STANLEY SYMMETRIC FUNCTIONS

MORSE SCHILLING CRYSTALS

$\Downarrow$

CRYSTAL STRUCTURES ON PIPEDREAMS.

1984 STANLEY COMPUTED THE NUMBER OF REDUCE EXPRESSIONS FOR $w = w_0$ IN S_n BY A METHOD APPLICABLE TO OTHER w.

STANLEY SYMMETRIC FUNCTIONS.

<u>THM</u>: THERE $\dfrac{\binom{n}{2}!}{1^{n-1} \cdot 3^{n-2} \cdot 5^{n-3} \cdots (2n-3)^1}$.

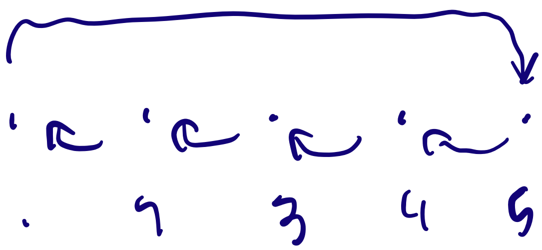
REDUCED EXPRESSIONS FOR w_0.

DEF: A PERMUTATION IS <u>DECREASING</u> IF IT HAS A REDUCED DECOMPOSITION

$$s_{i_1} s_{i_2} \cdots s_{i_k} \qquad i_1 > i_2 > \cdots > i_k$$

$\Delta_4 \, \Delta_3 \, \Delta_2 \, \Delta_1 \,:\; (45)(34)(23)(12) =$

$(1, 5, 4, 3, 2)$



DISJOINT PRODUCT OF CYCLES OF THIS TYPE.

A DECREASING FACTORIZATION OF w IS A REPRESENTATION

$$w = w_k \cdots w_2 w_1$$

w_k DECREASING PERMUTATIONS $w_i = 1$ ALLOWED

$$\ell(w) = \ell(w_k) + \dots + \ell(w_2) + \ell(w_1)$$

IN S_3 THERE ARE THE FOLLOWING DECREASING FACTORIZATIONS OF

w_0

INTO A PRODUCT OF 3 DECREASING FACTORS.

THERE ARE 8

$$\begin{array}{c} 1 \cdot s_1 \cdot s_2 s_1 \\ (\ )(1)(21) \end{array} = w_0$$

$$\begin{tikzcd}
& (\ )(1)(21) \arrow[dl] \arrow[dr] & \\
(1)(\ )(21) \arrow[d] & & (\ )(21)(2) \arrow[d] \\
(1)(2)(1) \arrow[d] & & (2)(1)(2) \arrow[d] \\
(1)(21)(1) \arrow[dr] & & (21)(\ )(2) \arrow[dl] \\
& (21)(2)(\ ) &
\end{tikzcd}$$

IF $w = w_n \cdots w_1$ IS A D.F. OF LENGTH k

$$wt(w) = (\ell(w_1), \ell(w_2), \cdots, \ell(w_n))$$

$$\begin{array}{c} 1 \cdot s_1 \cdot s_2 s_1 \\ (\ )(1)(21) \end{array}$$ HAS WEIGHT

$$(2, 1, 0).$$

THE WEIGHTS ARE:

$$\begin{tikzcd}
 & \underset{1\cdot s_1 \cdot s_2 s_1}{(\ )(1)(21)} = w_0 \arrow[dl] \arrow[dr] & \\
(\ )(\ )(21) \arrow[d] & & (\ )(21)(2) \arrow[d] \\
(1)(2)(1) \arrow[d] & & (2)(1)(2) \arrow[d] \\
(1)(21)(\ ) \arrow[dr] & & (21)(\ )(2) \arrow[dl] \\
 & (21)(2)(\ ) &
\end{tikzcd}$$

$(2,1,0)$

$(2,0,1)$ $(1,2,0)$

$(1,1,1)$ $(1,1,1)$

$(0,2,1)$ $(1,0,2)$

$(0,1,2)$

<u>THEOREM</u>: $\displaystyle\sum_{\substack{\text{DECREASING} \\ \text{FACT'N} \\ a = w_k \cdots w_1}} z^{wt(a)} = F_w(z)$

IS A SYMMETRIC POLYNOMIAL

STANLEY SYMMETRIC FUNCTIONS.

F_{w_0} IS A SCHUR POLYNOMIAL.

THEOREM: STANLEY, EDELMAN-GREENE, ...

F_w IS SCHUR POSITIVE $\displaystyle F_w = \sum_{\lambda \vdash \ell(w)} c_\lambda s_\lambda \qquad c_\lambda \overset{?}{\geq} 0$

THEOREM (MORSE, SCHILLING) DECREASING FACT'NS HAVE A CRYSTAL STRUCTURE. THIS IS A STEMBRIDGE CRYSTAL.

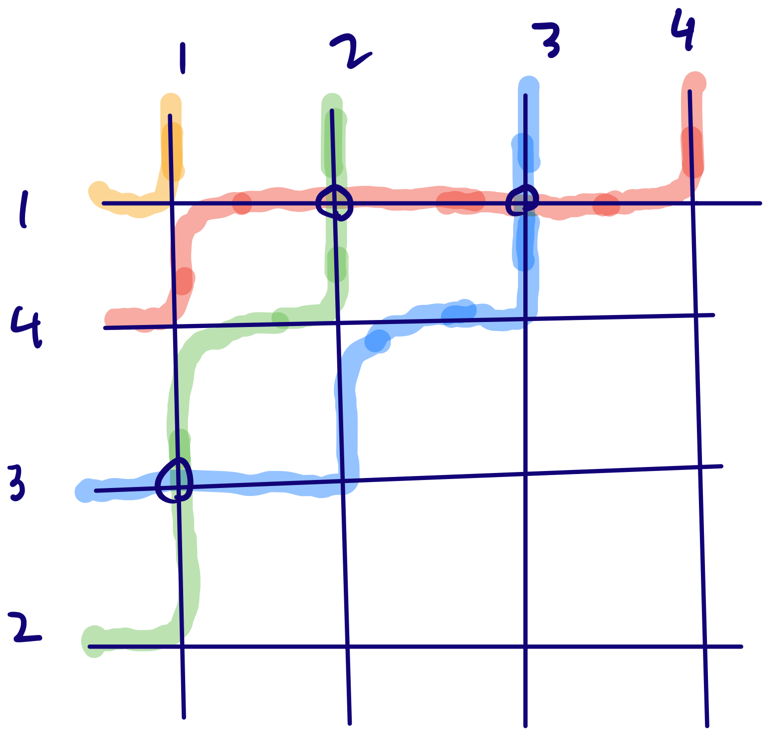
DIGRESSION: A SOURCE OF DECREASING FACTORIZATIONS DOES COME FROM PIPEDREAMS.

$$\ell(w_0) = \tfrac{1}{2}n(n-1) =$$

$$p = (n-1, n-2, \dots, 1, 0)$$

FOR S_n $\quad F_{w_0} = \Delta_p$.

FOR PIPEDREAMS, WE GET DECREASING FACTORIZATIONS AS FOLLOWS.



TO DRAW A PIPEDREAM CHOOSE LOCATIONS OF CROSSINGS. THEN FILL IT IN...

✓ HOPE NO PATH CROSS TWICE.

THEN WE CAN READ OF A REDUCED WORD OF LENGTH = # OF CROSSINGS.

FIRST ROW PRODUCES

$$\Delta_3 \quad \Delta_2 = \omega_1$$

$(34) \quad (23)$

SECOND ROW CAUSES NO CROSSING

$$1_W = \omega_2$$

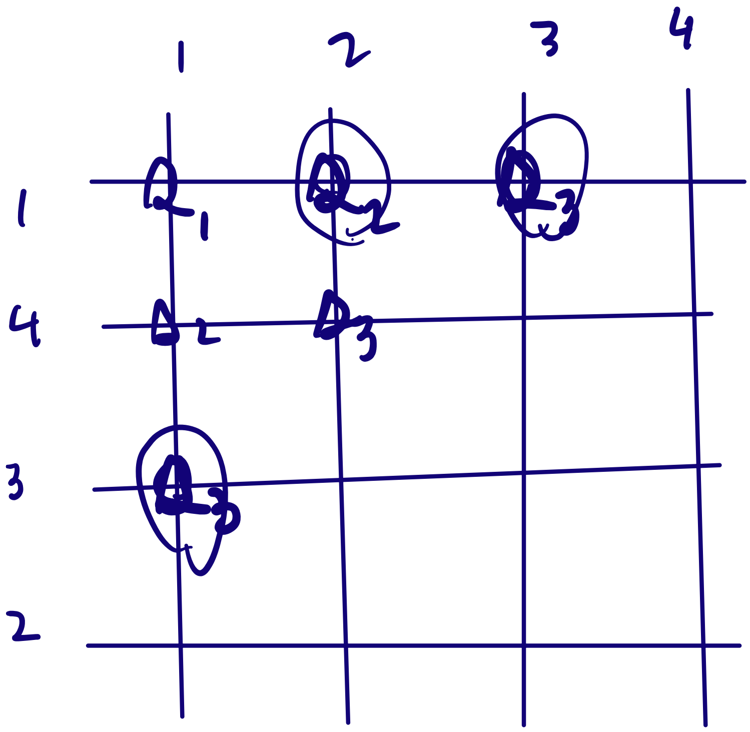
THIRD CROSSING SWITCHES GREEN & BLUE.

$$\Delta_3 = \omega_3.$$

THE RULE FOR WHICH SIMPLE REFLECTION IS PRODUCED BY A CROSSING:

IF CROSSING IS IN ROW i, COL. j

Δ_{i+j-1}. FOR THIRD ROW $i=3$, $j=1$ SO $\omega_3 = \Delta_3$



$$(\Delta_3)(\quad)(\Delta_2\ \Delta_3)$$

THIS IS A DECREASING FACTORIZATION OF A SPECIAL TYPE.

Δ_1 CAN'T APPEAR IN $\omega_2, \omega_3, \ldots$

Δ_2 CAN'T APPEAR IN $\omega_3, \omega_4, \ldots$

NO SUCH CONSTRAINT IN THE FACTORIZATIONS RELATED TO STANLEY S.F.

RETURN TO MORSE SCHILLING CRYSTALS.

WHAT DOES e_i OR f_i DO ?

ALGORITHM. e_i OR f_i ADJUSTS

w_i AND w_{i+1}

$1 \cdot s_1 \cdot s_2 s_1$

$$\begin{tikzcd}[column sep=tiny]
 & (\ )(1)(21) = w_0 \arrow[dl] \arrow[dr, "f_1"] & \\
(1)(\ )(21) \arrow[d] & & (\ )(21)(2) \arrow[d] \\
(1)(2)(1) \arrow[d] & & (2)(1)(2) \arrow[d] \\
(1)(21)(\ ) \arrow[dr, no head] & & (21)(\ )(2) \arrow[dl, no head] \\
 & (21)(2)(\ ) &
\end{tikzcd}$$

$f_i(a)$ HAS w_i REDUCED BY 1, w_{i+1} INCREASED BY 1

$\Uparrow$ D. FACT'N SOMETIMES IN A NONTRIVIAL WAY.

CONT(w_i) = SET OF SIMPLE REFL. IN i.

WE WANT TO BRACKET OR PAIR SOME ELEMENTS OF CONT(w_i) WITH SOME ELEMENTS OF w_{i+1}.

Example 10.5. *Let* $(s_3s_2)(s_3s_1)(s_2) \in \mathcal{W}_w^3$ *for* $w = s_3s_2s_3s_1s_2 \in S_4$. *To apply* e_2, *we need to first bracket the letters in* $\mathrm{cont}(w^3) = 32$ *with those in* $\mathrm{cont}(w^2) = 31$. *The letter* 3 *in* $\mathrm{cont}(w^3)$ *is unbracketed since there is no bigger letter in* $\mathrm{cont}(w^2)$, *but the letter* 2 *in* $\mathrm{cont}(w^3)$ *is bracketed with* 3 *in* $\mathrm{cont}(w^2)$. *Hence* $b = \min\bigl(L_2(w^3w^2w^1)\bigr) = 3$ *and* $t = \min\{j \geqslant 0 \mid b-j-1 \notin \mathrm{cont}(w^3)\} = 1$. *Therefore,* $e_2\bigl((s_3s_2)(s_3s_1)(s_2)\bigr) = (s_2)(s_3s_2s_1)(s_2)$. *Similarly,* $f_2\bigl((s_3s_2)(s_3s_1)(s_2)\bigr) = (s_3s_2s_1)(s_3)(s_2)$.

RULE: START WITH THE LARGEST ELEMENT OF CONT(w_{i+1}).

Let us begin by describing the bracketing procedure in analogy to the bracketing procedure in Section 2.4. Start with the largest letter b in $\mathrm{cont}(w^{i+1})$ and pair it with the smallest $a > b$ in $\mathrm{cont}(w^i)$. If there is no such a in $\mathrm{cont}(w^i)$, then b is unpaired. The pairing proceeds in decreasing order on elements of $\mathrm{cont}(w^{i+1})$, and with each iteration, previously paired letters of $\mathrm{cont}(w^i)$ are ignored. Define

$$L_i(w^\ell \cdots w^1) = \{b \in \mathrm{cont}(w^{i+1}) \mid b \text{ is unpaired in the } w^{i+1}w^i\text{-pairing}\}$$

and

$$R_i(w^\ell \cdots w^1) = \{b \in \mathrm{cont}(w^i) \mid b \text{ is unpaired in the } w^{i+1}w^i\text{-pairing}\}\,.$$

$w_{i+1} = (1)$ BRACKET 1

$w_i = (2\,1)$ $\downarrow$

2

WEDNESDAY: BRACKETING

FOR BOTH STANLEY S. F. AND FOR

PROEDREAMS.