Demazure operators and Demazure characters

DEMAZURE OPS WERE INTRODUCED INDEPENDENTLY BY BERNSTEIN, GELFAND & GELFAND AND DEMAZURE (EARLY 1970'S) IN CONNECT WITH SCHUBERT GEOMETRY.

LET S_n ACT ON $\overset{R}{\overset{\shortparallel}{\mathbb{C}}}[x_1, \dots, x_n]$ BY PERMUTING VARIABLES $f \in R$ THEN f ACTS BY MULTIPLICATION.

S_n, R BOTH ARE IN $\mathrm{END}(R)$

$$w f w^{-1} = {}^{w}f$$

<u>LEMMA</u>: IF $f \in R$ THEN $\dfrac{f - {}^{s_i}f}{x_i - x_{i+1}} \in R$

$s_i = (i, i+1)$

PROOF: NUMERATOR VANISHES WHERE $x_i = x_{i+1}$ SO ZEROS OF NUMERATOR AND DENOMINATOR CANCEL.

THE OPERATOR $D_i = (x_i - x_{i+1})^{-1}(1 - \Delta_i)$

MAPS $f \mapsto \dfrac{f - {}^{\Delta_i}f}{x_i - x_{i+1}}$

THESE DIVIDED DIFFERENCE OPS ARE ONE KIND OF DEMAZURE OPERATOR.

$g : D_i f$ SATISFIES $\Delta_i g = g$

(SYMMETRIC IN $x_i \leftrightarrow x_{i+1}$.)

IF f IS SYMMETRIC IN $x_i \leftrightarrow x_{i+1}$

THEN $D_i f = 0$ THEREFORE

$$D_i^2 = 0$$

ALSO D_i SATISFY BRAID RELATIONS;

$$D_i D_j = D_j D_i \quad \text{IF } |i-j| > 1$$

$$D_i D_{i+1} D_i = D_{i+1} D_i D_{i+1}$$

PROOF OF SECOND RELATION:

$D_1 D_2 D_1 = D_2 D_1 D_2$ LEFT SIDE =

$$(x_1 - x_2)^{-1}(1 - \Delta_1)(x_2 - x_3)^{-1}(1 - \Delta_2)(x_1 - x_2)^{-1}(1 - \Delta_1)$$

$$= (x_1 - x_2)^{-1}(x_1 - x_3)^{-1}(x_2 - x_3) \sum_{\omega \in S_3} \mathrm{sgn}(\omega) \cdot \omega$$

EXPAND AND COLLECT COEFS TO WRITE

$\sum_{\omega \in W} f_\omega \cdot \omega$ ALL TERMS EXCEPT f_1, f_{Δ_1} ARE EASY TO COMPUTE.

LET'S COMPUTE f_1: EXPAND $1 - \Delta_i$ INTO TWO TERMS. EITHER TAKE

$1, 1, 1$ OR $\Delta_1, 1, \Delta_1$

FIRST CONTRIBUTES $(x_1 - x_2)^{-1}(x_2 - x_3)^{-1}(x_1 - x_2)^{-1}$.

SECOND CONTRIBUTES

$$(x_1 - x_2)^{-1}(-\Delta_1)(x_2 - x_3)^{-1}(1)(x_1 - x_2)^{-1}(-\Delta_1)$$

$$= (x_1 - x_2)^{-1}(x_1 - x_3)^{-1}(x_2 - x_1)^{-1}(-1)^2$$

SO

$$\partial_1 = (x_1 - x_2)^{-1}(x_2 - x_3)^{-1}(x_1 - x_2)^{-1} - (x_1 - x_2)^{-1}(x_1 - x_3)^{-1}(x_1 - x_2)^{-1}$$

$$= (x_1 - x_2)^{-1}(x_1 - x_3)^{-1}(x_2 - x_3)^{-1}.$$

$D_2 D_1 D_2$ HAS SAME EVALUATION

PROVES BRAID RELATION

$$D_1 D_2 D_1 = D_2 D_1 D_2.$$

THE BRAID RELATION IS SOME CLOSE RELATIVE OF YANG-BAXTER EQUATION.

ARTIN BRAID GROUP $\mathcal{B}_{n-1}$

$u_1, \dots, u_{n-1}$ SUBJECT TO BRAID REL'NS

$$u_i u_{i+1} u_i = u_{i+1} u_i u_{i+1}$$

$$u_i u_j = u_j u_i \quad \text{IF } |i-j| > 1.$$

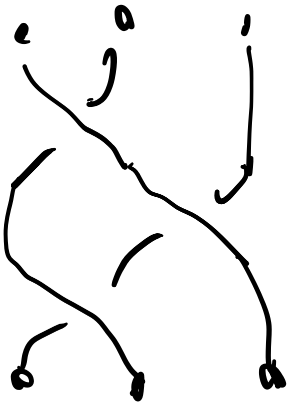
m_1

GENERATORS OF BRAID GROUP

m_2

BRAIDS ARE COMPOSED BY STACKING

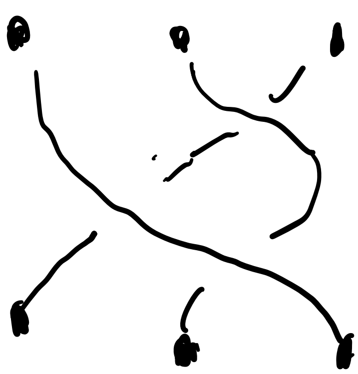
$m_1 m_2 m_1$



m_1

$\downarrow$

$m_2 m_1 m_2$



THESE BRAIDS ARE ISOTOPIC.

WE WILL RETURN TO THIS POINT LATER.

SYMMETRIC GROUP IS A COXETER GROUP.

Δ_i ALSO SATISFY BRAID RELATIONS AND ALSO A "QUADRATIC RELATION"

$$\Delta_i^2 = 1.$$

OTHER USEFUL QUADRATIC RELATIONS.

NILHECKE ALGEBRA (DEMAZURE OPS)

BRAID REL'NS

$$D_i^2 = 0$$

IF q IS AN INDETERMINATE GENS T_i

BRAID REL'N

$$T_i^2 = (q-1)T_i + q.$$

THIS IS THE $GL(n)$ "AFFINE HECKE ALGEBRA."

UBIQUITOUS TWO PLACES IT ARISES:

IWAHORI-MATSUMOTO: THE AFFINE HECKE ALGEBRA ARISES AS A CONVOLUTION RING OF FUNCTIONS ON $GL(n, F)$

$F = \mathbb{Q}_p$, $q = p$ (RESIDUE CARDINALITY)

T_i = CHARACTERISTIC FUNCTION OF

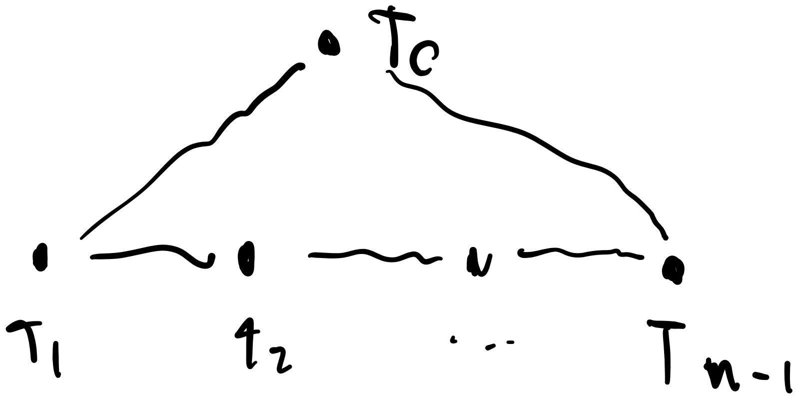
$$J \Delta_i J$$

$$J = \{ g \in GL(n, \mathbb{Z}_p) \mid g \equiv \text{UPPER } \Delta \text{ MODULO } p \}.$$

THIS RING HAS GENS

$T_1, \dots, T_{n-1}$ (QUADRATIC BRAID REL'NS)

AND ONE MORE T_0



$$T_0 T_1 T_0 = T_1 T_0 T_1$$

$$T_0 T_{n-1} T_0 = T_{n-1} T_0 T_{n-1}$$

$T_i T_j = T_i T_j$ IF i, j NOT ADJACENT IN THIS GRAPH.

$T_0, T_1, \dots, T_n$ SATISFY BRAID, QUADRATIC RELATIONS.

LUSZTIG PROVES (1978):

THIS SAME HECKE ALGEBRA CAN BE REALIZED AS A RING ACTING ON EQUIVARIANT K-THEORY OF COMPLEX FLAG VARIETY.

MATSUMOTO'S THEOREM.

RELATION BETWEEN DIFFERENT REDUCED WORDS FOR ELTS OF S_n (OR COXETER GROUPS)

$\omega = s_{i_1} \cdots s_{i_k}$ REDUCED IF k IS MINIMAL.

THEOREM: IF

$$s_{i_1} \cdots s_{i_k} = s_{j_1} \cdots s_{j_k}$$

ARE TWO REDUCED EXPRESSIONS THEIR EQUALITY CAN BE PROVED USING ONLY BRAID RELATIONS.

EXAMPLE: IN S_4

$$\omega_0 = s_1 s_2 s_1 s_3 s_2 s_1 = s_3 s_2 s_3 s_1 s_2 s_3$$

1 2 1 3 2 1   3 2 3 1 2 3

LONGEST ELT.

$(1,4)(2,3)$ $\ell(\omega_0) = 6$

1 2 1 3 2 1

1 2 3 1 2 1

1 2 3 2 1 2

1 3 2 3 1 2

3 1 2 1 3 2

3 2 1 2 3 2

3 2 1 3 2 3

3 2 3 1 2 3

$\Delta_1 \Delta_3 = \Delta_3 \Delta_1$

$\Delta_1 \Delta_2 \Delta_1 = \Delta_2 \Delta_1 \Delta_2$

THEOREM: IF

$$\Delta_{i_1} \cdots \Delta_{i_k} = \Delta_{j_1} \cdots \Delta_{j_k}$$

ARE TWO REDUCED EXPRESSIONS THEIR EQUALITY CAN BE PROVED USING ONLY BRAID RELATIONS. CONCLUSION REALLY MEANS

$$u_{i_1} \cdots u_{i_k} = u_{j_1} \cdots u_{j_k}$$

IN B_{n-1}.

APPLICATION TO NIL HECKE ALG

GENS D_i S.T. BRAID REL'N AND

$D_i^2 = 0$.

IF $w \in S_n$ LET $w = s_{i_1} \cdots s_{i_k}$. (REDUCED)

THEN WE CAN DEFINE

$$D_w = D_{i_1} \cdots D_{i_k}.$$

DEMAZURE OPERATORS. WELL-DEF'D BY MATSUMOTO'S THEOREM SINCE D_i SATISFY BRAID RELATIONS.

SCHUBERT POLYNOMIALS.

$\Lambda = \mathbb{Z}^n$ (THINK OF $GL(n)$ WEIGHT LATTICE)

IF $\lambda = (\lambda_1, \dots, \lambda_n) \in \Lambda$

$$x^\lambda = x_1^{\lambda_1} \cdots x_n^{\lambda_n}$$

$$\rho = (n-1, n-2, \ldots, 0)$$ "WEYL VECTOR".

A WEYL VECTOR SATISFIES

$$\Delta_i \rho = \rho - \alpha_i$$

$$\alpha_i = (0, \ldots, 0, \underset{\substack{\uparrow \\ i\text{TH} \\ \text{POSITION}}}{1}, -1, 0, \ldots)$$

SIMPLE ROOTS.

DEFINITION:

$$\mathfrak{S}_w(x) = \Delta_{w^{-1}w_0} \cdot x^\rho$$

SCHUBERT POLYNOMIALS

w	$\mathfrak{S}_w$
1234	1
1243	$x_1 + x_2 + x_3$
1324	$x_1 + x_2$
1342	$x_1x_2 + x_1x_3 + x_2x_3$
1423	$x_1^2 + x_1x_2 + x_2^2$
1432	$x_1^2x_2 + x_1^2x_3 + x_1x_2^2 + x_1x_2x_3 + x_2^2x_3$
2134	x_1
2143	$x_1^2 + x_1x_2 + x_1x_3$
2314	x_1x_2
2341	$x_1x_2x_3$
2413	$x_1^2x_2 + x_1x_2^2$
2431	$x_1^2x_2x_3 + x_1x_2^2x_3$
3124	x_1^2
3142	$x_1^2x_2 + x_1^2x_3$
3214	$x_1^2x_2$
3241	$x_1^2x_2x_3$
3412	$x_1^2x_2^2$
3421	$x_1^2x_2^2x_3$
4123	x_1^3
4132	$x_1^3x_2 + x_1^3x_3$
4213	$x_1^3x_2$
4231	$x_1^3x_2x_3$
4312	$x_1^3x_2^2$
4321	$x_1^3x_2^2x_3$

MACDONALD USES 1-LINE NOTATION FOR PERMUTATIONS

$$(1324) = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 3 & 2 & 4 \end{pmatrix} = \Delta_2 .$$