Lecture 1: Schubert Calculus

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Course Objectives

This course will cover the combinatorics of Schubert polynomials, including the theory of pipedreams, which can be described as solvable lattice models whose partition functions are Schubert polynomials.

Schubert polynomials were invented by Lascoux and Schützenberger to describe the cohomology of flag varieties. They have important two-variable versions related to the equivariant cohomology called *double Schubert polynomials*. It was shown by Billey and Bergeron that (double) Schubert polynomials can be expressed as sums over combinatorial gadgets that are now called classical *pipedreams*, and another class of pipedreams was introduced by Lee, Lam and Shimozono, called bumpless pipedreams. Knutson and Udell then showed that the classical and bumpless pipedreams can be combined into a unified theory, and other classes of pipedream models were introduced by Knutson and Zinn-Justin. There has been quite a bit of recent research activity in this area.

Motivation: Schubert Calculus

Schubert calculus can be defined as intersection theory for Grassmannians and Flag Varieties.

Let $n < N$ be positive integers. The *Grassmannian* $\mathrm{Gr}(n, N)$ can be defined as the space of n-dimensional subspaces of $\mathbb{C}^N$. It is a smooth projective variety of dimension $\binom{N}{n} - 1$ which come with a family of subvarieties called *Schubert varieties*.

A *complete flag* in $\mathbb{C}^n$ is a set of subspaces

$$0 = V_0 \subset V_1 \subset \cdots \subset V_n = \mathbb{C}^n \tag{1}$$

where $\dim(V_i) = i$. The (complete) *flag variety* is the set of complete flags. It is a smooth projective variety of dimension $\frac{1}{2}n(n-1)$.

We will not go into the geometric foundations of Schubert calculus, because our subject matter will be in combinatorics, not algebraic geometry. Yet combinatorial topics such as the Bruhat order on Weyl groups, the role of Demazure operators originate in the geometry, so in this introductory lecture we will discuss the geometry mostly without proofs. For proofs of the geometric results, see Fulton [4], Chapters 9 and 10.

If X is a manifold (such as a smooth projective variety) its cohomology ring $H^*(X)$ is an associative superalgebra, meaning a graded ring whose multiplication (cup product) satisfies

$$x \cup y = (-1)^{\deg(x)\deg(y)} y \cup x.$$

If X is a smooth projective variety over $\mathbb{C}$, its cohomology ring may contain elements of odd degree. For example, an algebraic curve of genus g has $\dim H^1(X) = 2g$. But Grassmannians and Flag varieties are special since they have cellular decompositions in which the cells are algebraic varieties. This has the consequence that the cohomology is all in even dimensions, and so the cohomology ring is commutative.

Let us see how this works for $\mathrm{Fl}(n)$. First we note that $G = \mathrm{GL}(n, \mathbb{C})$ acts transitively on the flags, and the stabilizer of a standard flag (in which V_i is the vector space spanned by the standard basis vectors $\mathbf{e}_1, \cdots, \mathbf{e}_i$) is the Borel subgroup B of upper triangular invertible matrices. So $\mathrm{Fl}(n) = G/B$. Let $W = S_n$ be the Weyl group, embedded in G as the group of permutation matrices. We have the Bruhat decomposition

$$G = \bigcup_{w \in W} BwB \qquad \text{(disjoint)}.$$

Consequently we have

$$\mathrm{Fl}(n) = \bigcup_{w \in W} X^\circ(w)$$

where $X^\circ(w) = BwB/B$.

Recall that W has a length function ([2, 3]). Let $s_1, \cdots, s_{n-1} \in W$ be the simple reflections. These are the generators $s_i = (i, i+1)$ in cycle notation. If $w \in W$ then $\ell(w)$ is the smallest integer k such that we may write $w = s_{i_1} \cdots s_{i_k}$; such a shortest expression is called *reduced*. Then $X^\circ(w)$ is homeomorphic to the affine space $\mathbb{C}^{\ell(w)} \cong \mathbb{R}^{2\ell(w)}$. There is an important partial order $\leqslant$ on W called the *Bruhat order* such that the closure of $X^\circ(w)$ is

$$X(w) = \bigcup_{y \leqslant w} X^\circ(y).$$

If $y < w$ then $\ell(y) < \ell(w)$, so $X(w)$ has a dense open set $X^\circ(w)$ that is an affine space of real dimension $2\ell(w)$. This decomposition implies that $H^*(X, \mathbb{Z})$ is a free $\mathbb{Z}$-module spanned by the classes $[X(w)]$. The degree of the class $[X(w)]$ is $\dim(X) - 2\ell(w)$. The spaces $X(w)$ are called *Schubert varieties*, and the $X^\circ(w)$ are *open Schubert varieties.*

Alternatively, let B^- be the "opposite" Borel subgroup of lower triangular matrices. Then we may define $\Omega^\circ(w) = B^- wB/B$. Its closure is

$$\Omega(w) = \bigcup_{y \geqslant w} \Omega^\circ(y).$$

Let w_0 be the longest Weyl group element. We have $w_0 B w_0^{-1} = B^-$ so in the action of G on $\mathrm{Fl}(n)$ the element w_0 maps $\Omega^\circ(w)$ to $X^\circ(w_0 w)$. The length of $w_0 w$ is $\dim(X) - \ell(w)$ where $\dim(X) = \ell(w_0) = \frac{1}{2}n(n-1)$. So $\Omega^\circ(w)$ has codimension $\ell(w)$ and the degree of $[\Omega^\circ(w)] = [X^\circ(w_0 w)] = 2\ell(w)$. The dual classes $[\Omega(w)]$ are perhaps a better set of generators since the degree is the length of $\ell(w)$.

Now the ring $H^*(\mathrm{Fl}(n))$ has another description (due to Borel) which represents it as a polynomial ring $\mathbb{C}[x_1, \cdots, x_n]$ modulo an ideal. To describe this, we interpret elements of $\mathrm{Fl}(n)$ as flags (1). Then V_i/V_{i-1} is a line bundle on $\mathrm{Fl}(n)$ and its first Chern class $x_i \in H^2(\mathrm{Fl}(n))$. These generate $H^*(\mathrm{Fl}(n))$ and indeed we have the following description of the kernel of the resulting homomorphism from the polynomial ring.

Theorem 1 (Borel [1]) *We have $H^*(\mathrm{Fl}(n)) \cong \mathbb{C}[x_1, \cdots, x_n]/I$ where I is the ideal generated by symmetric polynomials with zero constant term.*

One can ask for polynomials $\mathfrak{S}_w \in \mathbb{C}[x_1, \cdots, x_n]$ whose images in $H^*(\mathrm{Fl}(n))$ correspond to the classes $[\Omega(w)]$. These are the *Schubert polynomials* of Lascoux and Schützenberger. We will not define them today, but will study them at length in later lectures.

Schur polynomials

Let $\lambda = (\lambda_1, \cdots, \lambda_n)$ be a partition. The symmetric polynomial

$$s_\lambda(x_1, \cdots, x_n) = \frac{\det(x_i^{\lambda_j + n - j})}{\det(x_i^{n-j})}$$

is called a *Schur polynomial*. The denominator is a Vandermonde determinant and equals $\prod_{i<j}(x_i - x_j)$. If λ is regarded as a dominant weight for $\mathrm{GL}(n)$, and χ_λ is the

character of the corresponding irreducible representation, the Weyl character formula implies

$$\chi_\lambda(g) = s_\lambda(z_1, \cdots, z_n)$$

where z_i are the eigenvalues of $g \in \mathrm{GL}(n, \mathbb{C})$. Also by the Weyl character formula, the s_λ are an orthonormal basis for the ring $\mathbb{C}[x_1, \cdots, x_n]^{S_n}$ of symmetric functions with respect to the inner product derived from Haar measure on the compact subgroup $U(n) \subset \mathrm{GL}(n, \mathbb{C})$. See Macdonald [7] or the last part of Bump [3] for more information about Schur polynomials.

Special Schur functions are the *complete* symmetric functions

$$h_k = \sum_{i_1 \leqslant \cdots \leqslant i_k} x_{i_1} \cdots x_{i_k}$$

and the *elementary* symmetric functions

$$e_k = \sum_{i_1 < \cdots < i_k} x_{i_1} \cdots x_{i_k}.$$

These are Schur polynomials: $h_k = s_{(k)}$ and $e_k = s_{(1^k)}$ where (k) is the partition $(k, 0, \cdots, 0)$ and $(1^k) = (1, \cdots, 1, 0, \cdots, 0)$ with exactly k 1's. It is assumed for e_k that $k \leqslant n$.

The Littlewood-Richardson rule and Pieri's formula

The *Littlewood-Richardson coefficients* $c^\nu_{\lambda\mu}$ are the multiplicative structure constants on the ring of symmetric polynomials with respect to the Schur basis. Thus

$$s_\lambda s_\mu = \sum_\nu c^\nu_{\lambda\mu} s_\nu.$$

The coefficients $c^\nu_{\lambda\mu}$ have a combinatorial description that is slightly hard to state, but a special case, retro-historically called *Pieri's rule* is worth stating explicitly here. Let $\mathrm{YD}(\lambda)$ be the Young diagram of λ. If $\mathrm{YD}(\lambda) \subset \mathrm{YD}(\nu)$ and the set theoretic difference $\mathrm{YD}(\nu) - \mathrm{YD}(\lambda)$ is called a *skew shape* and denoted ν/λ. We say ν/λ is a *horizontal strip of length* k if $\mathrm{YD}(\nu) - \mathrm{YD}(\lambda)$ does not have more than one box in any given column, and if $|\nu| - |\lambda| = k$. Vertical strips are defined similarly.

Now take $\mu = (k)$, that is, the partition $(k, 0, 0, \cdots)$, so $s_\mu = h_k$. Then Pieri's formula states that

$$c^\nu_{\lambda,(k)} = \begin{cases} 1 & \text{if } \nu/\lambda \text{ is a horizontal strip of length } k, \\ 0 & \text{otherwise.} \end{cases}$$

So

$$s_\lambda h_k = \sum_\nu s_\nu$$

where the sum is over ν such that ν/λ is a horizontal strip of length k. There is a dual Pieri formula:

$$s_\lambda e_k = \sum_\nu s_\nu$$

where now the sum is over ν such that ν/λ is a vertical strip of length k.

The Jacobi-Trudi identity

Another formula worth mentioning is the *Jacobi-Trudi identity*. For convenience define $h_k = 0$ if $k < 0$. Then the Jacobi-Trudi identity asserts that

$$s_\lambda = \det \begin{pmatrix} h_{\lambda_1} & h_{\lambda_1+1} & h_{\lambda_1+2} & \cdots \\ h_{\lambda_2-1} & h_{\lambda_2} & h_{\lambda_2+1} & \cdots \\ h_{\lambda_3-2} & h_{\lambda_2-1} & h_{\lambda_3} & \cdots \\ \vdots & \vdots & \vdots & \end{pmatrix}.$$

If $\mu = \lambda'$ is the conjugate partition we also have

$$s_\lambda = \det \begin{pmatrix} e_{\mu_1} & e_{\mu_1+1} & e_{\mu_1+2} & \cdots \\ h_{\mu_2-1} & h_{\mu_2} & e_{\mu_2+1} & \cdots \\ h_{\mu_3-2} & h_{\mu_2-1} & e_{\mu_3} & \cdots \\ \vdots & \vdots & \vdots & \end{pmatrix}.$$

Proofs of the Pieri and Jacobi-Trudi identities may be found in Macdonald [7] or the last part of Bump [3].

Grassmannians

We have explained how Schubert polynomials (which we have not yet defined) the intersection theory of Flag varieties

A similar more classical result describes the cohomology of Grassmannians. Let $N = n + k$. Then $\mathrm{Gr}(n, N) \cong G/P$ where P is the maximal parabolic subgroup of $\mathrm{GL}(n, \mathbb{C})$ consisting of elements with the block matrix form

$$\begin{pmatrix} A & B \\ 0 & D \end{pmatrix}, \qquad A \in \mathrm{GL}(r), \qquad D \in \mathrm{GL}(k).$$

Now by the Bruhat decomposition for parabolic subgroups

$$G/P = \bigcup_{w \in (S_n \times S_k)/S_N} B w^{-1} P.$$

Now we may choose the representative w of minimal length in its coset. The choice of w^{-1} instead of w here is good for the combinatorial point of view. Then

$$w(1) < w(2) < \cdots < w(n), \qquad w(n+1) < w(n+2) < \cdots < w(n+k). \tag{2}$$

This w has a unique descent at n. A permutation with a unique descent is called *Grassmannian*. Given a Grassmann permutation satisfying this condition we may associate a partition $\lambda = (\lambda_1, \cdots, \lambda_n)$ by

$$\lambda_i = k + i - w(i).$$

Lemma 1 *This relationship gives a bijection between Grassmann permutations satisfying (2) and those partitions whose Young diagram fits in an $n \times k$ rectangle.*

Proof The condition on λ means that λ has $\leqslant n$ parts and that $\lambda_1 \leqslant k$. We leave this as an exercise for the reader. □

Now the intersection theory on Grassmannians is a historical topic due to Schubert, Pieri and Giambelli. Schubert cycles can be defined as with flag varieties: let σ_w be the closure of $B_- w^{-1} P/P$, where w is now a Grassmann permutation satisfying (2). If λ is the corresponding partition, we may alternatively denote σ_w as σ_λ.

Theorem 2 (Pieri) *We have*

$$\sigma_\lambda \sigma_{(k)} = \sum \sigma_\nu$$

where the sum is over ν such that $\mathrm{YD}(\nu)$ is contained in a $k \times n$ rectangle, and ν/λ is a horizontal strip of length k.

The analog of the Jacobi-Trudi identity is also true, and in this context is called the Giambelli formula.

Theorem 3 *The cohomology ring $H^*(\mathrm{Gr}(n, N))$ is isomorphic to the ring $\mathbb{C}[x_1, \cdots, x_n]^{S_n}$ modulo an ideal I that is spanned as a vector space by the Schur polynomials s_λ for partitions that do not fit in an $n \times k$ rectangle. In this isomorphism $\sigma_\lambda \longleftrightarrow s_\lambda$.*

Proof This follows from Borel's computation of the cohomology, though this exact formulation appeared later. □

This shows that the multiplicative structure of symmetric functions with the Schur polynomial basis exactly mirrors the cohomology of $\mathrm{Gr}(n, N)$. Lascoux and Schützenberger recognized the importance of finding polynomials in $\mathbb{C}[x_1, \cdots, x_n]$ that could play the role of Schur polynomials in this more general context. These are the *Schubert polynomials* [5, 6].

References

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