SCHUBERT POLYNOMIALS WERE INTRODUCED BY LASCOUX AND SCHÜTZENBERGER TO DESCRIBE COHOMOLOGY OF FLAG VARIETIES (LECTURE 1).

$$G = GL(n, \mathbb{C}) \qquad B = \begin{pmatrix} * & & * \\ & \ddots & \\ 0 & & * \end{pmatrix}$$

$X = G/B$ IS A PROJECTIVE VARIETY OF DIM $= \ell(w_0) = \frac{1}{2}n(n-1) = |\Phi^+|$.

IF $w \in W$ $\quad BwB/B = X_w^0$ IS $\cong \mathbb{C}^{\ell(w)}$ (AFFINE SPACE)

$w = w_0 \quad X_w^0$ OPEN

$\overline{X_w}$ = CLOSURE OF X_w^0

$$= \bigcup_{y \leq w} X_y^0.$$

(CELLULAR DECOMPOSITION)

BRUHAT ORDER

$[X_w]$ = COHOMOLOGY CLASS OF DEGREE

$$2\left(\underset{\frac{1}{2}n(n-1)}{\ell(w_0)} - \ell(w)\right)$$

REAL DIM $(X_w) = 2\ell(w)$

THESE FORM A BASIS OF $H^*(X)$ AS A VECTOR SPACE.

$$H^*(X) = \mathbb{C}[x_1, \dots, x_n] \Big/ \text{IDEAL GEN'D BY SYMMETRIC POLYNOMIALS.}$$

FIND A SET S_w OF POLYNOMIALS SUCH THAT THE IMAGE IN $H^*(X)$ IS $[X_w]$. THE SCHUBERT POLYNOMIALS ARE A GOOD CHOICE.

$X^{\rho} = x_1^{n-1} x_2^{n-2} \cdots x_{n-1}$ (DOESN'T INVOLVE x_n.)

PROPERTIES: $S_{w_0} = X^{\rho}$

IF $w\Delta_i < w$ THEN $S_{w\Delta_i} = D_i S_w$

WHERE $D_i = (x_i - x_{i+1})^{-1}(1 - \Delta_i)$.

LET US COMPUTE THESE FOR $GL(3)$.

$S_{\Delta_1\Delta_2\Delta_1} = x_1^2 x_2$

$w = w_0 = \Delta_1\Delta_2\Delta_1$

$w\Delta_1 = \Delta_1\Delta_2$

$S_{\Delta_1\Delta_2} = D_1(x_1^2 x_2)$

$= (x_1 - x_2)^{-1}(x_1^2 x_2 - x_1 x_2^2) = x_1 x_2$.

$S_{\Delta_2\Delta_1} = D_2(x_1^2 x_2) =$

$w_0 = \Delta_1\Delta_2\Delta_1 = \Delta_2\Delta_1\Delta_2$

$w = w_0$

$w\Delta_2 = \Delta_2\Delta_1$

$(x_2 - x_3)^{-1}(x_1^2 x_2 - x_1^2 x_3)$

$= x_1^2$.

$$
\begin{tikzcd}
 & \mathfrak{S}_{w_0} = x_1^2 x_2 & \\
\mathfrak{S}_{s_1 s_2} = x_1 x_2 \arrow[ur] & & \mathfrak{S}_{s_2 s_1} = x_1^2 \arrow[ul] \\
\mathfrak{S}_{s_1} = x_1 \arrow[u] & & \mathfrak{S}_{s_2} = x_1 + x_2 \arrow[u] \\
 & \mathfrak{S}_{\mathrm{id}} = 1 \arrow[ul] \arrow[ur] &
\end{tikzcd}
$$

GL(3)

1234	1
1243	$x_1 + x_2 + x_3$
1324	$x_1 + x_2$
1342	$x_1x_2 + x_1x_3 + x_2x_3$
1423	$x_1^2 + x_1x_2 + x_2^2$
1432	$x_1^2x_2 + x_1^2x_3 + x_1x_2^2 + x_1x_2x_3 + x_2^2x_3$
2134	x_1
2143	$x_1^2 + x_1x_2 + x_1x_3$
2314	x_1x_2
2341	$x_1x_2x_3$
2413	$x_1^2x_2 + x_1x_2^2$
2431	$x_1^2x_2x_3 + x_1x_2^2x_3$
3124	x_1^2
3142	$x_1^2x_2 + x_1^2x_3$
3214	$x_1^2x_2$
3241	$x_1^2x_2x_3$
3412	$x_1^2x_2^2$
3421	$x_1^2x_2^2x_3$
4123	x_1^3
4132	$x_1^3x_2 + x_1^3x_3$
4213	$x_1^3x_2$
4231	$x_1^3x_2x_3$
4312	$x_1^3x_2^2$
4321	$x_1^3x_2^2x_3$

COPIED FROM MACDONALD

MACDONALD USES 1-LINE NOTATION

2341 $x_1x_2x_3$

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \end{pmatrix} : (1234) = s_1 s_2 s_3$$

$$S_{s_1 s_2 s_3} = x_1 x_2 x_3 .$$

SYSTEMS WHOSE PARTITION FUNCTIONS WERE INTRODUCED BY BILLEY AND N. BERGERON CALLED RC-GRAPHS

RENAMED PIPEDREAMS BY MILLER-KNUTSON.

ANOTHER TYPE OF PIPEDREAMS WERE INTRODUCED BY LEE-LAM-SHIMOZONO. CALLED "BUMPLESS" PIPEDREAMS.

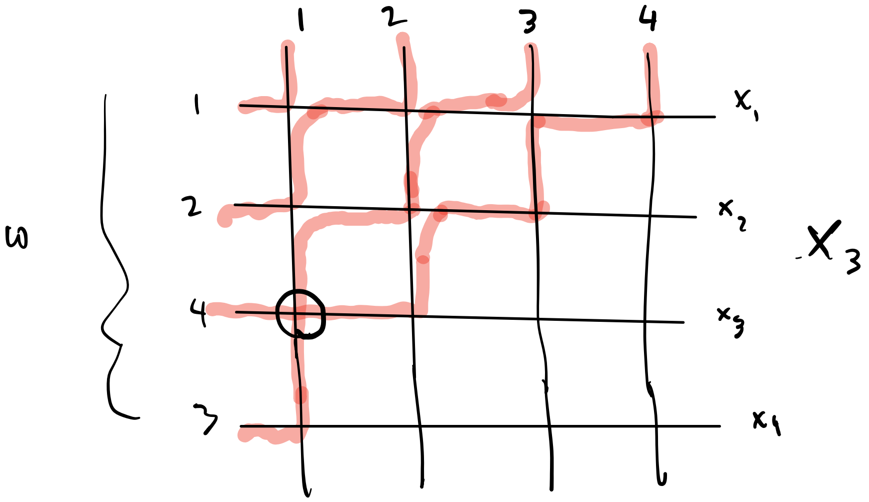
SIMILAR TO THE GAMMA/DELTA HYBRID MODELS, KNUTSON AND UDELL SHOWED IT IS PROFITABLE TO COMBINE THE TWO TYPES INTO HYBRID MODELS.

ORIGINALLY, PIPEDREAMS WERE NOT COLORED BUT COLOR IS IMPORTANT.

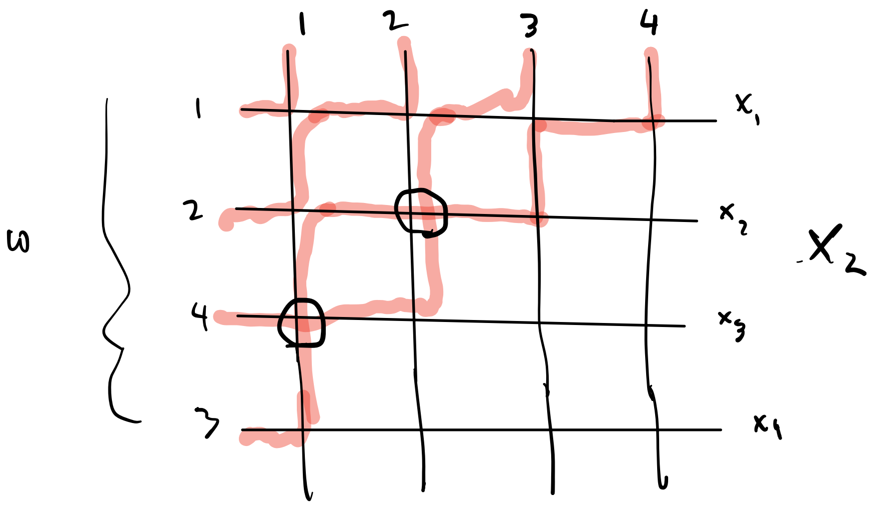
$\omega = \Delta_3$ IN S_4.

$\Delta_3 = (1243)$ IN 1-LINE NOTATION

$\mathfrak{S}_{\Delta_3} = x_1 + x_2 + x_3$

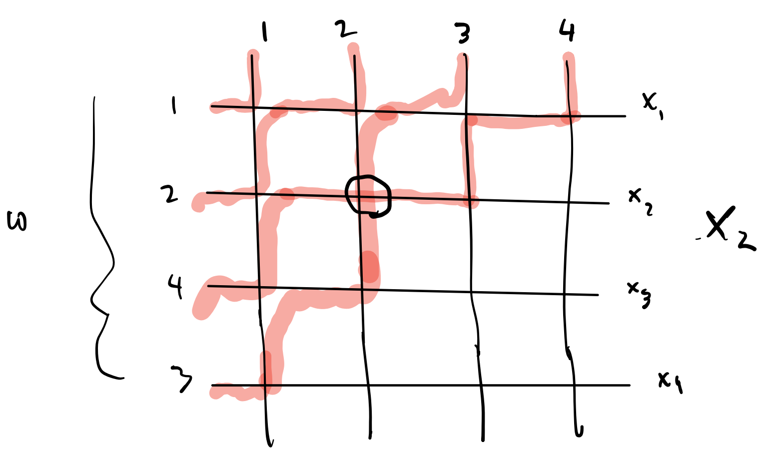


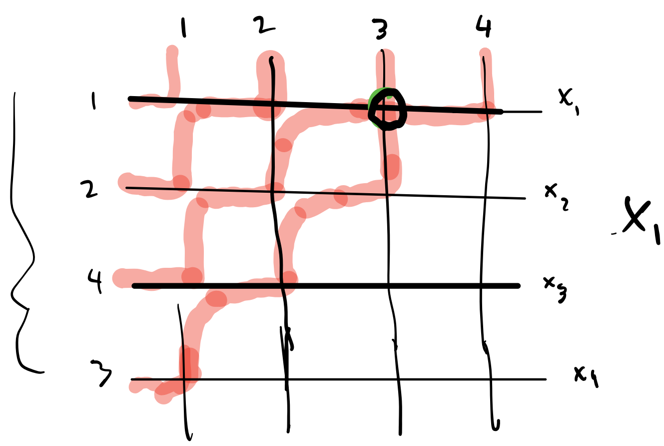
CONNECT THE LABELED EDGES
BY "PIPES."



NO!

PIPES ARE NOT ALLOWED TO CROSS TWICE!





THE TERM "REDUCED" IS USED TO MEAN A PIPEDREAM IN WHICH NO ROWS CROSS TWICE.

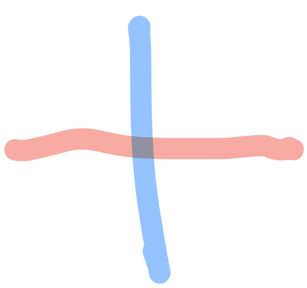
THIS UNFORTUNATELY A GLOBAL CONDITION

1	x_i	0	1	1

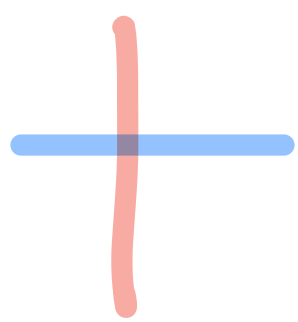
ABSENCE OF A COLOR $\mapsto$ ⊕ IN THE PREVIOUS LECTURES IS ANOTHER COLOR

⊕ < B < R

SOLUTION: WE COLOR THE PIPES AND ORDER THEM SO IF R > B

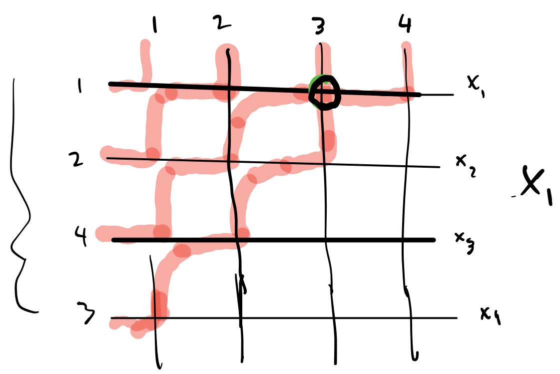


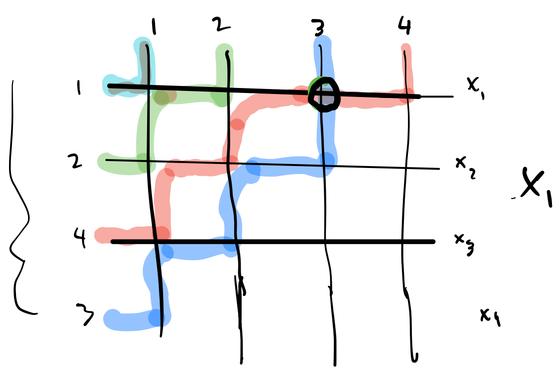
LEGAL



NOT LEGAL.

THE COLORING IS ALSO HELPFUL BECAUSE COLORS READ OFF THE PERMUTATION.





CONCLUSION: PIPEDREAMS SHOULD BE COLORED. THE "UNCOLOR" ⊕ IS ANOTHER COLOR.

WE CAN USE THE YBE TO VERIFY THE DEFINING PROPERTY.