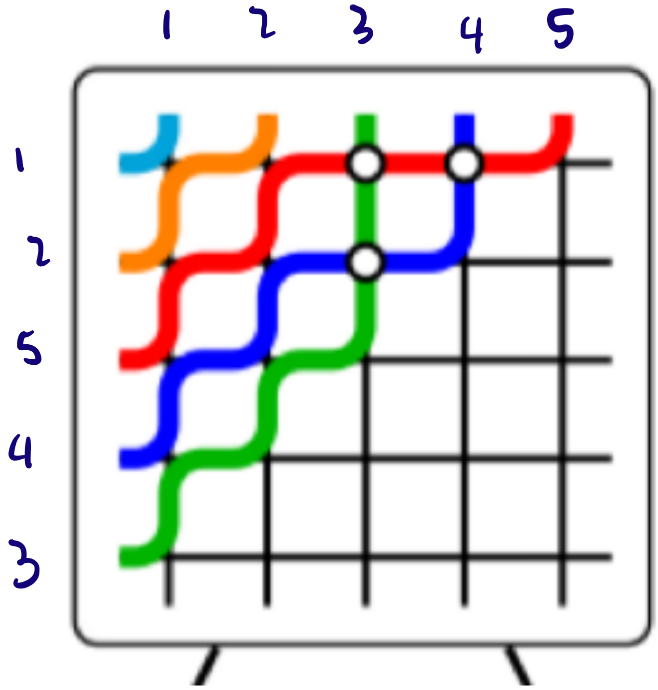
# CRYSTAL STRUCTURES ON CLASSICAL AND BUMPLESS PIPEDREAMS.

GIVEN $w \in S_n$

CLASSICAL PIPEDREAMS FOR $S_n$

$w = s_4 s_3 s_4 = 12543$






MARKED CROSSES SO

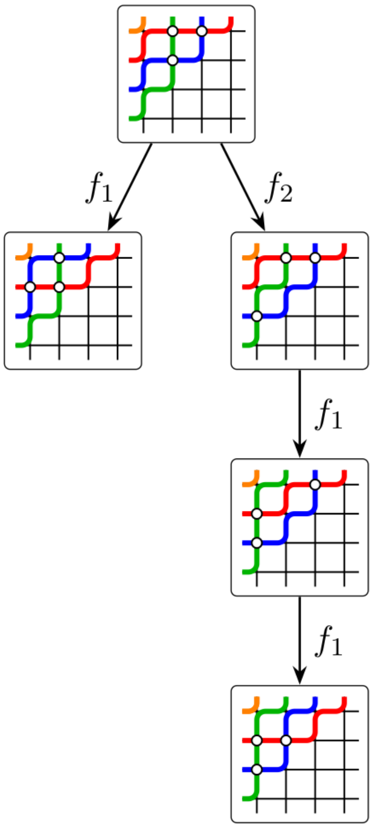
$$S_w = x_1^2 x_2 + \dots$$

$$S_w(x;y) = (x_1 - x_3)(x_1 - x_4)(x_2 - x_3) + \dots$$

THEOREM: ASAF-SCHILLING; GOLD-MILICEVIC-SUN

THE CLASSICAL PIPEDREAMS CAN BE ORGANIZED INTO A UNION OF DEMAZURE CRYSTALS.

IN THIS EXAMPLE, JUST ONE DEMAZURE CRYSTAL.



$$
\begin{tikzcd}
 & \boxed{(32)(3)()} \arrow[dl, "f_1"'] \arrow[dr, "f_2"] & \\
\boxed{(2)(32)()} \arrow[d, "f_2"'] & & \boxed{(32)()(3)} \arrow[d, "f_1"] \\
(2)(3)(2) \arrow[d, "f_2"'] & & \boxed{(3)(2)(3)} \arrow[d, "f_1"] \\
(2)()(32) \arrow[dr, "f_1"'] & & \boxed{()(32)(3)} \arrow[dl, "f_2"] \\
 & ()(2)(32) &
\end{tikzcd}
$$

$n = 4 \quad w = 1432 = s_2 s_3 s_2$

THE FULL CRYSTAL WOULD HAVE 8 ELEMENTS BUT ONLY 5 OF THESE CORRESPOND TO PIPEDREAMS.

RELATED: MORSE-SCHILLING CRYSTALS FOR STANLEY SYMMETRIC FUNCTIONS.

ONE APPROACH TO DESCRIBING THE CRYSTAL STRUCTURE: EXTRACT FROM THE PIPEDREAM A DESCENDING FACTORIZATION. THE CRYSTAL STRUCTURE ON DECREASING FACTORIZATIONS CAN BE DESCRIBED USING EITHER BRACKETING PROCEDURE OR; APPLY EDELMAN-GREENE INSERTION TO THE UNIQUE FACTORIZATION RECORDING TABLEAU GIVES AN ELEMENT OF $\mathcal{B}_x$ FOR SOME SHAPE, WHERE CRYSTAL OPERATIONS ARE KNOWN.

HOW TO OBTAIN A DECREASING FACTORIZATION FROM A PIPEDREAM.

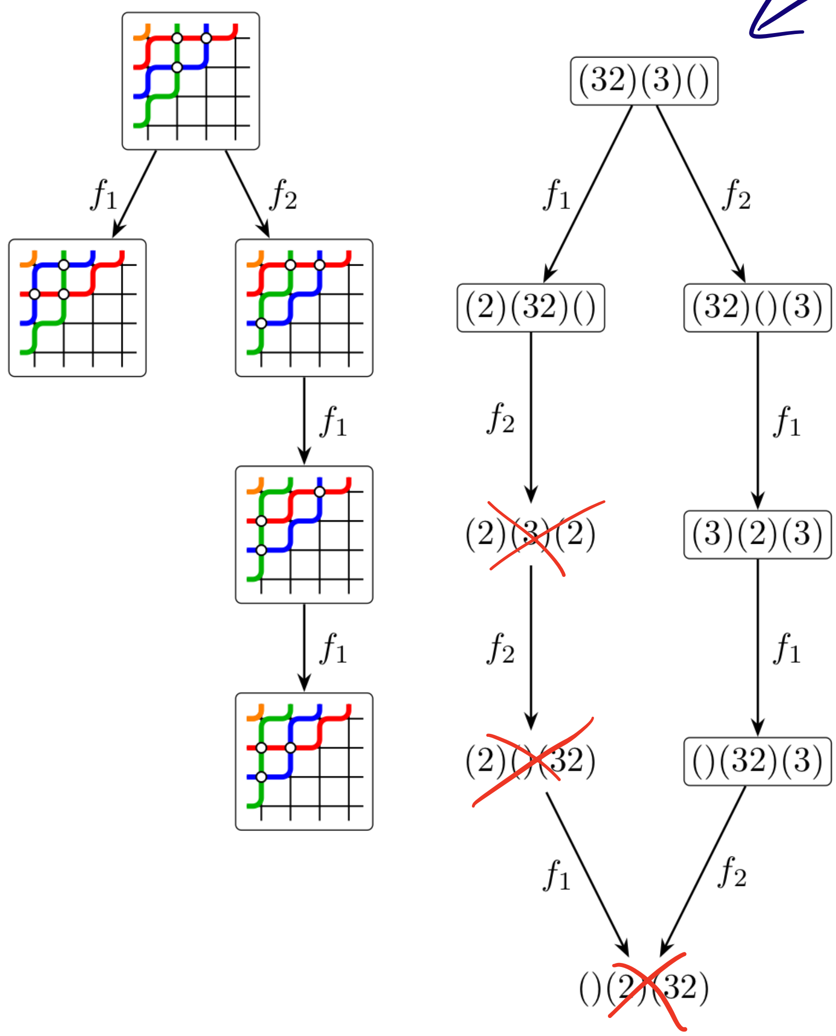
A PERMUTATION IS DECREASING IF IT HAS FORM $\Delta_{a_1}\Delta_{a_2}\cdots\Delta_{a_k}$

$a_1 > a_2 > \cdots > a_k$.

IF WE CONSIDER $w \in S_n$ SAME $k$

$w = w^1 w^2 \cdots w^k$   $w^i$ DECREASING.

DECREASING FACTORIZATIONS ORGANIZED INTO A MORSE-SCHILLING CRYSTAL.




$\ell(w) = \sum \ell(w^i)$

(32)(3)() MEANS $w^1 w^2 w^3$

$w^1 = s_3 s_2$

$w^2 = s_3$

$w^3 = 1$.

$\mathrm{wt}(w^1 w^2 w^3 \cdots) = (\ell(w^1), \ell(w^2), \ldots)$

$\mathrm{wt}((32)(3)()) = (2,1,0)$

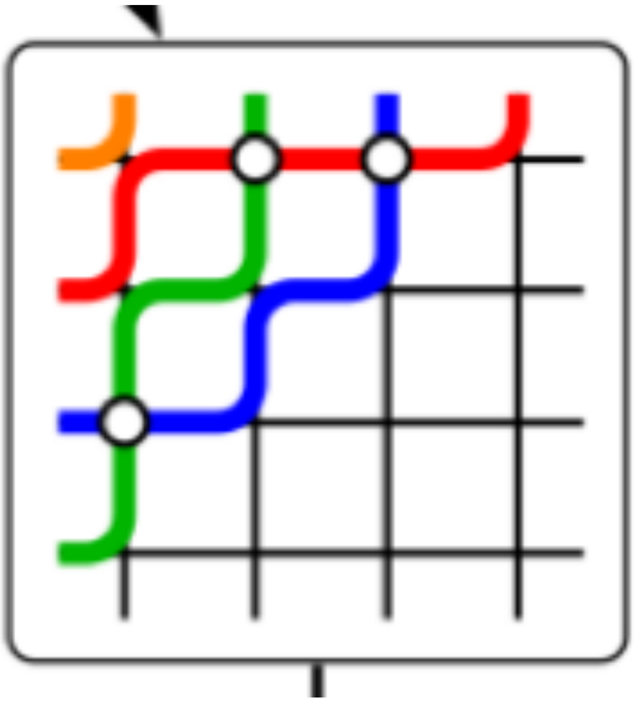
SUPPOSE THE PIPEDREAM $P$

HAS CROSSINGS IN $i$-TH ROW.

AT LOCATIONS $(i, j_1), (i, j_2), \ldots$

IN DECREASING ORDER

$$j_1 > j_2 > \ldots$$



$(1,3)$ $(1,2)$ $(3,1)$

$$w^i = (a_1, a_2, \ldots)$$

$$a_t = i + j_t - 1.$$

$$w^1 = s_3 s_2$$
$$w^2 = 1.$$
$$w^3 = s_3$$

THE PERMUTATION $w = w^1 w^2 w^3$

$w = s_3 s_2 s_3 = w^1 w^2 w^3$. $\quad (s_3 s_2)(\ )(s_3)$

STANLEY SYMMETRIC FUNCTION

$$F_w = \sum_{\substack{\text{ALL DECREASING} \\ \text{FACT'NS} \\ w = w^1 w^2 \cdots w^k}} x^{wt(w^1 \cdots w^k)} = x_1^{\ell(w_1)} x_2^{\ell(w_2)} \cdots$$

SYMMETRIC, A SUM OF SCHUR FUNCTIONS.

NOT EVERY DECREASING FACT'N COMES FROM A PIPEDREAM.

THE REASON IS

$P$ PIPEDREAM $\rightsquigarrow$ $w = w^1 w^2 w^3 \cdots$

$w^2$ CANNOT INVOLVE $s_1$

$w^3$ " $s_1, s_2$

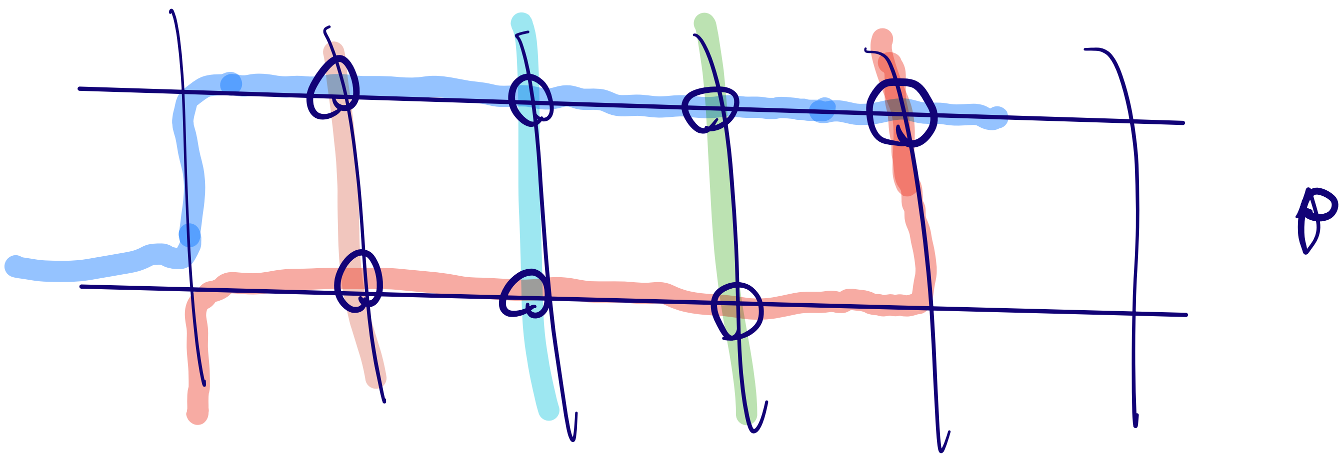
ETC.

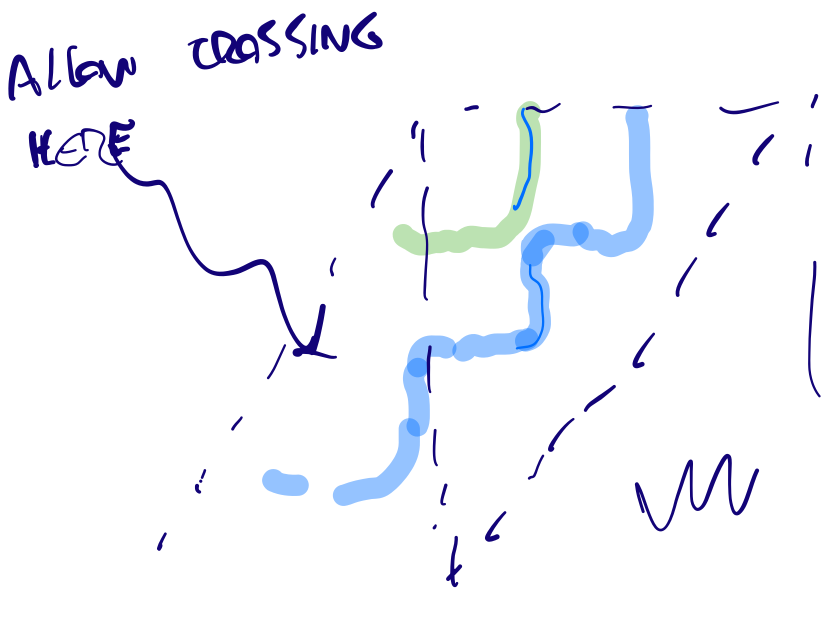
THIS IS THE REASON THE PIPEDREAM CRYSTAL

GIVES ONLY A PART OF THE MORSE-SCHILLING CRYSTAL.

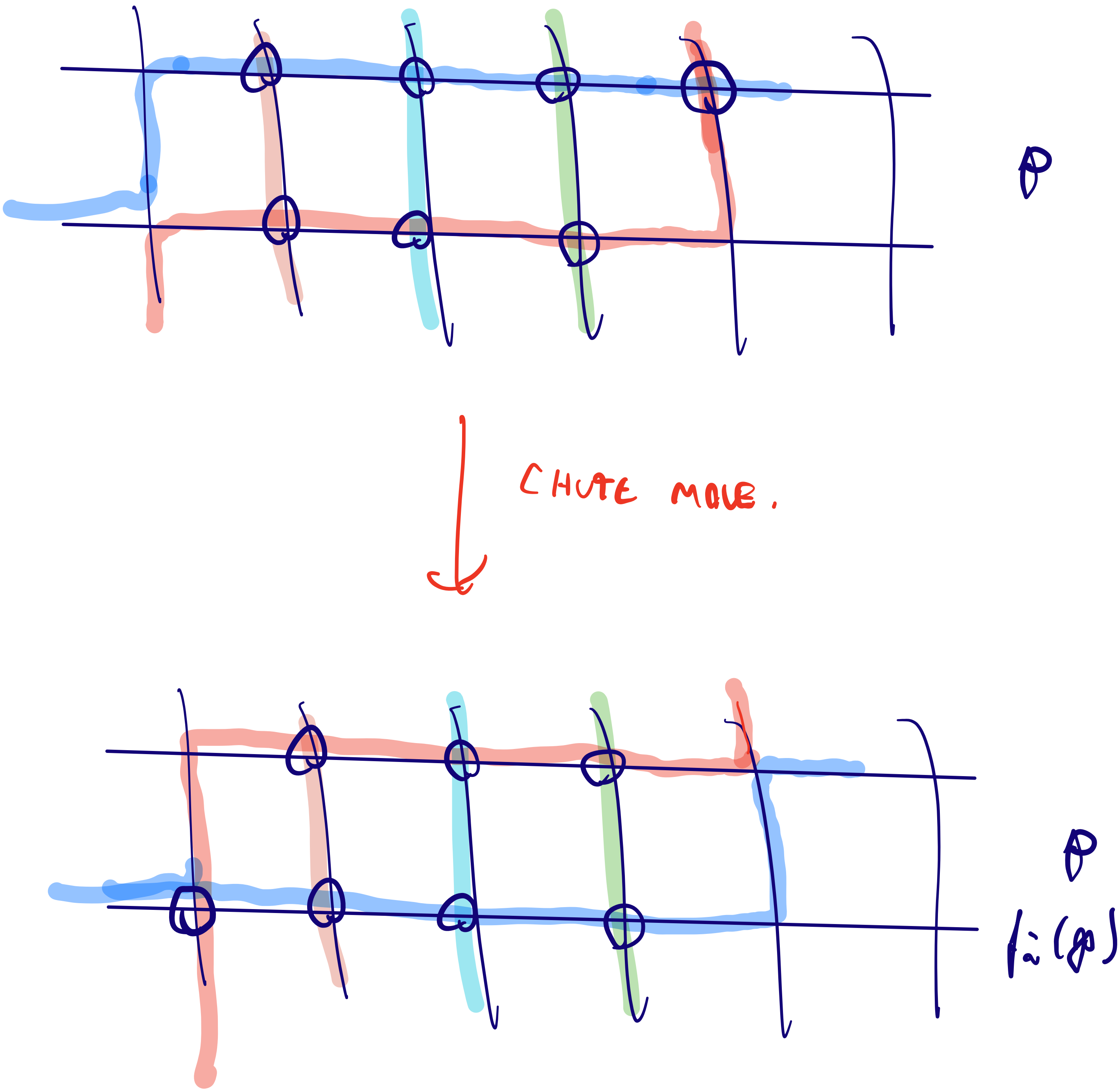
WHAT ARE THE CRYSTAL OPERATIONS ?

CHUTE MOVE



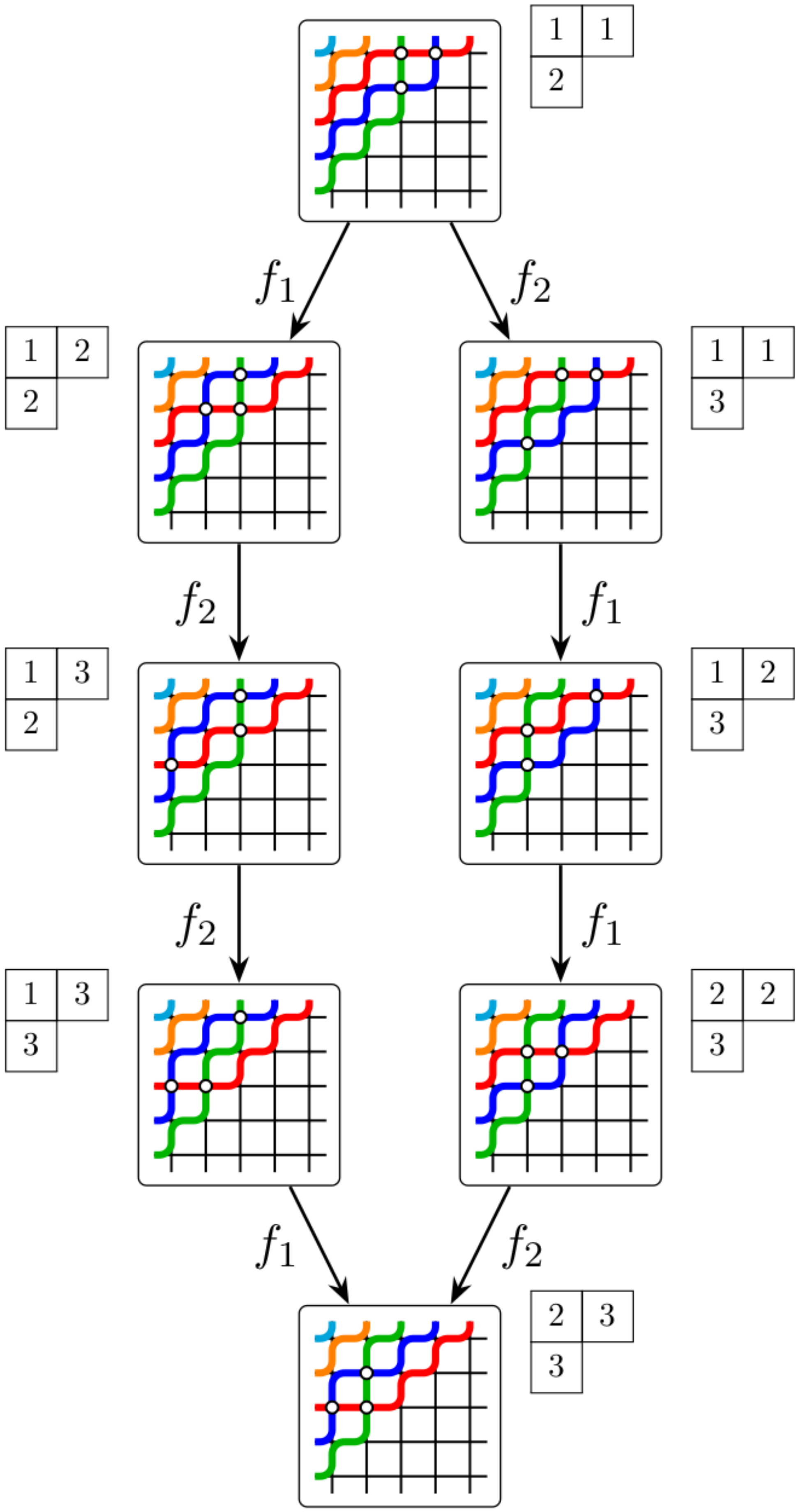




PIPEDREAM MODELS FOR STANLEY S.F. CAN BE MAD

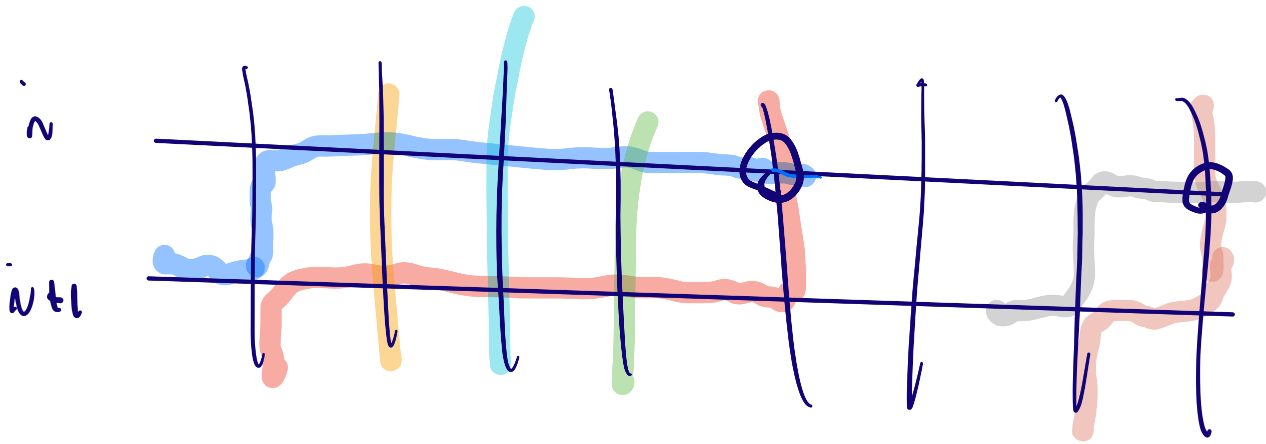



A CHUTE MOVE CHANGES ONE CROSSING FROM $i$-TH ROW TO $i+1$ ROW.

SUBTRACTS $(0, \dots, \underset{i}{1}, -1, 0, \dots) = \alpha_i$ FROM $wt(P)$:


1 1
2
$f_1$
$f_2$
1 2
2
1 1
3
$f_2$
$f_1$
1 3
2
1 2
3
$f_2$
$f_1$
1 3
3
2 2
3
$f_1$
$f_2$
2 3
3

QUESTION: SUPPOSE TWO CHUTE MOVES ARE POSSIBLE!




$f_i$ WILL MOVE ONE OF THESE CROSSINGS. WHICH WILL IT MOVE?

THIS IS DECIDED BY A BRACKETING PROCEDURE

**Definition 7.11.** Given a reduced classical pipedream $p$ for a permutation in $S_n$, we fix a row index $i \in [n]$. (Since crosses only occur in boxes $(i, j)$ such that $i + j \leqslant n$, $p$ has no crosses in row $n$.)

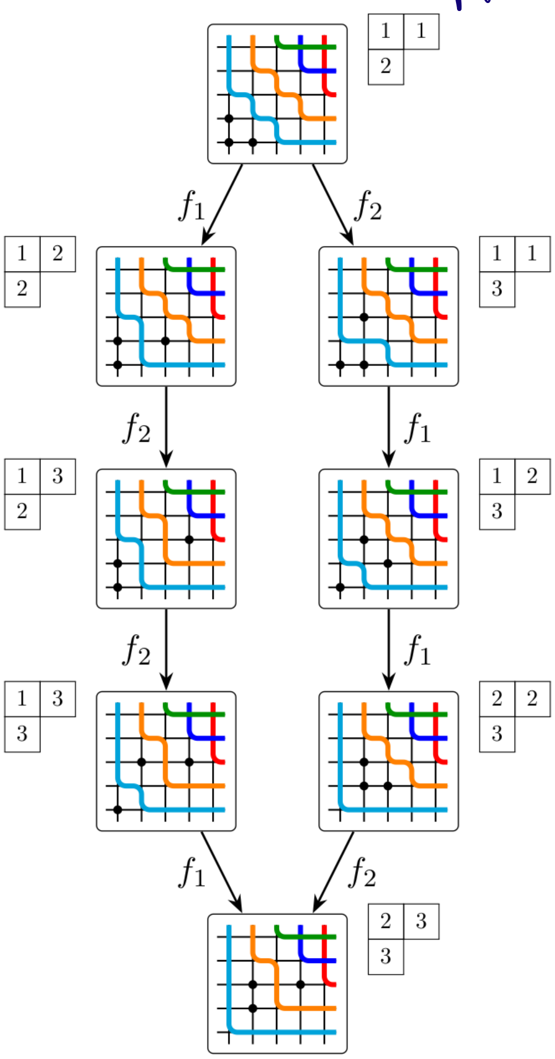
At the beginning of the algorithm all crosses in rows $i$ and $i+1$ are unpaired. Successively treat the crosses in row $i$ from right to left. Suppose the cross in row $i$ is in position $(i, a)$. Then find $b$ minimal such that $a \leqslant b$ and there is an unpaired cross in position $(i + 1, b)$. If such a $b$ exists, the crosses in positions $(i, a)$ and $(i + 1, b)$ are paired. Otherwise $(i, a)$ is unpaired. The pairing process is repeated until no further pairings can be found.

$$f_2\left(\begin{array}{|c|c|}\hline 1 & 2 \\ \hline 2 & \\ \cline{1-1}\end{array}\right) \qquad 2 \otimes 1 \otimes 2$$

$$: \quad \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 & \\ \cline{1-1}\end{array} \qquad \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & \\ \cline{1-1}\end{array}$$

WHICH ?

SIMILAR PROCEDURE ; BRACKET CROSSES IN $i$-TH AND $(i+1)$-ST ROW TO DECIDE WHAT $f_i$.




FOR BUMPLESS P.D. $f_i$ OPERATIONS ARE "DROOP MOVES"