GOAL: CRYSTAL STRUCTURES ON PIPEDREAM.

MORE ABOUT CRYSTALS AND CRYSTAL STRUCTURES OF $q=0$ TOKUYAMA MODELS.

(DEMAZURE CRYSTAL STRUCTURES ON CLOSED MODELS, PIPEDREAM MODELS.)

λ A PARTITION $\lambda = (\lambda_1, \dots, \lambda_n)$

$\mathcal{B}_\lambda$ = SEMISTANDARD YT IN $\{1, \dots, n\}$.

$$\mathrm{CH}_{\mathcal{B}_\lambda} = \sum_{T \in \mathcal{B}_\lambda} z^{\mathrm{wt}(T)} = \Delta_\lambda(z).$$

$\mathcal{B}_\lambda$ HAVE A TENSOR PRODUCT OPERATION.
EXACTLY MIRRORS TENSOR PRODUCT OF IRR REPS OF $GL(n, \mathbb{C})$

Δ_λ IS CHAR OF $\pi_\lambda^{GL(n)}$

$$\Delta_\lambda \cdot \Delta_\mu = \sum c_{\lambda\mu}^{\nu} \Delta_\nu \qquad \pi_\lambda^{GL(n)} \otimes \pi_\mu^{GL(n)} = \bigoplus c_{\lambda\mu}^{\nu} \pi_\nu$$

LITTLEWOOD RICHARDSON COEFS.

$$\mathcal{B}_\lambda \otimes \mathcal{B}_\mu = \bigsqcup_\nu c_{\lambda\mu}^{\nu} \mathcal{B}_\nu$$

(X) $e_i, f_i : \mathcal{B} \to \mathcal{B} \cup \{0\}$

$\varepsilon_i, \varphi_i : \mathcal{B} \to \mathbb{Z}$.

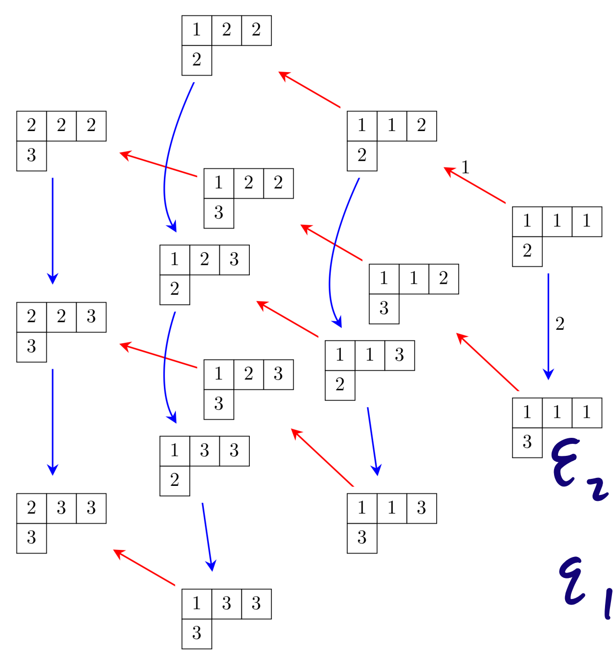
$wt : \mathcal{B} \to \Lambda = \mathbb{Z}^n$.

$wt(T) = (\mu_1, \dots, \mu_n)$

$wt(T)$ = # i IN ENTRIES

$\varepsilon_i(x)$ = # of times e_i CAN BE APPLIED.

$\varphi_i(x)$ = $\quad$ " $\quad$ f_i



$x \xrightarrow{i} y$

MEANS

$f_i(x) = y$

$\iff e_i(y) = x$.

$\varepsilon_2\left(\begin{smallmatrix}1 & 1 & 3\\ 2\end{smallmatrix}\right) = 1$

$\varepsilon_1\left(\begin{smallmatrix}1 & 1 & 3\\ 2\end{smallmatrix}\right) = 0$

TENSOR PRODUCT:

$$f_i(x \otimes y) = \begin{cases} f_i(x) \otimes y & \text{IF } \varepsilon_i(x) \geq \varphi_i(y) \\ x \otimes f_i(y) & \text{IF } \varepsilon_i(x) < \varphi_i(y). \end{cases}$$

THIS IS ASSOCIATIVE.

$$(\mathcal{B} \otimes \mathcal{C}) \otimes \mathcal{D} = \mathcal{B} \otimes (\mathcal{C} \otimes \mathcal{D})$$
$$= \mathcal{B} \otimes \mathcal{C} \otimes \mathcal{D}.$$

IF $f_2\left(\begin{array}{|c|c|}\hline 1 & 2 \\ \hline 2 \\ \cline{1-1}\end{array}\right) = \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 \\ \cline{1-1}\end{array}$ OR $\begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 \\ \cline{1-1}\end{array}$

WHICH?

f_2 CHANGES A 2 TO A 3

f_i: $\boxed{i} \to \boxed{i+1}$

e_i: $\boxed{i+1} \to \boxed{i}$

$\lambda = (k) = (k, 0, \cdots, 0)$ $\mathcal{B}_{(k)}$ IS A CRYSTAL OF ROWS.

$T = \boxed{v_1 \mid \cdots \mid v_k}$ RAW TABLEAU.

FOR CRYSTALS OF ROWS e_i AND f_i ARE CLEAR; THERE IS ONLY ONE POSSIBLE RESULT.

$f_i(T) = T'$ IF THERE IS SOME $i \in T$

T' IS OBTAINED BY CHANGING RIGHTMOST i TO $i+1$

$f_i(T) = 0$ IF $i \notin T$.

CRYSTAL OPERATIONS ARE KNOWN FOR ROW CRYSTALS.

$$\mathcal{B}_\lambda \hookrightarrow \mathcal{B}_{(\lambda_k)} \otimes \mathcal{B}_{(\lambda_{k-1})} \otimes \cdots$$

λ_k = LAST NON ZERO ENTRY.

$$T \to R_k \otimes R_{k-1} \otimes \cdots$$

$$\mathcal{B}_{(2,1)} \hookrightarrow \mathcal{B}_{(1)} \otimes \mathcal{B}_{(2)}$$

$$\begin{array}{ll} R_1 \rightsquigarrow & \begin{array}{|c|c|}\hline a & b \\ \hline c & \\ \cline{1-1}\end{array} \\ R_2 \rightsquigarrow & \end{array} \qquad R_2 \otimes R_1$$

$$\|$$

$$\boxed{c} \otimes \boxed{a \mid b} = \boxed{c} \otimes \boxed{a} \otimes \boxed{b}.$$

$$\mathcal{C} \otimes \mathcal{D} = \{ c \otimes d \mid c \in \mathcal{C},\ d \in \mathcal{D} \}.$$

WHAT IS $f_2\left(\begin{array}{|c|c|}\hline 1 & 2 \\ \hline 2 & \\ \cline{1-1}\end{array}\right)$

$= f_2\left(\boxed{2} \otimes \boxed{1 \mid 2}\right)$?

$$f_i(x \otimes y) = \begin{cases} f_i(x) \otimes y & \text{IF } \varepsilon_i(x) \geq \varphi_i(y) \\ x \otimes f_i(y) & \text{IF } \varepsilon_i(x) < \varphi_i(y). \end{cases}$$

$\varepsilon_2(\boxed{2}) = \#\, 3\text{'s} = 0$

$\varphi_2(\boxed{1 \mid 2}) = \#\, 2\text{'s} = 1$.

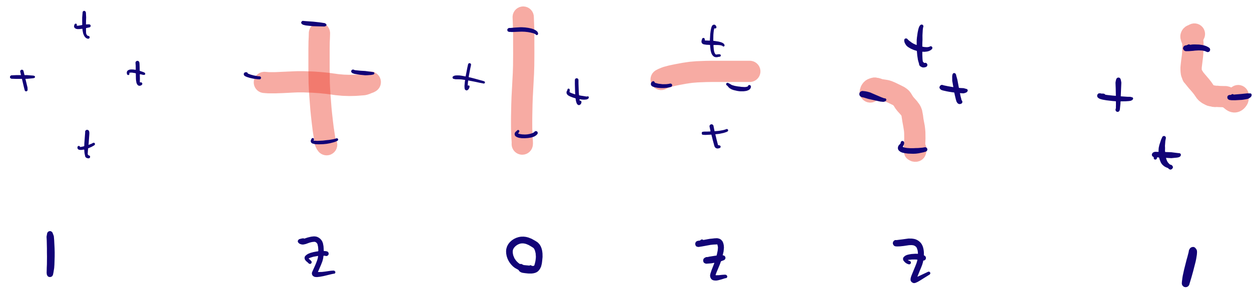
WE ARE IN SECOND CASE.

$$f_2\left(\boxed{2} \otimes \boxed{1\ 2}\right) = \boxed{2} \otimes f_2(\boxed{1\ 2})$$

$$= \boxed{2} \otimes \boxed{1\ 3} = \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 & \\ \cline{1-1}\end{array}$$

CRYSTAL STRUCTURES ON $q=0$ TOKUYAMA MODELS.

$\oplus$ $\ominus$



THEOREM: THE STATES OF THE $q=0$ TOKUYAMA MODEL WITH TOP BOUNDARY SPINS IN $\lambda_i + n - i$ COLUMNS HAS A CRYSTAL STRUCTURE $\cong \mathcal{B}_\lambda$ AND $\beta(s) = z^{p} \cdot z^{wt(T)}$.

$$s \mapsto T$$

STATE TABLEAU

DICTIONARY:

CREATE A GTP FROM VERTICAL EDGES THAT HAVE $-$ SPIN.

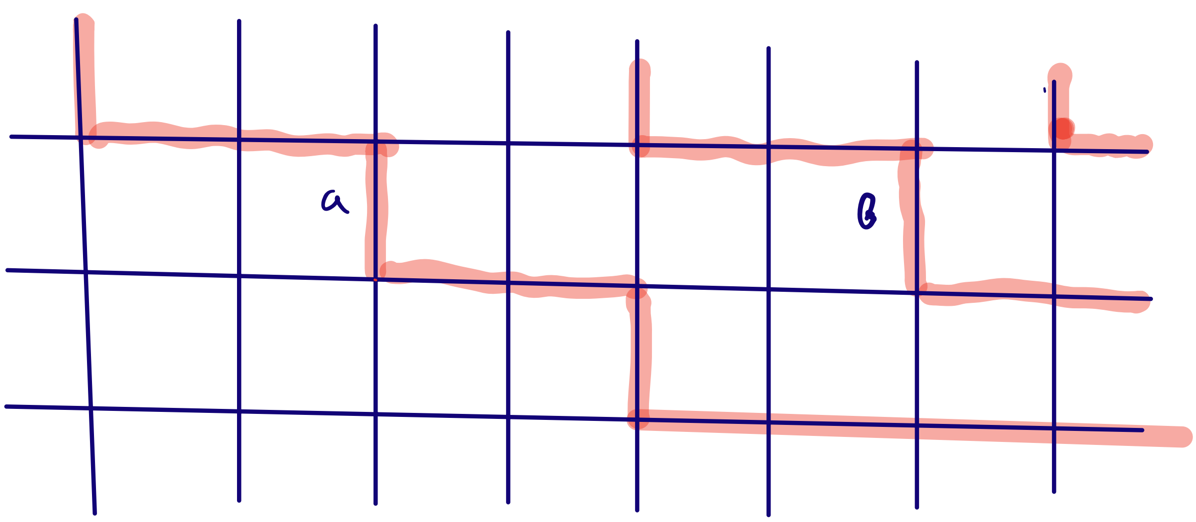
SUBTRACT $\left(\begin{array}{ccccc} 2 & & 1 & & 0 \\ & 1 & & 0 & \\ & & 0 & & \end{array}\right)$ GTP TOP ROW λ.

$n=3$

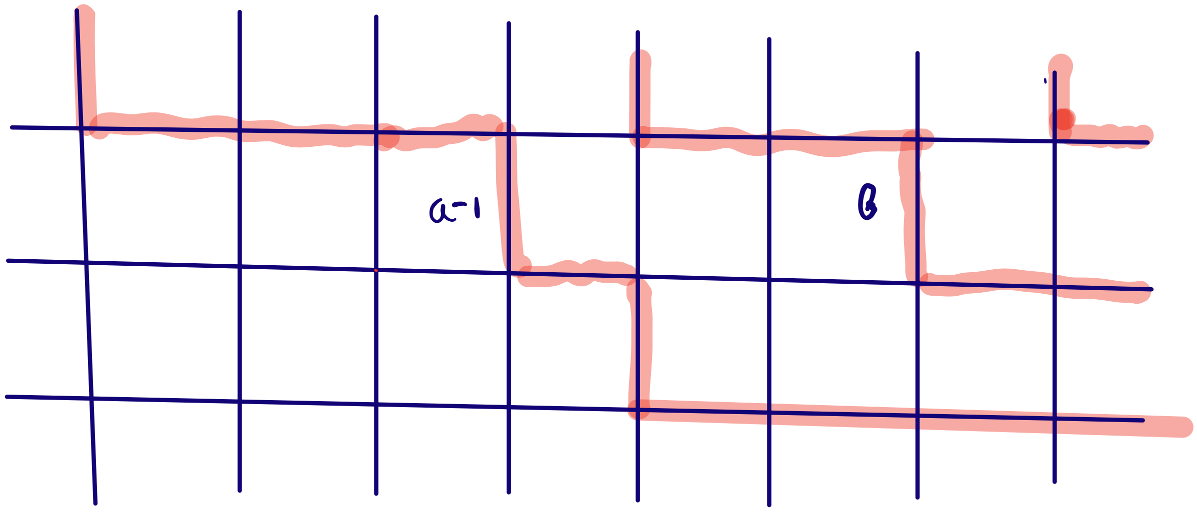
TURN THIS INTO A TABLEAU $\rightsquigarrow$ APPLY SCHÜTZENBERGER INVOLUTION TO GET T.

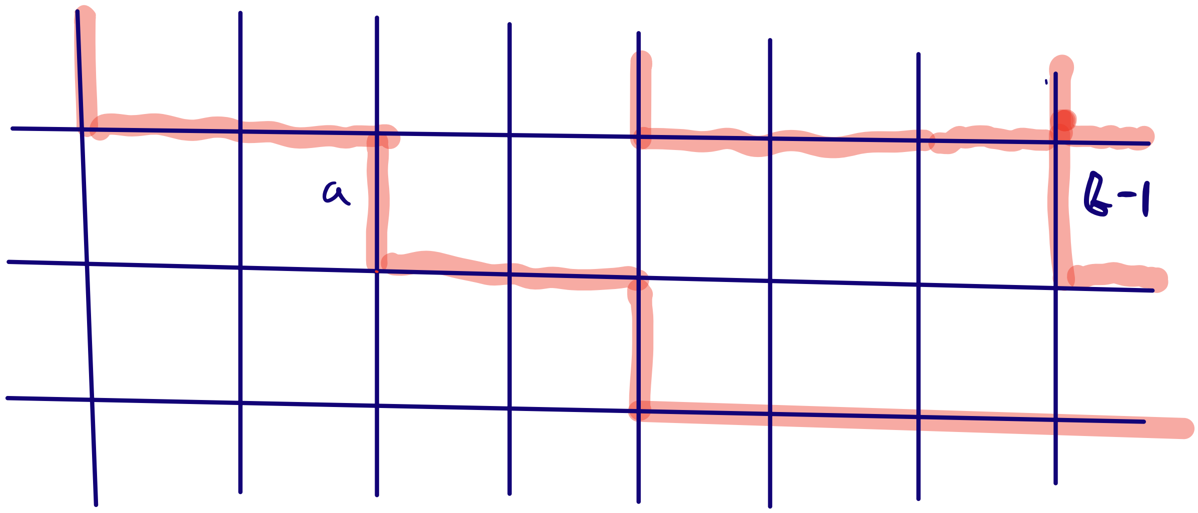
THE CRYSTAL OPERATOR e_i ON A STATE MOVES ONE SEGMENT BETWEEN ROWS i AND $i+1$ ONE STEP TO THE RIGHT.

WHICH ONE GETS MOVED?

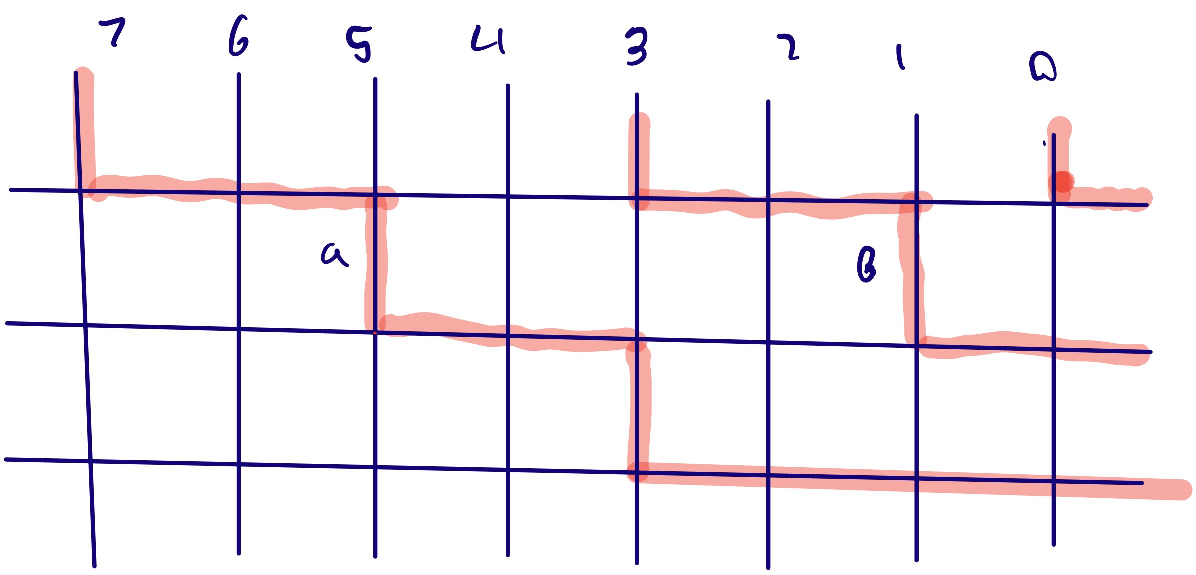


APPLYING e_1 WILL MOVE EITHER a OR b ONE STEP TO THE RIGHT TO GIVE ...





WHICH IS IT ?



$$\begin{pmatrix} 7 & & 3 & & 0 \\ & 5 & & 1 & \\ & & 3 & & \end{pmatrix} \to \begin{pmatrix} 5 & & 2 & & 0 \\ & 4 & & 1 & \\ & & 3 & & \end{pmatrix} \qquad \begin{array}{|c|c|c|c|c|} \hline 1 & 1 & 1 & 2 & 3 \\ \hline 2 & 3 \\ \cline{1-2} \end{array} \rightsquigarrow T$$

$$\Big\downarrow f_2 \qquad\qquad \Big\downarrow e_1$$

$$\begin{pmatrix} 7 & & 3 & & 0 \\ & 5 & & 0 & \\ & & 3 & & \end{pmatrix} \begin{pmatrix} 5 & & 2 & & 0 \\ & 4 & & 0 & \\ & & 3 & & \end{pmatrix} \qquad \begin{array}{ccccc} 1 & 1 & 1 & 2 & 3 \\ 3 & 3 \end{array} \qquad e_1(T)$$

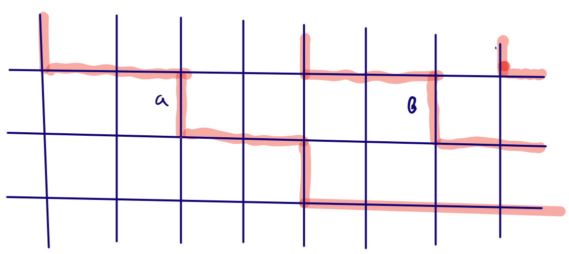
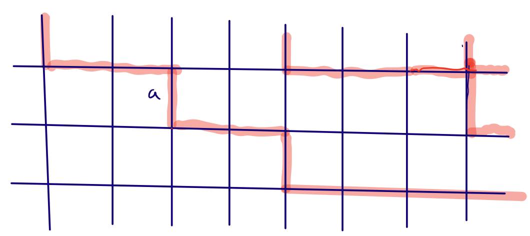
$$2 \otimes \left[3 \otimes \overset{m}{1} \otimes 1 \otimes 1 \otimes 2\right] \otimes 3$$

$$3 \otimes \left[3 \otimes 1 \otimes 1 \otimes 1 \otimes 2\right]$$

$f_2(m) =$
$e_2(m) = 0$

SO m CAN BE IGNORED

CONCLUSION:

$e_1\Big($  $\Big) =$ 

BUT DECIDING THE RESULT DEPENDS ON A CALCULATION.

BILLEY, STANLEY, JOCKUSZ
ASSAF AND SCHILLING, GOLD, SUN, MILICEVIC.

PIPEDREAMS RELATED TO CRYSTALS.