TWO STORIES

(1) USUAL THEORY OF TABLEAU COMBINATORICS

ROBINSON SCHENSTED KNUTH (RSK ALGORITHM)

PLACTIC MONOID (LASCOUX-SCHÜTZENBERGER)

$\mathcal{B}$ $\quad \boxed{1} \to \boxed{2} \to \cdots \to \boxed{n}$

GENERATORS OF PLACTIC MONOID.

CRYSTALS OF TABLEAUX

LATTICE MODELS $\quad q=0$ TOKUYAMA MODEL.

(2) ANOTHER KIND OF COMBINATORICS

EDELMAN GREENE VARIANT OF RSK.

WEYL GROUP S_n

GENERATORS $s_1, \dots, s_{n-1}$

MORSE-SCHILLING CRYSTALS (DECREASING FACTORIZATION)

LATTICE MODELS: PIPEDREAMS.

ELEMENTS OF PLACTIC MONOID ARE SSYT

IN $1, 2, \dots, n$

MULTIPLICATION: IF T, V ARE TABLEAUX

WE CAN CALULATE THEIR PRODUCT IN

PLACTIC MONOID AS FOLLOWS.

SUPPOSE $T = \boxed{i}$

$$\left(\begin{array}{llllll} 1 & 1 & 2 & 3 & 3 & 4 \\ 2 & 3 & 3 & 4 & & \\ 4 & & & & & \end{array}\right) \cdot \boxed{i} =$$

$T \leftarrow i$ COMPUTED AS FOLLOWS.

IF $i \geq$ LARGEST ENTRY IN FIRST ROW PLACE IT AT END.

$$\begin{array}{llllll} 1 & 1 & 2 & 3 & 3 & 4 \\ 2 & 3 & 3 & 4 & & \\ 4 & & & & & \end{array} \leftarrow 4 \; : \; \begin{array}{lllllll} 1 & 1 & 2 & 3 & 3 & 4 & 4 \\ 2 & 3 & 3 & 4 & & & \\ 4 & & & & & & \end{array}$$

OTHERWISE "SCHENSTED INSERT":

FIND LARGEST ENTRY IN FIRST ROW k LARGER THAN i AND REPLACE THE LEFTMOST INSTANCE OF k BY i.

THE k IS THEN INSERTED INTO SECOND ROW.

$$\begin{array}{l} \qquad\quad k \\ \begin{array}{cccccc} 1 & 1 & 2 & 3 & 3 & 4 \\ 2 & 3 & 3 & 4 & & \\ 4 & & & & & \end{array} \leftarrow 2 \end{array}$$

$$\begin{array}{cccccc} 1 & 1 & 2 & 2 & 3 & 4 \\ 2 & 3 & 3 & 4 & & \\ 4 & & & & & \end{array} \quad 3$$

$$\begin{array}{cccccc} 1 & 1 & 2 & 2 & 3 & 4 \\ 2 & 3 & 3 & 3 & & \\ 4 & 4 & & & & \end{array} \quad 4$$

NOW ABOUT PLACTIC MONOID DEFINE AN EQUIVALENCE RELATION ON WORDS

$a = i_1 \otimes \cdots \otimes i_k \qquad i_1, \dots, i_k \in B$

$b = j_1 \otimes \cdots \otimes j_k$

$a \equiv b$ IF IN DECOMPOSITION OF $\bigotimes^k B$

INTO IRREDUCIBLE CRYSTALS

$$\mathcal{C}, \mathcal{D} \subset \otimes^n \mathcal{B}$$

WHICH ARE ISOMORPHIC AND

$a \in \mathcal{C}$ MAPS TO $b \in \mathcal{D}$ UNDER

THE UNIQUE ISOMORPHISM.

THEOREM: PLACTIC EQUIVALENCE AMOUNTS

TO KNUTH EQUIVALENCE

(1) If $a < b \leqslant c$, then bac is Knuth equivalent to bca and vice versa.
(2) If $a \leqslant b < c$, then acb is Knuth equivalent to cab and vice versa.

EXAMPLE:

$$1\ 1\ 2\ 3\ 3\ 4 \leftarrow 2$$

$$\begin{array}{llllll} 1 & 1 & 2 & 2 & 3 & 4 \\ 3 \end{array}$$

THIS MEANS $1 \otimes 1 \otimes 2 \otimes 3 \otimes 3 \otimes 4 \otimes 2$

$\equiv 3 \otimes 1 \otimes 1 \otimes 2 \otimes 2 \otimes 3 \otimes 4.$

IF $a < b \leq c$ $\quad \vec{b}ac \doteq \vec{b}ca$

$a \leq b < c$ $\quad ac\overleftarrow{b} \doteq ca\overleftarrow{b}$

IT CAN BE CHECKED IN $\mathcal{B} \otimes \mathcal{B} \otimes \mathcal{B}$

THIS HAS 4 CONNECTED COMPONENTS :

$\mathcal{B}_{(3)}$, $\mathcal{B}_{(1,1,1)}$, 2 COPIES OF $\mathcal{B}_{(2,1)}$

IN THE ISO $\quad \mathcal{B}_{(2,1)} \to \mathcal{B}'_{(2,1)}$

$$\boxed{b} \otimes \boxed{a} \otimes \boxed{c} \rightsquigarrow \boxed{b} \otimes \boxed{c} \otimes \boxed{a}$$

IF $\vec{b}ac \doteq \vec{b}ca$

THIS IMPLIES WE CAN PERFORM THIS OPERATION IF bac APPEARS IN $\otimes^n \mathcal{B}$ FOR SOME LARGER n.

(1) If $a < b \leqslant c$, then bac is Knuth equivalent to bca and vice versa.
(2) If $a \leqslant b < c$, then acb is Knuth equivalent to cab and vice versa.

1 1 2 3 3 4   2

1 1 2 3 3 4 2

1 1 2 3 3 2 4

1 1 2 3 2 3 4

1 1 3 2 2 3 4

1 3 1 2 2 3 4

3 1 1 2 2 3 4

THE FACT THAT IF WORDS a, b ARE K.E. THEY ARE PLACTICAL EQUIV. FOLLOWS FROM A CALCULATION IN $\mathcal{B} \otimes \mathcal{B} \otimes \mathcal{B}$.

THIS IMPLIES THAT

$T \otimes n \equiv$ SCHENSTED INSERTION $T \leftarrow n$.

THIS IMPLIES AN ALGORITHM OF COMPUTING $T \otimes V$

TURN V INTO ITS READING WORD.
SUCCESSIVELY INSERT ENTRIES INTO T.

$$\begin{array}{ll} 1 & 1 \\ 2 & \end{array} \otimes \begin{array}{ll} 2 & 3 \\ 3 & \end{array}$$

$$\begin{array}{ll} 1 & 1 \\ 2 & \end{array} \otimes 3 \otimes 2 \otimes 3$$

$$\begin{array}{lll} 1 & 1 & 3 \\ 2 & & \end{array} \otimes 2 \otimes 3$$

$$\begin{array}{ll} 1 & 1 \\ 2 & \end{array} \leftarrow 3$$

$$\begin{array}{ccc} 1 & 1 & 2 \\ 2 & 3 & \end{array} \otimes 3 \equiv \begin{array}{cccc} 1 & 1 & 2 & 3 \\ 2 & 3 & & \end{array}.$$

SO I HAVE ENOUGH INFORMATION TO COMPUTE $T \otimes U$ I.E. FIND A TABLEAU EQUIVALENT.

EDELMAN GREENE INSERTION SIMILAR.

I'LL USE IT TO PROVE STANLEY'S THM COUNTING THE NUMBER OF REDUCED WORDS FOR S_n.

TABLEAUX WHOSE READING WORDS ARE REDUCED WORDS FOR ELEMENTS OF S_n

FOR $S_3 = \langle s_1, s_2 \rangle$

$w_0 = 121 = 212$

$1\,2 \leftarrow 1$ EDELMAN GREENE

FIND THE LARGEST ENTRY $> i$

THAT ENTRY IS BUMPED TO THE NEXT ROW EXCEPT:

INSTEAD OF REPLACING IT IS

IS UNCHANGED.

$$1\,2 \leftarrow 1 = \begin{array}{ll} 1 & 2 \\ 2 & \end{array} = 212$$

THIS ALWAYS WORKS:

IF THE READING FOR T IS w REDUCED AND

$w \cdot \Delta_i$ IS STILL REDUCED

$w \leftarrow i$ IS ANOTHER REDUCED WORD FOR $w\Delta_i$

$(\Delta_1 \Delta_2) \cdot \Delta_1$

$$1\,2 \leftarrow 1 = \begin{array}{ll} 1 & 2 \\ 2 & \end{array}$$