TWO STORIES THAT CAN BE TOLD IN PARALLEL.

I TOKUYAMA $q=0$ MODELS

PLACTIC MONOID: MULTIPLICATION OF CRYSTALS

RSK = ROBINSON-SCHENSTED-KNUTH

BASIC ALGORITHM IN TABLEAU COMBINATORICS

BRACKETING: IN APPLYING CRYSTAL OPERATIONS WHAT DOES e_i, f_i DO?

II PIPEDREAMS

WEYL GROUP (FOCUS ON REDUCED ELEMENTS)

EDELMAN-GREENE INSERTION.

BRACKETING: IN APPLYING CRYSTAL OPERATIONS WHAT DOES e_i, f_i DO?

GOALS: BRACKETING IN CASE II.

INTRODUCE EDELMAN-GREENE AND USE IT TO PROVE STANLEY'S THM:

\# OF REDUCED WORDS FOR $w_0 \in S_n$ = EXPLICIT NUMBER.

$\mathcal{B} : \boxed{1} \xrightarrow{1} \boxed{2} \xrightarrow{2} \boxed{3} \to \cdots \xrightarrow{n-1} \boxed{n}$

: STANDARD CRYSTAL FOR $GL(n)$

: COMBINATORIAL SUBSTITUTE FOR STANDARD MODULE $\mathbb{C}^n$ OF $GL(n)$.

$$\bigotimes^k \mathcal{B} = \bigoplus_{\lambda \vdash k} f_\lambda \mathcal{B}_\lambda$$

λ A PARTITION OF k,

IF $k = 3$ $\quad f_{(3)} = f_{(1,1,1)} = 1$

$f_{(2,1)} = 2$

IN GENERAL THERE IS AN IRR REP'N OF S_k DENOTED $\pi_\lambda^{S_k}$ OF DIM'N f_λ.

$1, 1, 2$: DIMS. OF IRR REPS OF S_3

IN REP'N THEORY OF $GL(n)$

SCHUR-WEYL DUALITY:

$$\otimes^k \mathbb{C}^n = \mathbb{C}^n \otimes \cdots \otimes \mathbb{C}^n$$

HAS COMMUTING REPS OF S_k AND $GL(n)$

$$\otimes^k \mathbb{C}^k = \bigoplus_{\substack{\lambda \vdash k \\ \ell(\lambda) \leq n}} \Pi_\lambda^{S_k} \otimes \Pi_\lambda^{GL(n)},$$

$n=3$ THIS SHOWS

$$\otimes^3 \mathbb{C}^n = \mathrm{SYM}^3(\mathbb{C}^n) \oplus \Lambda^3 \mathbb{C}^n \oplus 2\, \Pi^{GL(n)}_{(2,1,0,\dots)}$$

THIS EXPLAINS THE REASON THE REP'N THEORY OF S_k IS INVOLVED.

BY ANALOGY $$\otimes^k B = \bigoplus_{\substack{\lambda \vdash k \\ \ell(\lambda) \leq n}} \rho_\lambda \cdot B_\lambda .$$

$\mathcal{B}_\lambda \hookrightarrow \otimes^k \mathcal{B}$

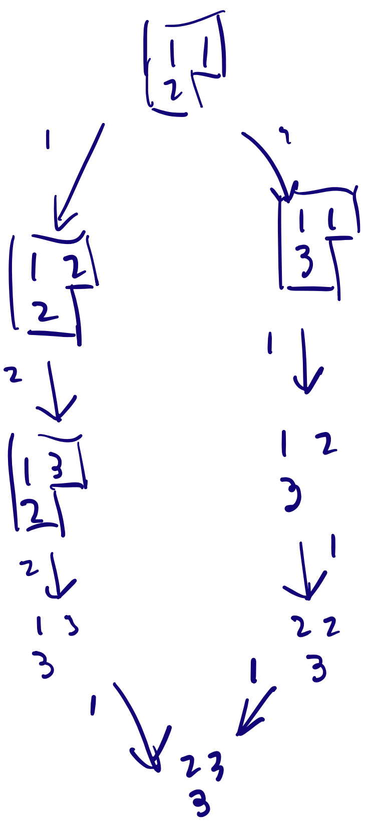
$T \rightsquigarrow R_N \otimes R_{N-1} \otimes \cdots \otimes R_1$

$R_1, \ldots, R_N$

Rows.

$R_i = \underbrace{\boxed{a_1}\boxed{a_2}\cdots\boxed{a_{\lambda_i}}} \hookrightarrow \boxed{a_1} \otimes \boxed{a_2} \otimes \cdots$

$a_1 \leq a_2 \leq \cdots \leq a_n.$



$\hookrightarrow \otimes^3 \mathcal{B}.$

$\begin{array}{|cc|} \hline a & b \\ c & \\ \hline \end{array} \rightsquigarrow \boxed{c} \otimes \boxed{a} \otimes \boxed{b}.$

$a \leq b$

$a < c.$

THIS GIVES 8 ELEMENTS.

BRACKETING OPERATION DESCRIBES PROMOTION EXACTLY.

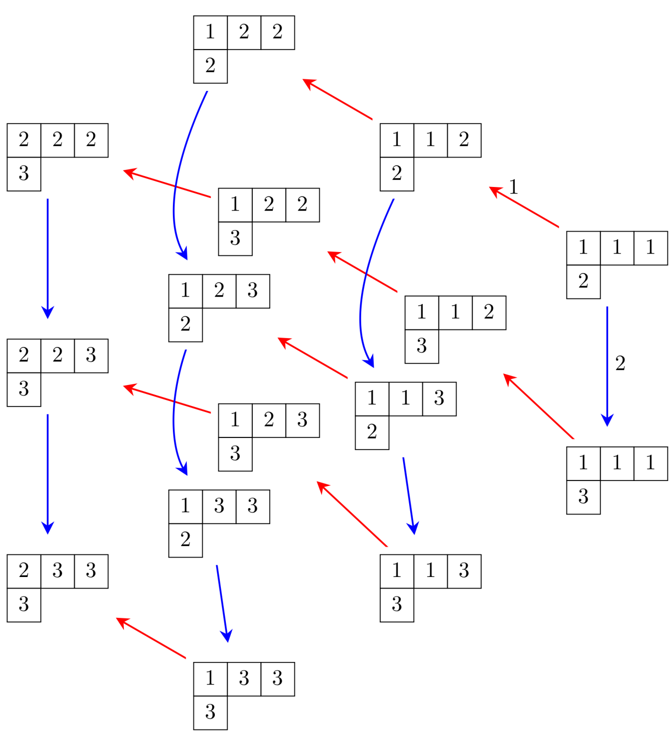
TO COMPUTE e_i OR f_i OF

$$T = \boxed{a_1} \otimes \boxed{a_2} \otimes \cdots \otimes \boxed{a_N}$$

f_i CHANGES SOME $i \to i+1$

e_i " $i+1 \to i$

EXCEPT SOMETIMES $f_i(T) = 0$



$GL(n)$

$\downarrow$

$U_q(\mathfrak{gl}(n))$

$$f_1 = \begin{pmatrix} 0 & 1 & \\ & 0 & \\ & & 0 \end{pmatrix}$$

$$e_1 = \begin{pmatrix} 0 & & \\ 1 & 0 & \\ & & 0 \end{pmatrix}$$

$$V(\mu) = \{ v \in V \mid H \cdot v = \mu(H) v \}.$$

$$V(\mu) \underset{e_i}{\overset{f_i}{\rightleftarrows}} V(\mu - \alpha_i)$$

$$\boxed{a_1} \otimes \boxed{a_2} \otimes \cdots \otimes \boxed{a_N}$$

f_i or e_i: IGNORE ALL ENTRIES $\neq i, i+1$

$$\boxed{3} \otimes \boxed{2} \otimes \boxed{2} \otimes \boxed{1} \otimes \boxed{1} \otimes \boxed{3} \otimes \boxed{1} \otimes \boxed{2}$$

$\boxed{m}$ IS INVISIBLE IF $m \neq i, i+1$

$\boxed{i+1} \otimes \boxed{i}$ IS INVISIBLE.

$$f_i(\boxed{m} \otimes x) = \boxed{m} \otimes f_i(x)$$
$$f_i(\boxed{i+1} \otimes \boxed{i} \otimes x) = \boxed{i+1} \otimes \boxed{i} \otimes f_i(x)$$

$\boxed{i+1} \otimes \boxed{m} \otimes \boxed{i}$ INVISIBLE $m \neq i,\ i+1$.

LOOK FOR $\boxed{i+1} \otimes \left(\text{INVISIBLE STUFF} \right) \otimes \boxed{i}$

BRACKET.

$i = 1$

$$f_1\left(\boxed{3}\otimes\boxed{2}\otimes\boxed{2}\otimes\boxed{1}\otimes\boxed{1}\otimes\boxed{3}\otimes\boxed{1}\otimes\boxed{2}\right)$$

BRACKET

BRACKET

UNBRACKETED

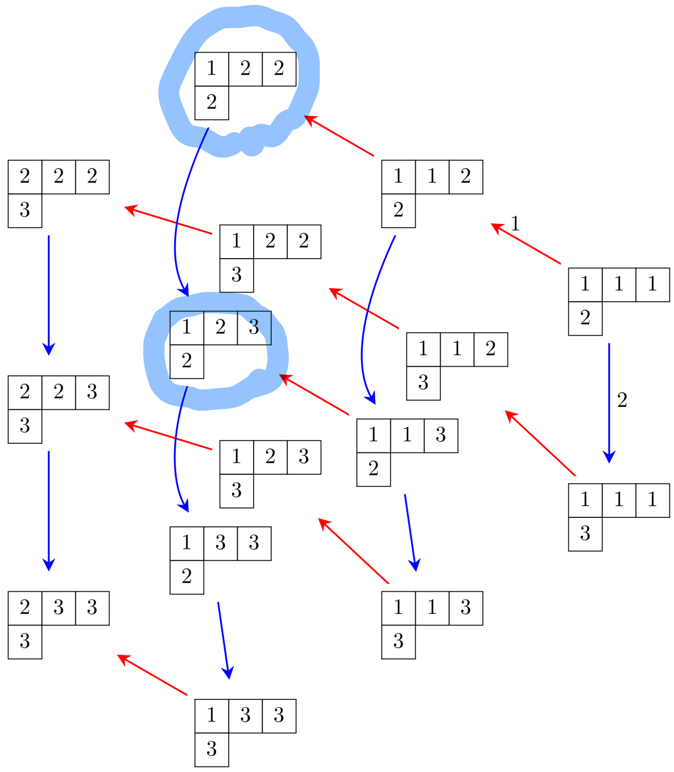
$$= \boxed{3}\otimes\boxed{2}\otimes\boxed{2}\otimes\boxed{1}\otimes\boxed{1}\otimes\boxed{3}\otimes\boxed{2}\otimes\boxed{2}$$

RULE: AFTER BRACKETING f_i CHANGES RIGHT MOST UNBRACKETED $i \to i+1$.

IF ALL i's BRACKETED $f_i(T) = 0$.

IF ALL $i+1$ ARE BRACKETED $e_i(T)=0$

OTHERWISE e_i SENDS LEFTMOST $\boxed{i+1} \to \boxed{i}$.



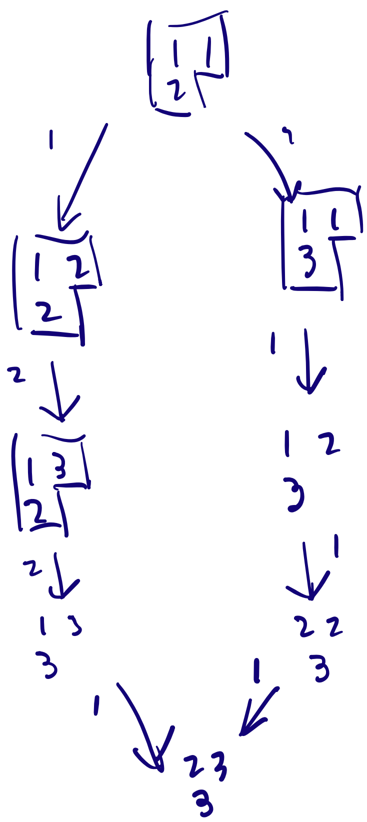
$$f_2\begin{pmatrix} 1 & 2 & 2 \\ 2 & & \end{pmatrix} = f_2(2 \otimes 1 \otimes 2 \otimes 2)$$

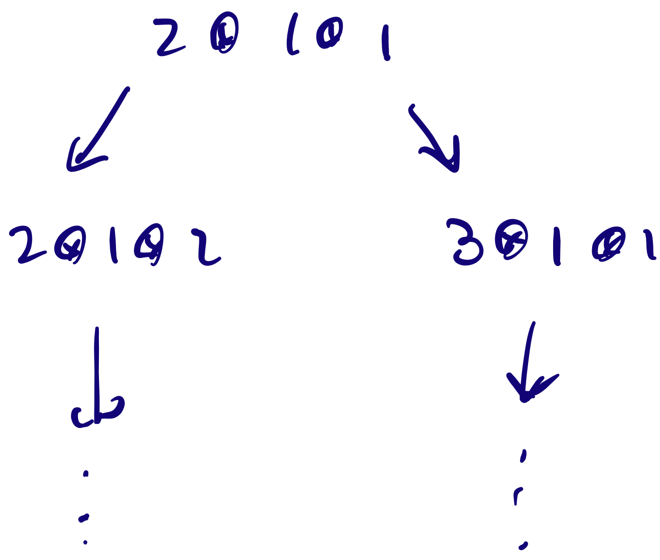
$$2 \otimes 1 \otimes 2 \otimes 3 = \begin{pmatrix} 1 & 2 & 3 \\ 2 & & \end{pmatrix}$$

THE PLACTIC MONOID

(LASCOUX & SCHUTZENBERGER — BASIC ALGORITHMS ARE DUE TO SCHENSTED AN KNUTH.)

DEFINE AN EQUIVALENCE RELATION ON $\bigotimes^n \mathcal{B} = \bigoplus$ IRREDUCIBLE CRYSTALS.





THERE MAY BE OTHER SUBCRYSTALS OF $\otimes^k \mathcal{B}$ THAT ARE ISOMORPHIC

BUT NOT IN THE IMAGES OF THE EMBEDDINGS $\mathcal{B}_\lambda \to \otimes^k \mathcal{B}$.

$$1 \otimes 2 \otimes 1$$

$e_1(1 \otimes \underbrace{2 \otimes 1}_{\text{BRACKET}}) = 0$ SINCE NO UNBRACKETED 2

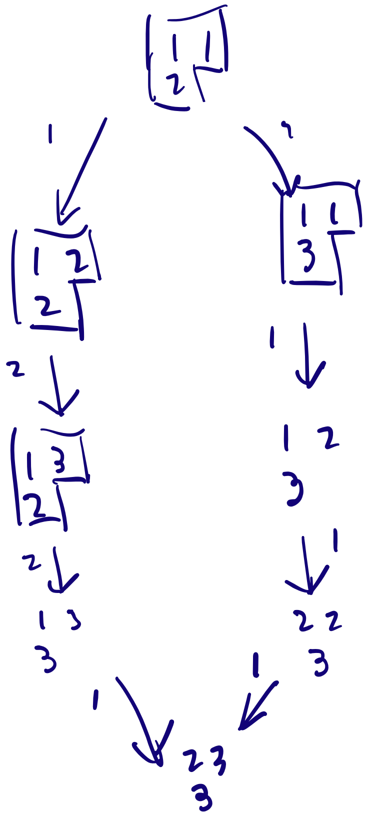
$e_2(1 \otimes 2 \otimes 1) = 0$ NO UNBRACKETED 3.

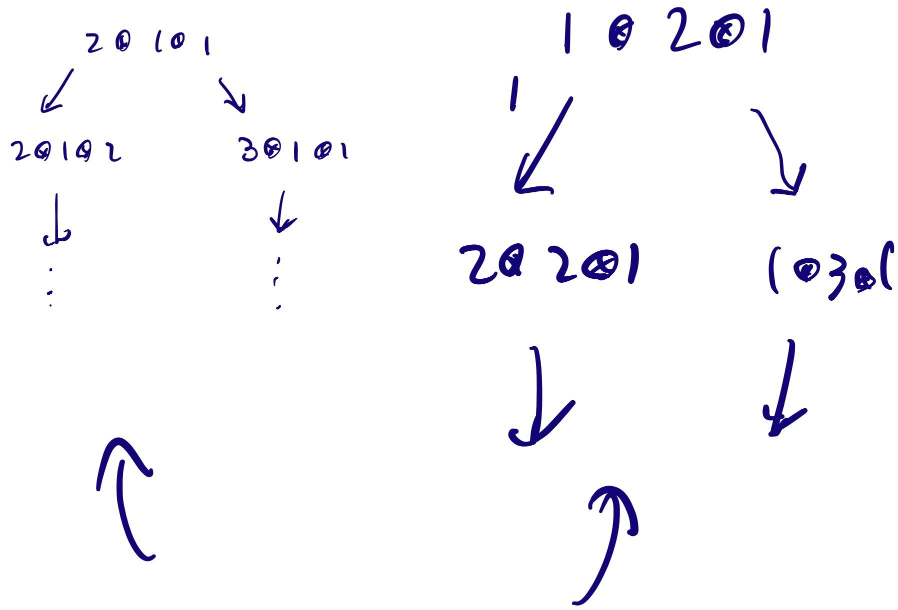
$e_i(1 \otimes 2 \otimes 1) = 0$ ALL i

SO $1 \otimes 2 \otimes 1$ IS A HIGHEST WEIGHT ELT.

IT GENERATES A CRYSTAL OF HIGHEST WEIGHT $(2,1,0)$

I.E. A CRYSTAL ISOMORPHIC TO $\mathcal{B}_{(2,1,0)}$





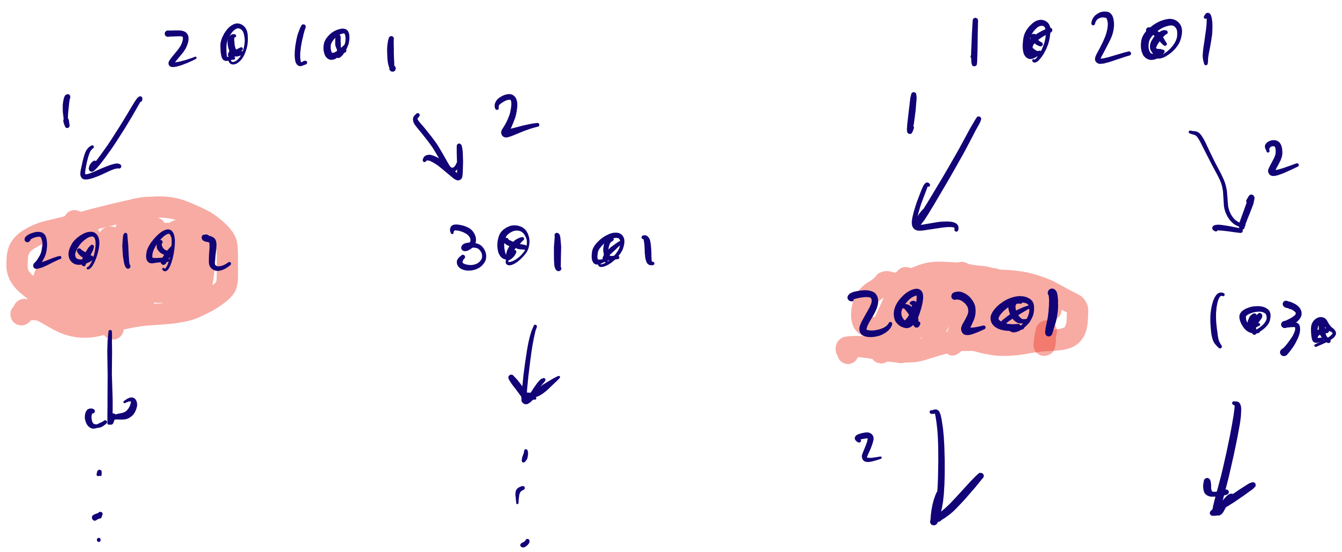
Two copies of the same crystal.

Both embedded in $\bigotimes^3 \mathcal{B}$.

Def: $T, T' \in \bigotimes^k \mathcal{B}$ are plactically equivalent if they are in isomorphic subcrystals $\mathcal{C}, \mathcal{C}' < \bigotimes^k \mathcal{B}$ and the unique

T, T'

ISOMORPHISM $\mathcal{C} \to \mathcal{C}'$ TAKES $T \to T'$.



$$\overset{b}{2}\otimes\overset{a}{1}\otimes\overset{c}{2} \equiv 2\otimes 2\otimes 1.$$

DONALD KNUTH DETERMINED ALGORITHMICALLY WHAT PLACTIC EQUIVALENCE AMOUNTS TO

(1) If $a < b \leqslant c$, then bac is Knuth equivalent to bca and vice versa.
(2) If $a \leqslant b < c$, then acb is Knuth equivalent to cab and vice versa.