OPEN AND CLOSED MODELS (CONTINUED)

$\lambda = (\lambda_1, \dots, \lambda_n)$ A PARTITION $\quad z = \begin{pmatrix} z_1 & & \\ & \ddots & \\ & & z_n \end{pmatrix}$

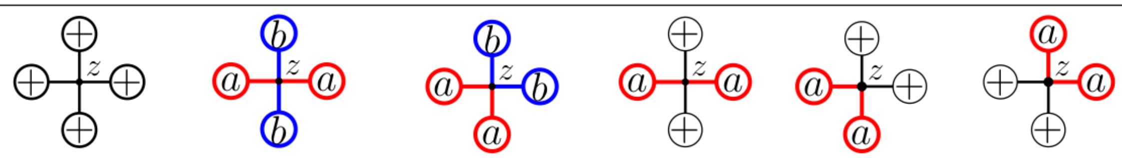
$\partial_\lambda(z)$ = PARTITION FUNCTION OF

$q = 0$ TOKUYAMA MODEL.

COLORED VARIANTS (OPEN AND CLOSED) REFLECT THE DIVISION OF THE CRYSTAL $\mathcal{B}_\lambda$ WHOSE CHARACTER IS ∂_λ INTO DEMAZURE ATOMS (OPEN) OR DEMAZURE CHARACTERS (CLOSED).

THE YANG-BAXTER EQUATION CAN BE USED TO COMPUTE THE PARTITION FUNCTIONS.

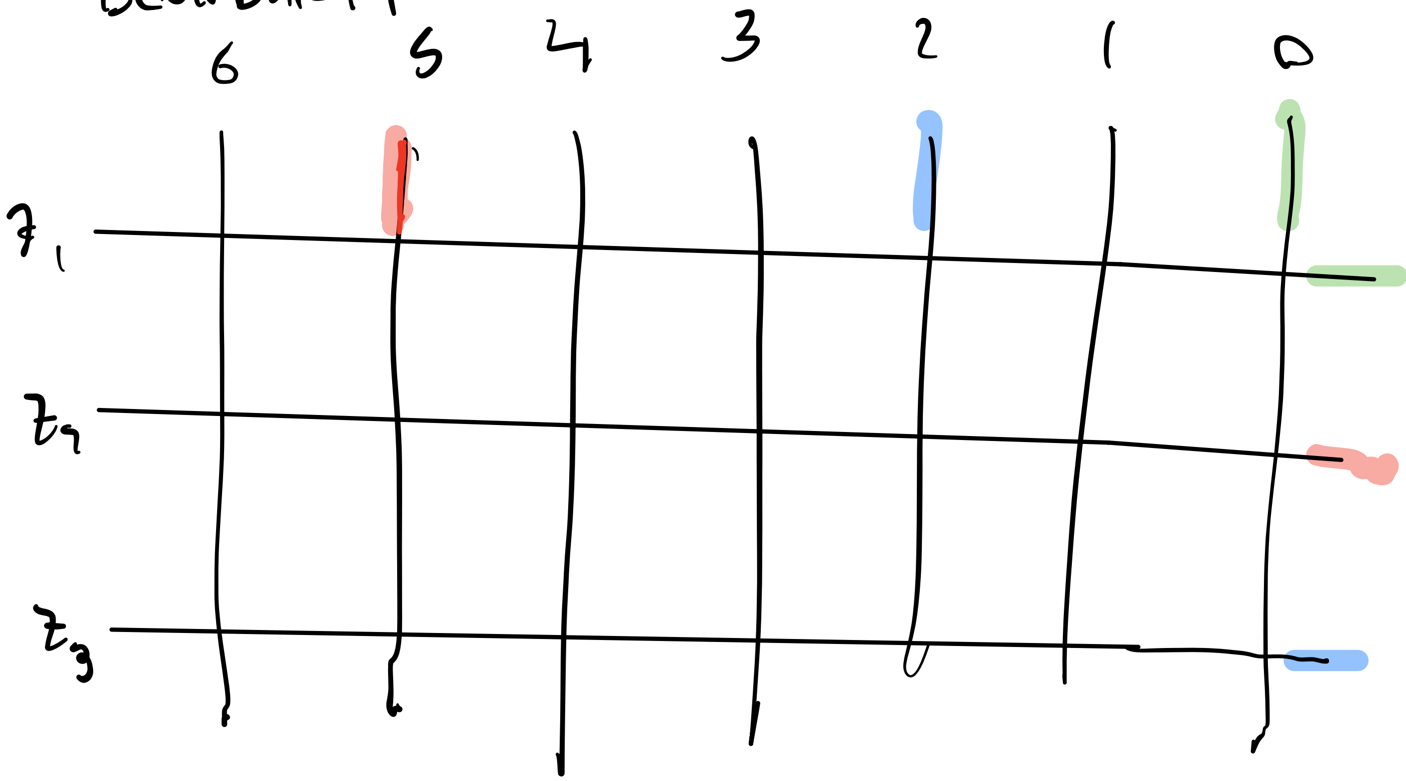
Open T-weights



1	$z \quad a \geqslant b$ $0 \quad a < b$	$0 \quad a > b$ $z \quad a < b$	z	z	1

ORDER COLORS $c_1 < c_2 < \cdots < c_n$

PUT THEM IN REVERSE ORDER ON TOP BOUNDARY:



AT COL. $\lambda_i + n - i$ PUT c_{n-i}

$\lambda = (3,1,0)$ $\lambda + \rho = (5,2,0)$

AND FOR RIGHT BOUNDARY IN ROW i

PUT $w_0 c_0$ $w_0 c_0 = (c_n, c_{n-1}, \dots, c_1)$

$$w(d_1, \dots, d_n) = (d_{w^{-1}(1)}, \dots, d_{w^{-1}(n)})$$

$w_1 = \Delta_1 \Delta_2 = (1\,2\,3)$

c_3 c_2 c_1

THEOREM:

$$Z(S_{\lambda,w}) = z^{\rho} \cdot \underbrace{\partial^{\circ}_{w} z^{\lambda}}_{\text{DEMAZURE ATOM.}}$$

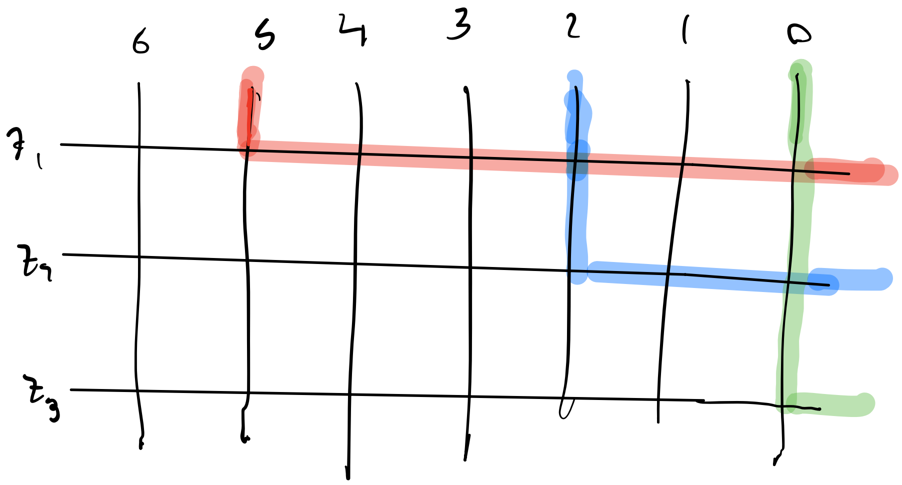
$$\partial^{\circ}_{w} = \partial^{\circ}_{i_1} \cdots \partial^{\circ}_{i_n} \qquad w = s_{i_1} \cdots s_{i_k} \text{ REDUCED.}$$

$$\partial_i = (z^{\alpha} - 1)^{-1}(1 - s_i)$$

BRAID.

PROVED BY INDUCTION ON $\ell(w)$.

IF $w = 1$, THIS IS THE SYSTEM $S_{\lambda,1}$ WITH EXACTLY ONE STATE.



$$z_1^5 z_2^2 = \prod z_i^{\lambda_i + n - i} = z^{\lambda + \rho}.$$

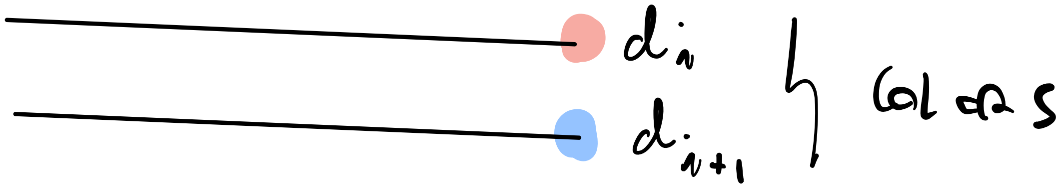
SO WHAT WE NEED TO PROVE IS

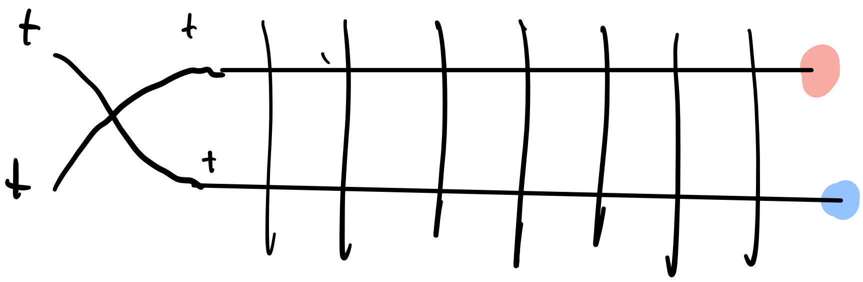
IF $\Delta_i w > w$

$$Z(\mathfrak{S}_{\lambda, \Delta_i w}) = \partial_i^{a} Z(\mathfrak{S}_{\lambda, w}).$$

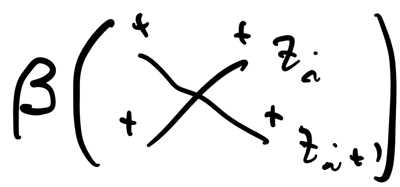
WE CAN PROVE THIS USING YBE.

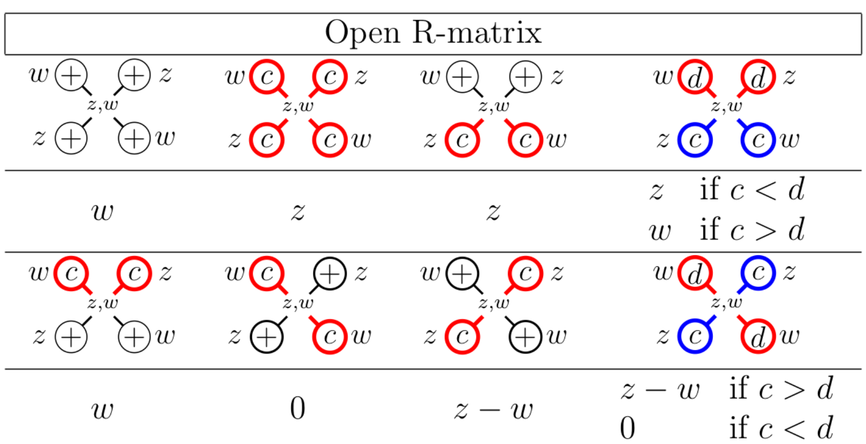
THE CONDITION $\Delta_i w > w$ AMOUNTS $d_i > d_{i+1}$.





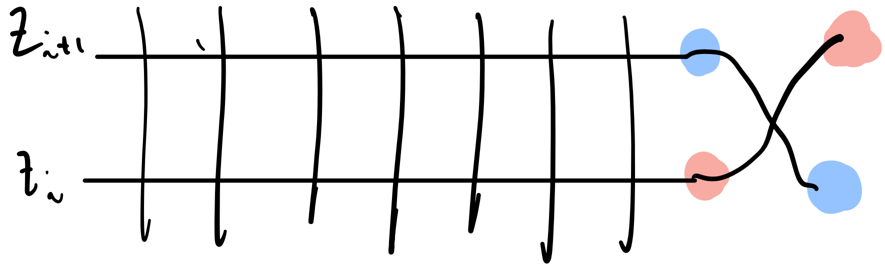
$a(z_i, z_{i+1})$:





ATTACHING R-MATRIX GIVES

$$a(z_i, z_{i+1})\, Z(S_\lambda(z)) =$$



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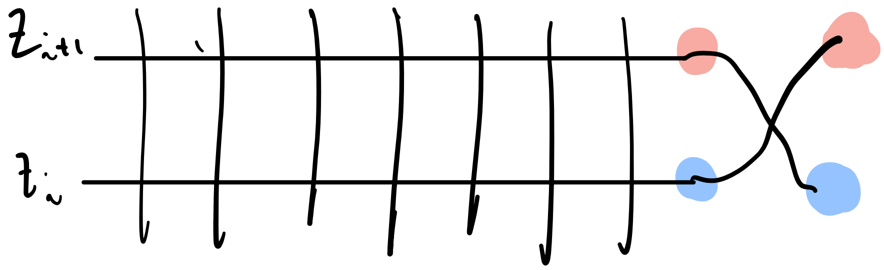
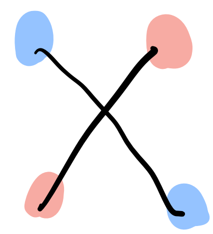
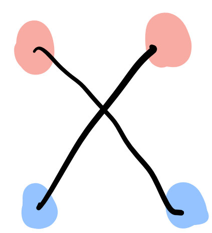


$B(z_i, z_{i+1}) = \beta_{z_i, z_{i+1}}\Big($  $\Big)$

R > B

$C(z_i, z_{i+1}) = \beta_{z_i, z_{i+1}}\Big($  $\Big)$

$$B(z_i, z_{i+1})\, Z(S_{\lambda, s_i w}(s_i z)) + C(z_i, z_{i+1})\, Z(S_{\lambda, w}(s_i z))$$

$$a(z_i, z_{i+1})\, Z(\mathfrak{S}_{\lambda,w}(z)) =$$

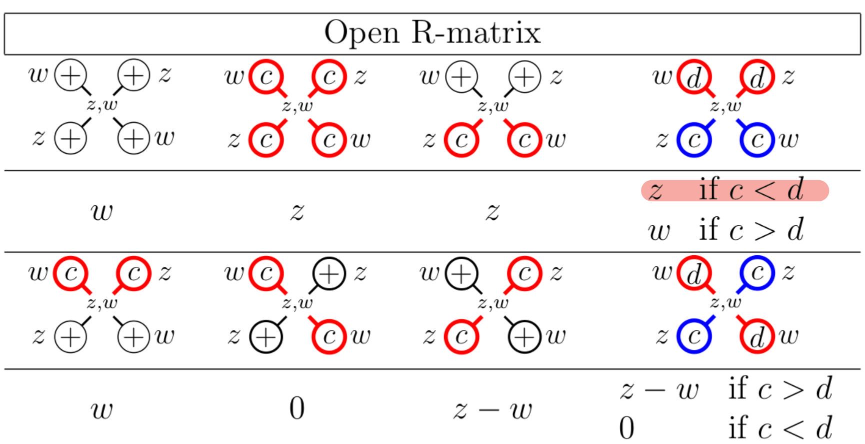
$$b(z_i, z_{i+1})\, Z(\mathfrak{S}_{\lambda, s_i w}(s_i z)) +$$

$$c(z_i, z_{i+1})\, Z(\mathfrak{S}_{\lambda,w}(s_i z)) .$$

REPLACE $z \to s_i z$ $\qquad z_i \leftrightarrow z_{i+1}$

$$Z(\mathfrak{S}_{\lambda, s_i w}(z)) = \frac{c(z_{i+1}, z_i)\, Z(\mathfrak{S}_{\lambda, w}(z)) - a(z_{i+1}, z_i)\, Z(\mathfrak{S}_{\lambda, w}(s_i z))}{-b(z_{i+1}, z_i)}$$

$$\mathfrak{T}_i^{+} f(\mathbf{z}) = \frac{c^{\pm}(z_{i+1}, z_i) f(\mathbf{z}) - a(z_{i+1}, z_i) f(s_i \mathbf{z})}{-b^{\pm}(z_{i+1}, z_i)} \qquad (6.18)$$



$$c(z_i, z_{i+1}) = z_i \qquad z_{i+1} = a(z_i, z_{i+1})$$

$$b(z_i, z_{i+1}) = z_i - z_{i+1}$$

$$Z(S_{\lambda, \Delta_i \omega}(t)) = \frac{z_{i+1} - z_i \Delta_i}{z_i - z_{i+1}} Z(S_{\lambda,\omega}(t))$$

$$z^{\alpha_i} = z_i / z_{i+1}$$

$$\frac{1 - z^{\alpha} \Delta_i}{z^{\alpha} - 1} = z^{\rho} \partial_i^0 z^{-\rho}.$$

$$\partial_i^0 = (z^{\alpha} - 1)^{-1} (1 - \Delta_i)$$

$$\Delta_i \rho = \rho - \alpha_i$$

$$\Delta_i z^{-\rho} = z^{-\rho} \cdot z^{\alpha} \Delta_i$$

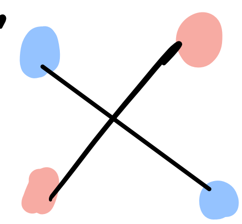
$$z^{\rho} \Delta_i z^{-\rho} = z^{\alpha} \Delta_i .$$

APPLYING TO z^p. $\partial_w z^p = \mathcal{Z}(S_{\lambda, w}(z))$

GIVES $z^p \mathcal{Z}(S_{\lambda, s_i w})$.

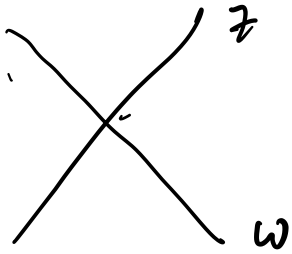
THIS IS A MODEL FOR MANY SITUATIONS WHERE

PARAMETRIZED YBE $\rightsquigarrow$ DEMAZURE OPERATOR RECURSION FORMULA.

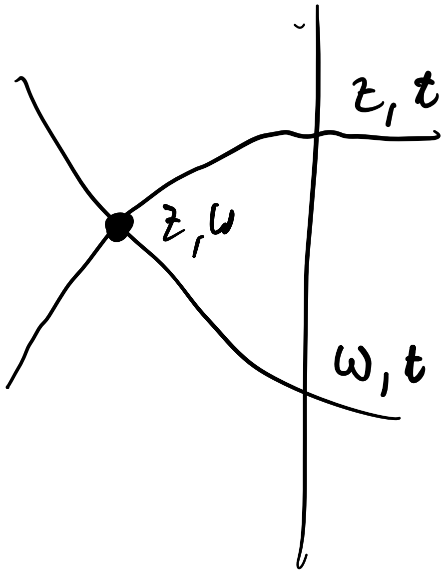
KEY POINT: $\beta\Big($  $\Big)$ = CONSTANT $\times\ z_i - z_{i+1}$

BECAUSE THIS SHOWS UP IN THE DENOMINATOR.

KIRILLOV
BRUBAKER-DASHER } MANY EXAMPLES
PIPEDREAMS ...



z, w



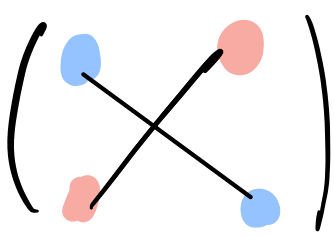
$$R_{z,w} : V \otimes V \to V \otimes V$$

$$(R_{z,w} \otimes I)(I \otimes R_{z,t})(R_{w,t} \otimes I) =$$

$$(I \otimes R_{v,t})(\cdots)(\quad)$$

GENERAL SHAPE.

IF $z = w$, $R_{z,w}$ SHOULD BE A MULTIPLE OF $I_{V \otimes V}$.

β  IS AN OFF DIAGONAL TERM!

$$\langle w \otimes z \,|\, R \,|\, z \otimes w \rangle$$

SHOULD VANISH IF $z = w$.

THIS MEANS THIS GENERAL CALCULATION ALWAYS PRODUCES A DIVIDED DIFFERENCE OPERATOR.