1 A brief history of metaplectic forms

If $G$ is a group, a *central extension* of $G$ by $A$ is a short exact sequence

$$1 \to A \to \tilde{G} \to G \to 1$$

where the image of $A$ is in the center of $\tilde{G}$. If $G$ and $\tilde{G}$ are topological groups, and if $A$ is a discrete subgroup of the center of $\tilde{G}$, then we may think of $\tilde{G}$ as a *cover* of $G$, in the topological sense. So we will sometimes use the term *covering group*. Very often for us, $A$ will be the group $\mu_n(F)$ of $n$-th roots of unity, in a given field $F$. We will use this notation only if $|\mu_n(F)| = n$.

By a *Metaplectic group* we mean a central extension $\tilde{G}(F)$ of the group $G(F)$ of $F$-rational points of a reductive algebraic group over a local field $F$, by $\mu_n$ where $\mu_n = \mu_n(F)$. When we consider complex representations of a metaplectic group we will also encounter $\mu_n(\mathbb{C})$ and we will chose fix an isomorphism to identify $\mu_n(F) = \mu_n(\mathbb{C})$. We will usually need $\mu_n \subset F$ and often it will be convenient to assume $\mu_{2n} \subset F$.

Alternatively, let $F$ be a global field of characteristic zero or prime to $n$. Let $\mathbb{A} = \mathbb{A}_F$ be the adele ring. We then look for an extension $\tilde{G}(\mathbb{A})$ of the adele group $G(\mathbb{A})$ by $\mu_n$. Typically the cover will split over $G(F)$. This means that we may consider $G(F)$ to be a subgroup of $\tilde{G}(\mathbb{A})$, and so we may consider automorphic forms on metaplectic groups.

To give a famous example, consider the *Jacobi theta function*

$$\theta(z) = \sum_{n=-\infty}^{\infty} e^{2\pi i n^2 z},$$

where $z$ lies in the upper half-plane $\mathcal{H} = \{x + iy | y > 0\}$. This is a modular form of weight $\frac{1}{2}$ for the group $\Gamma_0(4)$.

Now consider what happens when we transfer this form to the adele group by the usual procedure for modular forms. First recall that if $f$ is a modular form of integer weight $k$, this procedure produces an automorphic form on $GL(2, \mathbb{A}_\mathbb{Q})$. If one tries to repeat this procedure with $\theta$, one ends up with a function not on $GL(2, \mathbb{A}_\mathbb{Q})$ but rather on a central extension of $SL(2, \mathbb{A}_\mathbb{Q})$. 

The adelic viewpoint on modular forms of half integral weight began in 1964, when Weil (1964) constructed such central extensions of $\text{Sp}(2n, A)$, where $A = A_F$ is the adele ring of a global field $F$, and used their representation theory to reformulate much of Siegel’s work on quadratic fields. To oversimplify slightly Weil’s central extensions were double covers, that is central extensions

$$1 \longrightarrow \mu_2 \longrightarrow \widetilde{\text{Sp}}(2n, A) \longrightarrow \text{Sp}(2n, A) \longrightarrow 1.$$

He recognized a connection with the Hilbert symbol of class-field theory, noting a proof of the quadratic reciprocity law as a biproduct of his theory. Indeed, the quadratic reciprocity law is equivalent to the fact that the cover is trivial over the subgroup $\text{Sp}(2n, F)$. In other words, the embedding $\text{Sp}(2n, F) \longrightarrow \text{Sp}(2n, A)$ may be lifted to a homomorphism $\text{Sp}(2n, F) \longrightarrow \widetilde{\text{Sp}}(2n, A)$, and naturally occurring representations may be regarded as spaces of functions on $\text{Sp}(2n, F) \backslash \text{Sp}(2n, A)$. These are the metaplectic automorphic forms.

Weil coined the term “metaplectic” to describe $\widetilde{\text{Sp}}(2n, A)$. Although the etymology of metaplectic is thus derived from symplectic, we will use the term to describe central extensions of other reductive groups.

Then Kubota (1967, 1969) constructed similar central extensions

$$1 \longrightarrow \mu_n \longrightarrow \widetilde{\text{GL}}(2, A) \longrightarrow \text{GL}(2, A) \longrightarrow 1.$$

As in the $n = 2$ case Hilbert reciprocity law from class field theory implies a splitting over $\text{GL}(2, F)$, and based on the spectral theory of automorphic forms, Kubota constructed Eisenstein series, and interpreted their residues as “generalized theta series.” Thus $\theta$ appears as a member of this family of objects.

Matsumoto (1966) generalized Kubota’s work in constructing central extensions of $G(A)$, where $G$ is a split, simply-connected semisimple algebraic group. (The group $\text{GL}(2)$ is not semisimple, but one may apply Matsumoto’s results by embedding it in $\text{SL}(3)$ and restricting the cover.) Eventually Brylinski and Deligne (2001) reconsidered the metaplectic groups, and recent papers tend to use their foundations.

Kazhdan and Patterson (1984) generalized Kubota’s construction of generalized theta series to the higher metaplectic covers of $\text{GL}(n)$. As in Kubota’s work, the generalized theta series are constructed as residues of Eisenstein series. This paper contained much important foundational material that we will discuss later.

Meanwhile, there were important developments in the theory of metaplectic forms on the double cover of $\text{SL}(2)$. This is the natural domain for modular forms of half-integral weight. Shimura (1973) showed how such modular forms may be “lifted” to ordinary modular forms, an important phenomenon known as the Shimura correspondence. Moreover in very important work Waldspurger proved that if $\tilde{f}$ is a modular form of weight $k + \frac{1}{2}$ and $f$ is the modular form of weight $2k$ then the Whittaker (Fourier) coefficients of $\tilde{f}$ are related to the quadratic twists of $L(s, f)$ at the central point $s = k$. They are also related to $O(2)$ periods of $f$. A key point in Waldspurger’s work is a generalization of the Siegel-Weil formula, a relationship between theta series and Eisenstein series at special points, going back to Weil (1965).
The Shimura correspondence and many other examples of functorial liftings, special value results, inner product formulae and other phenomena can be subsumed in the following picture, described by Howe (1979) and known as *Howe duality*. The original metaplectic group \( \widetilde{Sp}(2n, F) \) has a particular representation \( \omega \) on \( L^2(F^n) \), or its dense subspace of Bruhat-Schwartz functions. This representation, sometimes called the *Weil representation* or *oscillator representation* has remarkable properties. The corresponding adelic representation of \( \tilde{\text{Sp}}(2n, \mathbb{A}) \) is automorphic.

The work of Weil (1964) was motivated by work of Siegel which in the modern view concerned *theta liftings* which are transfers of automorphic forms between orthogonal groups and symplectic groups. Thus if \( \pi \) is an automorphic form on an orthogonal group \( O_V \), where \( V \) is a quadratic space, the theta lifting produces an automorphic form (possibly zero) on any symplectic group \( \text{Sp}_W \) if \( \dim(V) \) is even or its metaplectic double cover if \( n \) is odd.

The theta lift can be represented by means of a *theta kernel*. This means a function \( \theta(g, h) \) on \( O_V(\mathbb{A}) \times \text{Sp}_W(\mathbb{A}) \) (if \( \dim(V) \) is even) or \( O_V(\mathbb{A}) \times \tilde{\text{Sp}}_W(\mathbb{A}) \) (if \( \dim(V) \) is odd) that is automorphic in both variables such if \( f \) is an automorphic form on \( O_V(\mathbb{A}) \) then

\[
\theta(f)(g) = \int_{O_V(F) \backslash O_V(\mathbb{A})} \theta(g, h) f(h) \, dh
\]

is an automorphic form on \( \text{Sp}_W \). The lift can also go in the other direction. For example if \( \dim(V) = 3 \) and the quadratic form is split, and \( \dim(W) = 2 \), we may identify \( O_V = \text{PGL}(2) \) and \( \text{Sp}_W = \tilde{\text{SL}}_2 \), and the theta lift implements the Shimura correspondence. See Shintani (1975) and Niwa (1975). More trivially, the example the Jacobi theta function (1) can be understood as the case where \( \dim(V) = 1 \), and the automorphic representation of \( O(1) \) is the trivial representation, and again \( \tilde{\text{Sp}}_W = \tilde{\text{SL}}(2) \).

The remarkable fact that theta lifts take Hecke eigenforms to Hecke eigenforms has its root in uniqueness principles, which we come to next.

Howe (1979) isolated the notion of a *dual reductive pair*. Let \( F \) be a local field of characteristic not equal to 2. We are given a pair of reductive subgroups \( G, H \) in \( \text{Sp}(2n, F) \) that centralize each other. For example, let \( V \) be a quadratic space, that is, a vector space equipped with a nondegenerate symmetric bilinear form, and let \( W \) be a symplectic vector space, equipped with a nondegenerate skew-symmetric bilinear form. Then \( V \otimes W \) is naturally a symplectic vector space, and if \( O(V) \) and \( \text{Sp}(W) \) are the orthogonal and symplectic groups stabilizing the given bilinear forms, then \( O(V) \times \text{Sp}(W) \) is a dual reductive pair in \( \text{Sp}(V \otimes W) \). Other examples of dual reductive pairs take \( G, H \) to be a pair of general linear groups or unitary groups.

Now let \( \tilde{G} \) and \( \tilde{H} \) be the preimages of \( G \) and \( H \) in \( \tilde{\text{Sp}}(2n, F) \). The covers may trivialize; so \( \tilde{G} \) and \( \tilde{H} \) may be double covers, or they may be ordinary reductive groups over \( F \). Howe (1979) conjectured that the restriction of the oscillator representation \( \omega \) to \( \tilde{G} \times \tilde{H} \) is multiplicity-free in a very strong sense. That is, let \( \pi_{\tilde{G}} \) and \( \pi_{\tilde{H}} \) be irreducible representations of \( \tilde{G} \) and \( \tilde{H} \). Then \( \pi_{\tilde{G}} \times \pi_{\tilde{H}} \) can occur in \( \omega \) with multiplicity at most one. Moreover, this representation is determined by the isomorphism class of either \( \pi_{\tilde{G}} \) or \( \pi_{\tilde{H}} \). This *Howe duality conjecture* was proved by Howe (1989), Mœglin, Vignéras, and Waldspurger (1987), Howe...
(1990) and Waldspurger (1990). The bijection \( \pi_{\tilde{G}} \leftrightarrow \pi_{\tilde{H}} \) between those representations of \( \tilde{G} \) that occur and those of \( \tilde{H} \) is known as the Howe correspondence.

As Howe explained, the Howe correspondence is related to classical topics such as Schur-Weyl duality and invariant theory. Howe (1989) pointed out that the theory can be advanced by use of Lie superalgebras (then called graded Lie algebras), and generalizations of Howe duality to Lie superalgebras were found by Sergeev (1984, 1999). See Cheng and Wang (2012) for a good treatment of this topic.

Although Howe duality is primarily a local theory, it has global implications. For automorphic forms, the uniqueness principle in Howe duality implies theta lifts take Hecke eigenforms to Hecke eigenforms.

### 2 The Stone-Von-Neumann theorem I: Finite 2-Step Nilpotent Groups

A Heisenberg group has a unique irreducible representations. This leads to central extensions of the symplectic group, and representations of that. The underlying uniqueness principle is the Stone-Von Neumann Theorem. This strategy works over a finite field though in odd characteristic the central extensions are trivial. However it seems good to first demonstrate this construction in this simple case. We will prove a very strong version of the Stone-Von Neumann Theorem for an arbitrary 2 step nilpotent groups.

If \( G \) is a group, then a projective representation of \( G \) is a homomorphism \( \pi : G \rightarrow \text{PGL}(V) \), where \( V \) is a vector space. The following Lemma shows how projective representations are related to central extensions:

**Lemma 2.1.** Let \( \pi : G \rightarrow \text{PGL}(V) \) be a projective representation. Then there exists a central extension

\[
1 \rightarrow \mathbb{C}^* \rightarrow \tilde{G} \rightarrow G \rightarrow 1
\]

such that \( \pi \) lifts to a true representation \( \tilde{\pi} \) of \( \tilde{G} \).

**Proof.** Choose a lifting \( \pi' : G \rightarrow \text{GL}(V) \), so for every \( g \in G \), \( \pi'(g) \in \text{GL}(V) \) represents the coset \( \pi(g) \) of in \( \text{PGL}(V) = \text{GL}(V)/Z_{\text{GL}} \) where \( Z_{\text{GL}} \) is the center of \( \text{GL}(V) \) consisting of scalar matrices. Then \( \pi(gh) = \pi(g)\pi(h) \) implies that \( \pi'(gh) = c(g,h)\pi'(g)\pi'(h) \) for some nonzero complex number \( c(g,h) \). Then the identity \( \pi(g(hk)) = \pi((gh)k) \) implies the “cocycle condition”

\[
c(g,hk)c(h,k) = c(gh,k)c(g,h).
\]

Now define \( \tilde{G} \) to be the set \( G \times \mathbb{C}^* \) with the group law

\[
(g,\varepsilon)(h,\varepsilon') = (gh,c(g,h)\varepsilon\varepsilon').
\]

The cocycle condition implies that this multiplication is associative, so \( \tilde{G} \) is a group. Then we define

\[
\tilde{\pi}(g,\varepsilon) = \varepsilon^{-1}\pi'(g)
\]

and it is easy to see that this is a representation lifting \( \pi \).
If $H$ is any group and $\omega : H \to \text{GL}(V)$ an irreducible representation of $H$, then by Schur’s lemma there is a character $\psi$ of the center $Z_H$ such that $\omega_H(z) = \psi(z)I_V$ for $z \in Z_H$. This $\psi$ is called the central character of $H$.

One way that we can obtain a projective representation of $G$ is to find another group $H$ and a character $\psi$ of the center $Z_H$ such that $H$ has a unique irreducible representation $\omega : H \to \text{GL}(V)$ with central character $\chi$. Let $G$ be the group of automorphisms of $H$ that fix the center $Z_H$.

One way that we can obtain a projective representation of $G$ is to find another group $H$ and a representation $\pi : H \to \text{GL}(V)$ where $V$ is some (possibly infinite-dimensional) vector space, with central character $\psi : Z_H \to \mathbb{C}^\times$ such that $\pi$ is the unique irreducible representation of $H$ with central character $\psi$. Let $G$ be the group of automorphisms of $H$ that fix $Z_H$.

**Proposition 2.2.** In this setting, there exists a projective representation $\omega : G \to \text{GL}(V)$ such that if $g \in G$ and $h \in H$, then

$$\omega(g)\pi(h) = \pi(g^\ast h)\omega(g).$$  \hfill (2)

**Proof.** Define a representation $\pi^g$ of $H$ by $(\pi^g)(h) = \pi(g^\ast h)$. This irreducible representation has the same central character as $\pi$, so by assumption it is equivalent. This means that there is an intertwining operator $\omega(g)V \to V$ that intertwines the two actions and so (2) is satisfied. If $g_1, g_2 \in G$, then applying (2) to $g_1$ and $g_2$ separately we see that

$$\omega(g_1g_2)\pi(h) = \pi(g_1^\ast g_2^\ast h)\omega(g_1^\ast)\omega(g_2) = \omega(g_1)\omega(g_2)\pi(h).$$

By Schur’s Lemma, this implies that $\omega(g_1g_2)$ and $\omega(g_1^\ast)\omega(g_2)$ differ by a constant and so $\omega$ is a projective representation. \hfill \Box

To give an example, let $F$ be a field of characteristic $\neq 2$. In this section we assume that $F$ is a finite field. (But the same principle will apply with $F$ local.) Let $V$ be a symplectic vector space, that is, a vector space equipped with a nondegenerate symplectic bilinear form over the ground field $F$. Let $H = V \oplus F$. We make this into a group with the multiplication

$$(v,x)(v',x') = \left(v + v', x + x' + \frac{1}{2}\langle v, v' \rangle \right).$$  \hfill (3)

This is the Heisenberg group. The center $Z_H$ consists of the group $(0,z)$ with $z \in F$. We will show that if $\psi$ is a nontrivial character of $Z_H$ then $H$ has a unique irreducible representation with central character $\psi$. This is the (finite) Stone-von Neumann theorem.

Now Proposition 2.2 is applicable, and we get projective representations of the the group of automorphisms of $H$ that fix $Z_H$. But this group is just the symplectic group $\text{Sp}(V)$ of symplectic automorphisms of $V$, and so we obtain a projective representation of $\text{Sp}(V)$.

With this motivation in mind, we classify the irreducible representations of an arbitrary finite 2-step nilpotent group. This is a version of the Stone-von Neumann theorem. Let $H$ be a finite 2-step nilpotent group; that is, the derived group is contained in the center $Z$. I will
write \( H \) multiplicatively but the abelian quotient \( W = H/Z \) additively, so let \( \tau : H \rightarrow W \) be the projection map.

Pick a unitary character \( \psi : Z \rightarrow T \). Here \( T \) is the subgroup of \( \mathbb{C}^\times \) of elements of absolute value 1. Define \( \beta = \beta_\psi : H \times H \rightarrow \mathbb{C} \) by \( \beta(x, y) = \psi([x, y]) \) where the commutator \( [x, y] = xyx^{-1}y^{-1} \).

**Proposition 2.3.** The pairing \( \beta \) is skew-symmetric and bilinear; that is,

\[
\beta(x, y) = \overline{\beta(y, x)}, \quad \beta(xy, z) = \beta(x, z)\beta(y, z), \quad \beta(x, yz) = \beta(x, y)\beta(x, z).
\]

Moreover \( \beta(xy^{-1}, z) = \beta(y, z) \).

**Proof.** We have \( \beta(x, y) = \psi([x, y]) = \psi([y, x]^{-1}) = \overline{\psi([y, x])} \) since \( \psi \) is assumed to be unitary proving the first identity. As for the second, observe that \( [y, z] \) is central, so

\[
\beta(x, z)\beta(y, z) = \psi(xzx^{-1}zy^{-1}[y, z]) = \psi(x[y, z]zx^{-1}y^{-1})
\]

and \( x[y, z]zx^{-1}y^{-1} = [xy, z] \), proving the second identity. As for the third,

\[
\beta(xy^{-1}, z) = \psi(xy^{-1}zy^{-1}x^{-1}z^{-1}).
\]

The argument of \( \psi \) can be written \( [x, y]yzxy^{-1}x^{-1}z^{-1} \) and since \( [x, y] \) is central this equals

\[
\psi(yzxy^{-1}x^{-1}y^{-1}[x, y]z) = \psi([y, z]) = \beta(y, z).
\]

\[ \square \]

We observe that \( \beta(x, y) \) depends only on the cosets of \( x \) and \( y \) mod \( Z \), so we may define a skew-symmetric bilinear pairing \( B = B_\psi \) of \( W \times W \) to \( \mathbb{T} \) by

\[
B(\tau x, \tau y) = \beta(x, y).
\]

Here skew-symmetric **skew-symmetric** and **bilinear** mean

\[
B(x, y) = \overline{B(y, z)}, \quad B(xy, z) = B(x, z)B(y, z),
\]

which together imply linearity in the second variable. Moreover \( B \) is **invariant** meaning

\[
B(xy^{-1}, z) = B(y, z).
\]

These properties follow from the corresponding properties of \( \beta \).

Let \( Q = Q_\psi = \{ x \in H | \beta(x, y) = 1 \ \text{for all} \ y \in H \} \). Clearly \( Q \) contains \( Z \) and \( \bar{Q} = Q/Z \) is the kernel of the skew-symmetric form \( B \) on \( W \).

A closed subgroup \( \bar{L} \) of \( W \) is called **isotropic** or **\( \psi \)-isotropic** if \( B(w, w') = 1 \) for all \( w, w' \in \bar{L} \). A maximal isotropic subgroup of \( W \) is called **Lagrangian** or (to remind ourselves that the definition of Lagrangian depends on \( \psi \)) **\( \psi \)**-**Lagrangian**.

If \( L \) is a closed subgroup of \( H \), we call \( L \) **isotropic** (resp. **Lagrangian**) if \( L \supset Z \) and \( \bar{L} = L/Z \) is isotropic (resp. Lagrangian). Clearly a Lagrangian subgroup of \( H \) contains \( Q \). It is normal since \( H/Z \) is abelian. We will also denote by \( \bar{Z}_1 \) the kernel of \( \psi \) in \( Z \).
Lemma 2.4. Let $L$ be a Lagrangian subgroup of $H$. Then $L/Z_1$ is abelian and $\psi$ may be extended to a character of $L$. Let $\chi$ any such character of $L$ extending $\psi$. Suppose that $x \in H$ such that

$$\chi(xux^{-1}) = \chi(u)$$

for all $u \in L$. Then $x \in L$.

Proof. If $x, y \in L$ then $B(x, y) = 1$ so $\psi(xyx^{-1}y^{-1}) = 1$. This implies that $L/Z_1$ is abelian. Since $\psi$ may be regarded as a character of $Z/Z_1$, it may be extended to a character of $L/Z_1$; that is, $\psi$ may be extended to $L$.

For the second part, observe that $\chi(xux^{-1}u^{-1}) = 1$ for all $u \in L$. But since $xux^{-1}u^{-1} \in Z$ this equals $\psi([x, u]) = B(x, u)$. Therefore the subgroup obtained by adjoining $x$ to $L$ (and taking the closure) is isotropic. Since $L$ is maximal isotropic, $x \in L$. \hfill \Box

Let $L$ be a Lagrangian subgroup of $H$, and let $\chi$ be a unitary character of $L$ that extends $\psi$. We may consider the $H$-module induced from the character $\chi$ of $L$. This is the space $V_\chi$ of all functions $f : H \rightarrow \mathbb{C}$ such that

$$f(xh) = \chi(x)f(h), \quad x \in L, h \in H.$$

The group $H$ acts by right translation: $\pi(h)f(x) = f(xh)$.

We recall Mackey’s theorem on intertwining operators of induced representations in its geometric form, computing the intertwining operators between two induced modules. Let $\chi'$ be another character of $L$ and let $\Delta$ be a function on $H$ that satisfies

$$\Delta(ugv) = \chi'(u)\Delta(g)\chi(v), \quad u, v \in L.$$  (4)

Using (4) it is easy to see that convolution with $\Delta$ is an intertwining operator $T : V_\chi \rightarrow V_{\chi'}$. That is, if $f \in V_\chi$ then

$$Tf(g) = (\Delta * f)(g) = \sum_{x \in L \setminus H} \Delta(x^{-1})f(xg).$$

The term $\Delta(x^{-1})f(xg)$ depends only on the coset $Lx$. Using (4) one may check that $\Delta * f \in V_{\chi'}$, and it is clear that $T$ commutes with the actions of $H$ on $V_\chi, V_{\chi'}$ by right translation. According to Mackey theory every intertwining operator is of this form. See Bump (2013) Theorem 32.1 for a proof of this version of Mackey theory.

This leads to a classification of the irreducible representations that is a version of the Stone-von Neumann theorem. Every element induced representation with central character $\psi$ is induced from a character of any $\psi$-Lagrangian subspace.

Theorem 2.5. Let $\chi$ be a unitary character of $L$ extending $\psi$. Then $(\pi_\chi, V_\chi)$ is irreducible. Moreover every irreducible unitary representation is of this type. If $\chi, \chi'$ are distinct unitary characters of $L$ extending $\psi$, then $\pi_\chi$ and $\pi_{\chi'}$ are equivalent if and only if there exists $x \in H$ such that

$$\chi'(u) = B(u, x)\chi(u)$$  (5)

for $u \in L$. This implies that $\chi$ and $\chi'$ agree on $Q_\psi$. \hfill \Box

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Proof. We compute $\text{Hom}_H(V_\psi, V_{\psi'})$. By Mackey theory, if $T : V_\psi \rightarrow V_{\psi'}$ is an intertwining operator then there exists a distribution $\Delta$ satisfying (4) such that $T(f) = \Delta \ast f$. We will show that $\Delta$, if it exists, is supported on a single coset of $L$, which will prove that $\dim \text{Hom}_H(V_\psi, V_{\psi'}) \leq 1$.

Let $x \in H$. Using the normality of $L$ we have

$$\Delta(x)\chi(u) = \Delta(xu) = \Delta(xux^{-1}) = \chi'(xux^{-1}) \Delta(x).$$

Thus

$$\Delta(x) = 0 \text{ unless } \chi(u) = \chi'(xux^{-1}) \quad (6)$$

Now if $x, y$ are given such that $\Delta(x), \Delta(y)$ are both nonzero then we see that

$$\chi(x^{-1}ux) = \chi'(u) = \chi(y^{-1}uy)$$

which by Lemma 2.4 implies that $xy^{-1} \in L$. Since $\Delta$ (if nonzero) is supported on a single coset, $\dim \text{Hom}_H(V_\psi, V_{\psi'}) \leq 1$.

Taking $\chi = \chi'$, this shows that $V_\psi$ is irreducible. Moreover, we may now determine when $V_\psi$ and $V_{\psi'}$ are isomorphic. By (6) there must exist some $x$ such that $\chi'(u) = \chi(x^{-1}ux)$. Now

$$B(u, x) = \psi([u, x]) = \chi(uxu^{-1}x^{-1}) = \chi(u)\chi'(u^{-1})$$

so the necessary and sufficient condition is (5).

It remains to be argued that every irreducible $\pi$ of $H$ with central character $\psi$ is a $V_\chi$. Observe that $Z_1$ acts trivially so $\pi$ factors through $H/Z_1$. Now $L/Z_1$ is abelian, so the restriction of $\pi$ to $L/Z_1$ decomposes as a direct integral of characters, each of which must match $\psi$. By Frobenius reciprocity, $\pi$ contains a $V_\psi$, but since both are irreducible they are the same. \qed

Corollary 2.6. The number of nonisomorphic irreducible modules of $H$ with central character $\psi$ is $|Q_\psi|$. The degree of such an irreducible is $\sqrt{|W : Q_\psi|}$.

Proof. We define a homomorphism $\theta : W \rightarrow W^*$ (the dual group) by letting $\theta(x)$ be the character $u \mapsto B(u, x)$. Such a character can be pulled back to a character of $H$ that is trivial on $Z$. By the theorem, two extensions of $\psi$ to $L$ induce to isomorphic $H$-modules if and only if they differ by an element of $\theta(W)$. Therefore the number of such extensions equals $|W^*/\theta(W)|$. We have a short exact sequence:

$$0 \rightarrow \bar{Q} \rightarrow W \rightarrow \theta^* W \rightarrow W^*/\theta(W) \rightarrow 0.$$

Since $|W| = |W^*|$ we have $|W^*/\theta(W)| = |\bar{Q}|$.

The degree of an irreducible with central character $\psi$ is the index in $L$ in $W$, since it is induced from a one-dimensional character of $L$. Now the restriction of $B$ to $L \times W$ induces a dual pairing $L/Q \times (W/L) \rightarrow \mathbb{T}$ which is nondegenerate (since otherwise $L$ would not be maximal isotropic. Therefore $|L/Q| = |W/L|$ and so $|L| = \sqrt{|W||Q|}$. This implies that $|W : L| = \sqrt{|W||Q|}$. \qed
An important special case:

**Corollary 2.7.** Suppose that the bilinear form $B$ on $W$ is nondegenerate, meaning that the kernel $Q_{\psi} = 1$. Let $L$ be a Lagrangian subspace and let $\chi, \chi'$ be extensions of $\psi$ to $L$. Then $V_\chi \cong V_{\chi'}$.

**Proof.** With $\theta$ as in the last Corollary, if the kernel $\bar{Q} = 1$ then $\theta$ is injective, and so it is surjective. This means that we may always find $x$ such that (5) is true. \qed

**Corollary 2.8.** Elements of $Q_{\psi}$ act by scalars in any irreducible representation with central character $\psi$.

**Proof.** If $u \in Q$ then for any $x \in H$ we have $\pi(u)\pi(x)\pi(u)^{-1}\pi(x)^{-1} = \psi([u,x]) = \beta(u,x) = 1$. Thus $\pi(u)$ commutes with the action of $H$ and is a scalar transformation by Schur’s Lemma. \qed

**Corollary 2.9.** If $\theta$ is the character of an irreducible representation with central character $\psi$, then $\theta$ vanishes off $Q_{\psi}$.

**Proof.** Let $h \in H - Q$. We claim that there exists a Lagrangian subspace $L$ such that $h \notin L$. Indeed, by definition of $Q$ there exists $x \in H$ such that $\beta(h,x) \neq 1$. Now the group generated by $x$ and $Q$ is isotropic since $\beta(x,x) = 1$, so let $L$ be a maximal isotropic subgroup containing $x$ and $Q$. We cannot have $h \in L$ since otherwise $L$ would not be isotropic.

We may construct $\theta$ by extending $\psi$ to $L$ and inducing. Since $L$ is normal, $\theta$ vanishes off $L$, and in particular $\theta(h) = 0$. \qed

Now we return to the special case of the Heisenberg group. Let $q$ be the residue cardinality of $F$. We assume that $q$ is odd.

**Theorem 2.10 (“Finite Stone-von Neumann”).** Let $H = V \oplus F$ be the Heisenberg group with multiplication given by (3). Let $\psi$ be any nontrivial character of $Z_H \cong F$. Then $H$ has a unique irreducible representation with central character $\psi$. If the dimension of $V$ is $2r$ then the degree of this representation in this equivalence class equals $q^r$.

**Proof.** If we identify $Z_H$ with $F$, then it is easy to check that $\beta(v,w) = \psi(\langle v,w \rangle)$ for $v, w \in V$. Because $\langle \rangle$ is nondegenerate and $\psi$ is nontrivial we have $Q_{\psi} = Z_F$. The statement now follows from Corollary 2.6. \qed

Now Theorem 2.2 produces a projective representation of $\text{Sp}(2r,F)$ and hence a central extension. In this case where $F$ is finite, it may be shown that this central extension is trivial. Hence this method produces representations of $\text{Sp}(2r,F_q)$ when $q$ is odd.

Theorem 2.10 may be found in Gérardin (1977) as well as its consequent representations of the finite symplectic groups.

However the Stone-Von Neumann theorem remains true if $F$ is a local field, and so we get a representation of $\text{Sp}(2r,F)$ when $F$ is local. This is the oscillator representation. The corresponding central extension is essentially a double cover. We will construct the oscillator representation in the next section.

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3 Weil 1964

Although the construction of the oscillator representation is a consequence of the Stone-Von Neumann theorem, Weil (1964) found a construction the oscillator representation that uses some of the ideas of a proof of Stone-Von Neumann, but which produces the representation directly. We will follow Weil to a large extent. Some parts of this section are adapted from Bump (1997) Section 4.8.

A proof of the Stone Von-Neumann theorem (over archimedan fields) using the same tools may be found in Lion and Vergne (1980) Section 1.3.

Let $G$ be a locally compact abelian group, and let $G^*$ be its Pontriagin dual (defined below). We will write $G$ and $G^*$ additively. Let $T \subset \mathbb{C}^*$ be the group of complex numbers of unit norm, and let $\langle , \rangle$ denote the dual pairing $G \times G^* \rightarrow \mathbb{T}$. We introduce the group $A(G)$, which as a topological space is $G^* \times G \times T$, with the group law

$$\left( v_1^*, v_1, t_1 \right) \left( v_2^*, v_2, t_2 \right) = \left( v_1^* + v_2^*, v_1 + v_2, t_1 t_2 \langle v_1, v_2^* \rangle \right).$$

(7)

If $w = (v^*, v) \in G^* \times G$ and $t \in \mathbb{T}$, we may write $(w, t)$ instead of $(v^*, v, t)$ for the corresponding element of $A(G)$. If $w_1 = (v_1^*, v_1)$ and $w_2 = (v_2^*, v_2) \in G^* \times G$, we will write $[w_1, w_2] = \langle v_1, v_2^* \rangle$, so that the group law (7) may be written

$$\left( w_1, t_1 \right) \left( w_2, t_2 \right) = \left( w_1 + w_2, t_1 t_2 [w_1, w_2] \right).$$

(8)

The group $A(G)$ is a two-step nilpotent group.

We have a unitary representation $\pi$ of $A(G)$ on $L^2(G)$, namely

$$\left( \pi(v^*, v, t) \Phi \right)(u) = t \langle u, v^* \rangle \Phi(u + v).$$

(9)

Let $B(G)$ be the group of automorphisms of $G$, and let $B_0(G)$ be the subgroup of $B(G)$ consisting of elements which act trivially on the center $Z(A(G)) = \{(0, 0, t) | t \in \mathbb{T}\}$.

Our goal is to prove:

**Theorem 3.1.** The unitary representation $\pi$ is irreducible. Let $\sigma \in B_0(G)$. Then there exists a unitary operator $\omega(\sigma)$ on $L^2(G)$, determined up to scalar multiple, such that (for $h \in A(G)$)

$$\pi(\sigma h) = \omega(\sigma) \pi(h) \omega(\sigma)^{-1}.$$

(10)

Let $\phi \in C_c(G^* \times G)$, the space of compactly supported continuous functions on $G^* \times G$. Let $\pi(\phi)$ be the endomorphism of $L^2(G)$ defined by

We recall the essentials of Fourier analysis on locally compact abelian groups. It was Weil who first described Fourier analysis in this generality. Proofs may be found, for example in Rudin (1962) or Loomis (1953).

If $G$ is a locally compact group, then $G$ has a regular left Borel measure (unique up to constant) that is invariant under left translation. Here *regularity* means that for a measurable set $X$

$$\mu(X) = \sup \mu(K) = \inf \mu(E)$$

where $K \subseteq X$.
where \( K \) runs through compact subsets of \( X \) and \( E \) runs through open sets containing it. Similarly there is a right Haar measure; often the two coincide, in which case the group is called \textit{unimodular}.

Let \( \mathbb{T} \) be the group of complex numbers of absolute value 1, and let \( G \) be a locally compact abelian group. The Pontryagin dual \( G^* \) is the group of all characters \( G \to \mathbb{T} \), with the topology of uniform convergence on compact sets. It is also a locally compact group. We will write the groups \( G \) and \( G^* \) additively. The groups \( G, G^* \) are given additive Haar measure. Then if \( f \in L^2(G) \), the \textbf{Fourier transform} \( \hat{f} = \mathcal{F}f \in L^2(G^*) \) is defined by the formula

\[
\hat{f}(x^*) = \int_G f(x) \langle x, x^* \rangle \, dx.
\]

More precisely, this definition (using the Haar integral) is correct on the dense subspace \( L^2(G) \cap L^1(G) \), and is an isometry in the sense that we may choose the Haar measures on \( G \) and \( G^* \) so that the \textit{Plancherel formula}

\[
\|f\|_2 = \|\hat{f}\|_2
\]

is satisfied. Then the Fourier transform extends by continuity to all of \( L^2(G) \). The Fourier transform is an isometry \( L^2(G) \to L^2(G^*) \), and we have the \textbf{Fourier inversion formula}

\[
\mathcal{F}^2 \hat{f}(x) = f(-x).
\]

Let \( G \) be a locally compact abelian group. The \textbf{Bruhat-Schwartz space} \( S(G) \) was defined in Weil (1964), Section 11. A useful reference is Osborne (1975). To recapitulate Weil’s definition, first assume that \( G \) is of the form \( \mathbb{R}^n \times \mathbb{T}^q \times \mathbb{Z}^p \times F \) where \( F \) is a finite abelian group. Such a group will be called \textit{elementary}. Then a function on \( G \) is called \textit{polynomial} if it can be written as a polynomial with respect to the coordinates in the \( \mathbb{R} \) and \( \mathbb{Z} \) coordinate functions. Also if \( \alpha = (k_1, \ldots, k_{n+q}) \) is any tuple of nonnegative integers then let \( \partial_\alpha = \partial^{k_1}_{x_1} \cdots \partial^{k_{n+q}}_{x_{n+q}} \) be the corresponding derivative in the \( \mathbb{R} \) and \( \mathbb{T} \) coordinates. Then \( S(G) \) will be the space of smooth functions \( f \) such that \( P \partial_\alpha(f) \) is bounded for all polynomial functions \( P \) and derivatives \( \partial_\alpha \).

Now if \( G \) is a general locally compact abelian group, a subgroup \( H \) generated by a compact open neighborhood of the identity will be called \textit{compactly generated}. Thus it is an open subgroup. We consider pairs \((H, K)\) where \( H \) is a compactly generated subgroup of \( G \), and \( K \) is a closed subgroup of \( H \) such that \( H/K \) is elementary. If \( f \in S(H/K) \), we may pull it back to a function on \( H \), then extend it by zero to the complement of \( H \) in \( K \). Thus \( f \) gives rise to a function on \( G \). The set of functions \( f \) on \( G \) that arise in this way (for some \( H \) and \( K \) depending on \( f \)) is \( S(G) \).

The Bruhat-Schwartz space is dense in \( L^2(G) \). If \( f \in S(G) \) then the Fourier transform \( \hat{f} \in S(G^*) \). If \( F \) is the additive group of a nonarchimedean local field, then \( S(F) \) consists of functions that are compactly supported and locally constant.

We next review the theory of \textit{Hilbert-Schmidt operators}. Let \( H \) be a Hilbert space, and let \( T : H \to H \) be a bounded operator. Let \( \phi_i \) be an orthonormal basis of \( H \) and let

\[
T\phi_i = \sum_j a_{i,j} \phi_j.
\]
Then the \textit{Hilbert-Schmidt norm} is
\[
|T|_{\text{HS}} = \sum_j |A\phi_j|^2 = \sum_{i,j} |a_{i,j}|^2.
\]
It does not depend on the choice of orthonormal basis. The operator is called \textit{Hilbert-Schmidt} if \(|T|_{\text{HS}} < \infty\). A Hilbert-Schmidt operator is compact.

If \(H = L^2(X)\), where \(X\) is a measure space, then we can define an integral kernel
\[
K(x, y) = \sum_{i,j} a_{i,j} \phi_i(x) \overline{\phi_j(y)}
\]
and
\[
T\phi(x) = \int_X K(x, y) \phi(y) \, dy.
\]
Then
\[
|T|_{\text{HS}} = \int_{X \times X} |K(x, y)|^2 \, dx \, dy.
\]
Hence the space of Hilbert-Schmidt operators may be identified with \(L^2(X \times X)\). We call elements of \(L^2(X \times X)\) \textit{Hilbert-Schmidt kernels}.

We remind the reader that \(A(G)\) is the group \(G^* \times G \times \mathbb{T}\), where the multiplication is
\[(x^*, x, t)(y^*, y, u) = (x^* + y^*, x + y, \langle x, y^* \rangle tu).\]
Then \(A(G)\) is a 2-step nilpotent group, the center \(Z_{A(G)}\) being the subgroup of elements of the form \((0, 0, t)\) isomorphic to \(\mathbb{T}\). The group \(A(G)\) is easily seen to be unimodular. We have a unitary representation \(\pi\) of \(A(G)\) on \(L^2(G)\) defined by
\[
\pi((u^*, u, t)) \Phi(x) = t \Phi(x + u) \langle x, u^* \rangle.
\]
Let \(C^\psi_c(A(G))\) be the ring (under convolution) of compactly supported continuous functions on \(G\) that satisfy
\[
\phi((u, u^*, t)) = t^{-1} \phi(u, u^*, 1).
\] (11)
The space of such functions may be identified with \(C_c(G^* \times G)\), so we will denote
\[
\phi(u^*, u) = \phi((u^*, u, 1)).
\]
We also define \(L^2_c(A(G))\) to be the subspace of \(L^2(A(G))\) determined by the condition (11).

With \(\phi \in C^\psi_c(A(G))\) and \(\Phi \in L^2(G)\), we have
\[
\pi(\phi)\Phi(x) = \int_{A(G)} \phi(w)(\pi(w)\Phi)(x) \, dw = \int_{G \times G^*} \phi(u^*, u)\pi(u^*, u, 1)\Phi(x) \, du \, du^*.
\]
Then
\[
\pi(\phi)\Phi(x) = \int_G K_\phi(x, y)\Phi(y) \, dy
\]

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where
\[ K_\phi(x, y) = \int_{G^*} \phi(u^*, y - x) \langle x, u^* \rangle du^*. \]

Note that \( \phi \) can be recovered from the kernel \( K_\phi \) by the Fourier inversion formula:
\[
\phi(x^*, x) = \int_{G} K_\phi(u, u + x) \overline{\langle u, x^* \rangle} du.
\] (12)

**Proposition 3.2.** If \( \phi \) is in \( C_c(G \times G^*) \) then \( \pi(\phi) \) is Hilbert-Schmidt and
\[
|\pi(\phi)|_{HS} = \int_{G \times G^*} |\phi(u^*, u)|^2 du^* du.
\] (13)

The map \( \phi \mapsto \pi(\phi) \) extends to an isomorphism between \( L^2(G^* \times G) \) and the space of Hilbert-Schmidt operators on \( L^2(G) \).

**Proof.** We have
\[
|\pi(\phi)|_{HS} = \int_{G \times G^*} |K_\phi(x, y)|^2 dx dy = \int_{G} \int_{G^*} \left| \int_{G^*} \phi(u^*, y - x) \langle x, u^* \rangle du^* \right|^2 dy dx.
\]

With \( x \) fixed, we make the variable change \( y \mapsto y + x \). We obtain
\[
\int_{G} \int_{G^*} \left| \int_{G^*} \phi(u^*, y) \langle x, u^* \rangle du^* \right|^2 dy dx.
\]

By the Plancherel formula
\[
\int_{G^*} \left| \int_{G^*} \phi(u^*, y) \langle x, u^* \rangle du^* \right|^2 dx = \int_{G} |\phi(u^*, u)|^2 du.
\]

Now (13) follows. Since \( C_c(G^* \times G) \) is dense in \( L^2(G^* \times G) \) and \( \phi \mapsto \pi(\phi) \) is an isometry for the \( L^2 \) and HS norms, it can be extended to all of \( L^2(G^* \times G) \). From the Fourier inversion formula, the inverse construction from Hilbert-Schmidt kernels to \( L^2(A(G)) \) is given by (12).

More generally, if \((\Pi, H)\) is any unitary representation of \( A(G) \) we may define
\[
\Pi(\phi) \Phi(x) = \int_{A(G)} \phi(w)(\Pi(w)\Phi)(x) dw
\]
for \( \phi \in C_c^\psi(A(G)) \) and \( \Phi \in L^2(G) \). It is straightforward to check that
\[
\Pi(\phi_1) \Pi(\phi_2) = \Pi(\phi_1 \ast \phi_2)
\]
so \( \Pi \) is a representation of the convolution ring \( C_c^\psi(A(G)) \). If we identify \( C_c^\psi(A(G)) \) with \( C_c(G^* \times G) \) then the convolution is given by the formula
\[
(\phi_1 \ast \phi_2)(u^*, u) = \int_{G^* \times G} \phi_1(v^*, u_2)\phi_2(u^* - v^*, u - v) \langle v^* - u^*, v \rangle dv^* dv.
\]
Since \( \pi(\phi_1 * \phi_2) = \pi(\phi_1)\pi(\phi_2) \) we have
\[
K_{\phi_1 * \phi_2} = K_{\phi_1} \times K_{\phi_2},
\]
(14)
where the composition law in \( L^2(G \times G) \) is defined by
\[
(K_1 \times K_2)(u, v) = \int_G K_1(u, x) K_2(x, v) \, dx.
\]
(15)
It is easily deduced from the Cauchy-Schwarz inequality that
\[
||K_1 \times K_2||_2 \leq ||K_1||_2 \cdot ||K_2||_2,
\]
so this composition law makes \( L^2(G \times G) \) into a Banach algebra (without unit). Similarly if \( P \) and \( Q \) \( \in \) \( L^2(G) \), \( K \) \( \in \) \( L^2(G \times G) \), we define
\[
(K \times P)(u) = \int_G K(u, v) P(v) \, dv, \quad (Q \times K)(v) = \int_G Q(u) K(u, v) \, du,
\]
and the Cauchy-Schwarz inequality implies that
\[
||K \times P||_2 \leq ||K||_2 \cdot ||P||_2, \quad ||Q \times K||_2 \leq ||Q||_2 \cdot ||P||_2.
\]
Thus \( L^2(G) \) is both a left and right module for \( L^2(G \times G) \).

We now prove the first statement in Theorem 3.1.

**Proposition 3.3.** The representation \((\pi, L^2(G))\) is irreducible.

**Proof.** If \( V \subset L^2(G) \) is a closed invariant subspace we may consider the orthogonal projection \( p_V \) on \( V \). This commutes with the action of \( \pi \), hence it commutes with the operators \( \pi(\phi) \) and indeed all Hilbert-Schmidt operators. The identity \( \pi(\phi)p_V = p_V \pi(\phi) \) implies that \( V \) is invariant under all Hilbert-Schmidt operators from which it is clear that either \( V = 0 \) or \( V = L^2(G) \). \( \square \)

If \( P \) and \( Q \) \( \in \) \( L^2(G) \), \( K \) \( \in \) \( L^2(G \times G) \), we define
\[
(K \times P)(u) = \int_G K(u, v) P(v) \, dv, \quad (Q \times K)(v) = \int_G Q(u) K(u, v) \, du,
\]
and the Cauchy-Schwarz inequality implies that
\[
||K \times P||_2 \leq ||K||_2 \cdot ||P||_2, \quad ||Q \times K||_2 \leq ||Q||_2 \cdot ||P||_2.
\]
Thus \( L^2(G) \) is both a left and right module for \( L^2(G \times G) \).

Let \( P, Q \in L^2(G) \). We define an element \( P \otimes Q \) of \( L^2(G \times G) \) by \((P \otimes Q)(u, v) = P(u) \otimes Q(v)\). By abuse of language, we will refer to elements of this type as **pure tensors**. If \( K \in L^2(G \times G) \) we then have
\[
(P \otimes Q) \times K = P \otimes (Q \times K), \quad K \times (P \otimes Q) = (K \times P) \otimes Q,
\]
(16)
and (with $S$ and $T$ ∈ $L^2(G)$)

\[(P \otimes Q) \times (\tilde{S} \otimes T) = \left\{ \int_G Q(u) \overline{S(u)} \, du \right\} P \otimes T = (Q, S)_2 P \otimes T,\]

(17)

where $(, )_2$ denotes the inner product in $L^2(G)$.

**Lemma 3.4.** (i) An element $K$ of $L^2(G \times G)$ has the form $P \otimes Q$ if and only if $K \times K' \times K$ is proportional to $K$ for all $K'$.

(ii) Let $K_1 = P_1 \otimes Q_1$ and $K_2 = P_2 \otimes Q_2$. Then $P_1$ and $P_2$ are proportional if and only if $K_1 \times K$ and $K_2 \times K$ are proportional for all pure tensors $K \in L^2(G \times G)$, and similarly $Q_1$ and $Q_2$ are proportional if and only if $K \times K_1$ and $K \times K_2$ are proportional for all pure tensors $K$.

(iii) Suppose that $s : L^2(G \times G) \to L^2(G \times G)$ is a unitary transformation respecting the composition law $\times$. Then there exists a unitary transformation $s_0 : L^2(G) \to L^2(G)$ such that $s(P \otimes \bar{Q}) = s_0(P) \otimes s_0(\bar{Q})$. If $s$ is invertible, then so is $s_0$.

**Proof.** For (i), if $K$ is nonzero, find $P'$ and $Q'$ in $L^2(G)$ such that both $P = K \times P'$ and $Q = Q' \times K$ are nonzero. Then if $K' = P' \otimes Q'$, we find by (16) that $K \times K' \times K = P \otimes Q$, and if $K$ is proportional to this, we must have (after adjusting $P$ and $Q$ by a constant if necessary) $K = P \otimes Q$. Conversely, if $K = P \otimes Q$, by (16) and (17) we have $K \times K' \times K = (Q, \bar{K} \times \bar{P})_2 K$, which is proportional to $K$, so the condition of (i) is both necessary and sufficient.

For (ii), if $K = \tilde{S} \otimes T$, we have by (17)

\[K_1 \times K = (Q_1, S)_2 P_1 \otimes T, \quad K_2 \times K = (Q_2, S)_2 P_2 \otimes T.\]

These are proportional if $P_1$ and $P_2$ are, and since (if $Q_1 \neq 0$) $S$ and $T$ may be chosen so that $(Q_1, S)$ and $T$ are nonzero, the proportionality of $K_1 \otimes K$ and $K_2 \otimes K$ also implies the proportionality of $P_1$ and $P_2$.

For (iii), it is a consequence of the intrinsic characterization (i) of the pure tensors $P \otimes Q$ in $L^2(G \times G)$ that if $K$ is a pure tensor so is $s(K)$. We choose $P_0$ of unit norm; since $s$ is unitary $s(P_0 \otimes \bar{P}_0)$ has the form $P'_0 \otimes \bar{Q}'_0$ for some $P'_0$ and $Q'_0$, both of norm one. Now if $P \in L^2(G)$, we may write $s(P \otimes \bar{P}_0)$ in the form $P' \otimes \bar{Q}'$, and it follows from the intrinsic criterion (ii) that $Q_0$ and $Q$ are proportional. Thus we may write $s(P \otimes \bar{P}_0) = \mu(P) \otimes \bar{Q}_0'$ for some uniquely determined $\mu(P) \in L^2(G)$. It is clear that $\mu$ defined this way is a unitary linear operator on $L^2(G)$; similarly $s(P_0 \otimes \bar{Q}) = P'_0 \otimes \nu(\bar{Q})$, for a unitary linear operator $\nu$. We have in particular $\mu(P_0) = P'_0$ and $\nu(P_0) = Q'_0$. By (17), we have $P \otimes \bar{Q} = (P \otimes \bar{P}_0) \times (P_0 \otimes \bar{Q})$. Therefore $s(P \otimes \bar{Q}) = c\mu(P) \times \nu(\bar{Q})$ where $c = (P'_0, Q'_0)_2$, and and since $s$ is unitary, applying this relation with $P = Q = P_0$ gives $c = 1$. We have

\[(P \otimes \bar{Q}) \times (P \otimes \bar{Q}) = (P, Q)_2 (P \otimes \bar{Q}),\]

and applying $s$ to this identity gives

\[(\mu(P), \nu(\bar{Q}))_2 (\mu(P) \otimes \nu(\bar{Q})) = (P, Q)_2 (\mu(P) \otimes \nu(\bar{Q})),\]
so \((\mu(P), \nu(Q))_2 = (P, Q)_2\) for all \(P\) and \(Q\). This (together with the unitarity of \(\mu\) and \(\nu\)) implies that \(\mu = \nu\), since expanding \((\mu(P) - \nu(P), \mu(P) - \nu(P))_2\) gives zero. We may then take \(s_0 = \mu = \nu\). If \(s\) is invertible, then we may apply the same reasoning to \(s^{-1}\), and construct an inverse to \(s_0\).

Since \(A(G)/Z(A(G)) \cong G^* \times G\), any \(\sigma \in B_0(G)\) induces an automorphism \(s\) of \(G^* \times G\), and so \(\sigma(w, t) = (s(w), f(w) t)\) where \(f : G^* \times G \to \mathbb{T}\) is some mapping. In order that \(\sigma\) be a homomorphism, we must have

\[
f(w_1 + w_2) = f(w_1) f(w_2) [s(w_1), s(w_2)] [w_1, w_2]^{-1}.
\]

(18)

We recall that if \(H\) is a locally compact abelian group and if \(\alpha : H \to H\) is an automorphism, then \(\alpha\) takes Haar measure into a constant multiple of itself, and the module \(|\alpha|\) of \(\alpha\) is defined to be that constant. We need to know that the module of \(s\) is one. To prove this, we note that the commutator

\[
(w_1, 1) (w_2, 1) (w_1, 1)^{-1} (w_2, 1)^{-1} = (0, (w_1, w_2)),
\]

lies in the center of \(A(G)\), where we define \(\langle w_1, w_2 \rangle = [w_1, w_2] [w_2, w_1]^{-1}\). If \(w_1 = (v_1^*, v_1)\) and \(w_2 = (v_2^*, v_2)\), then

\[
\langle w_1, w_2 \rangle = \langle v_1, v_2 \rangle \langle v_2, v_1^* \rangle^{-1}.
\]

It follows from this that the pairing \(\langle , \rangle\) of \(G^* \times G\) with itself is skew-symmetric, bilinear and nondegenerate. Since by definition of \(B_0(G)\) it is assumed that \(\sigma\) acts trivially on the center of \(A(G)\), it follows that \(s\) preserves the pairing \(\langle , \rangle\). Thus if we identify \(G \times G\) with its dual by means of this pairing then \(s = s^{-1}\). Thus \(|s| = |s|^{-1}\), and so \(|s| = 1\).

We now define a unitary transformation \(\Sigma\) of \(L^2(G^* \times G)\) by

\[
(\Sigma \phi)(w) = f(w)^{-1} \phi(s(w)), \quad \phi \in L^2(G^* \times G).
\]

(19)

It is easily deduced from (18) and (3) that

\[
\Sigma(\phi_1 \ast \phi_2) = \Sigma \phi_1 \ast \Sigma \phi_2.
\]

(20)

This is a straightforward verification, but it involves a change of variables, so it uses that fact that the module of \(s\) is one; and the unitaricity of \(\Sigma\) also uses the unimodularity of \(s\). Now by means of the isometry \(\lambda : L^2(G \times G) \to L^2(G^* \times G)\), we may transfer \(\Sigma\) to a unitary transformation of \(L^2(G \times G)\), which we also denote \(\Sigma\). In view of (14), \(\Sigma\) preserves the multiplication \(\times\) on \(L^2(G \times G)\), and by Lemma 3.4 (iii) there exists a unitary map \(\omega : L^2(G) \to L^2(G)\) (the inverse of \(s_0\) in that lemma) such that

\[
\Sigma(P \otimes Q) = \omega^{-1}(P) \otimes \omega^{-1}(Q).
\]

(21)

Next we prove that if \(P, Q \in L^2(G)\), \(w \in G^* \times G\) and \(t \in \mathbb{T}\), then

\[
\tilde{t} \lambda(P \otimes Q)(w) = (P, \pi(w, t) Q)_2.
\]

(22)
Indeed, by (12) with \( \phi = \lambda(P \otimes Q) \), \( K_\phi = P \otimes Q \) and \( w = (v^*, v) \), the left side of (22) equals
\[
\int_G P(u) t Q(u + v) \langle u, v^* \rangle \, du,
\]
whence (22). Now consider
\[
\bar{t} \lambda(P \otimes Q)(w).
\]
On the one hand, by (19) and (22), this equals
\[
\bar{t} f(w) \lambda(P \otimes Q)(s(w)) = (P, \pi(s(w), f(t)) Q)_2 = (P, \pi(\sigma(w, t)))_2.
\]
On the other hand, using the fact that \( \Sigma \circ \lambda = \lambda \circ \Sigma \) and (22) again, (23) equals
\[
\bar{t} \lambda(\omega^{-1}(P) \otimes \omega^{-1}(Q))(w) = (\omega^{-1}P, \pi(w, t) \omega^{-1}Q)_2 =
(P, \omega \pi(w, t) \omega^{-1})_2.
\]
Since these are equal for all \( P \) and \( Q \), we have \( \pi(\sigma(w, t)) = \omega \pi(w, t) \omega^{-1} \). The proof of Theorem 3.1 is now complete.

4 The Stone-Von Neumann Theorem II: The case of a local field

Let \( F \) be a local field and let \( \psi \) be a nontrivial additive character.

Let \( W \) be symplectic with respect to the alternating form \( B : W \times W \to F \). Let \( V \) and \( V' \) be complementary Lagrangian (maximal isotropic subspaces) so that \( W = V^* \oplus V \). Then we may write
\[
B(w, w') = \beta(w, w') - \beta(w', w)
\]
where \( \beta \) is a bilinear map \( V' \times V \to \mathbb{C} \) defined by
\[
\beta(x + x', y + y') = B(x', y)
\]
for \( x, y \in V, x', y' \in V' \). The map \( \beta \) induces a dual pairing \( V' \times V \to \mathbb{C} \), and also we may identify \( V' = V^* \) (Pontriagin dual) via the pairing
\[
\langle x^*, x \rangle = \psi(\beta(x^*, x))
\]
for \( x \in V, x^* \in V' \). Then we may define both the Heisenberg group \( H(W) = W \oplus F \) and the group \( A(V) \). The multiplications are:
\[
(w, a)(w', b) = \left( w + w', a + b + \frac{1}{2} B(w, w') \right)
\]
for $H(W)$ and
\[(w, t)(w', t') = (w + w', tt'\psi(\beta(w, w')))\]
for $A(V)$. Define
\[\lambda : H(W) \longrightarrow A(V) \]
\[\lambda(w, a) = (w, f(w)\psi(a)).\]

This is to be a homomorphism.

\[\lambda(w, 0)\lambda(w', 0) = (w, f(w))(w', f(w')) = (w + w', f(w)f(w')\psi(\beta(w, w')))\]
\[\lambda((w, 0)(w', 0)) = \lambda \left( w + w', \frac{1}{2}B(w, w') \right) = \left( w + w', f(w + w')\psi \left( \frac{1}{2}B(w, w') \right) \right).\]

Thus we need
\[\frac{f(w + w')}{f(w)f(w')} = \psi \left( \beta(w, w') - \frac{1}{2}\beta(w, w') + \frac{1}{2}\beta(w', w) \right) = \psi \left( \frac{1}{2}\beta(w, w') + \frac{1}{2}\beta(w', w) \right).\]

We may take
\[f(w) = \psi \left( \frac{1}{2}\beta(w, w') \right).\]

**Theorem 4.1.** Let $(\Pi, \mathcal{H})$ be a unitary representation of $H(W)$ on a Hilbert space $\mathcal{H}$ such that
\[\Pi((0, t)) = \psi(t)I_{\mathcal{H}}.\]

Then $\mathcal{H}$ decomposes as a direct sum of irreducible subspaces each isomorphic to $(\pi, L^2(V))$.

**Proof.** The representations $\pi$ and $\Pi$ factor through $A(V)$. Let $\theta \in S(V)$, the Bruhat-Schwartz space, normalized so $|\theta|^2 = 1$. The projection $p$ onto the one-dimensional space spanned by $\theta$ is Hilbert-Schmidt, so
\[p = \pi(\phi_{\theta})\]
for some $\phi_{\theta} \in L^2(W)$. It is given by a kernel $K_{\phi_{\theta}}(x, y) = \theta(x)\overline{\theta(y)}$, that is
\[p(f)(x) = \int_V K_{\phi_{\theta}}(x, y)f(x)\,dx, \quad f \in L^2(V).\]

As before
\[K_{\phi_{\theta}}(x, y) = \int_{V^*} \phi(u, y - x)\langle x, u^* \rangle du^*,\]
\[\phi_{\theta}(x^*, x) = \int_V K_{\phi_{\theta}}(u, u + x)\psi(-B(u, x^*))\,du.\]
\[\phi_{\theta}(x^*, x) = \int_V \theta(u)\overline{\theta(u + x)}\psi(-B(u, x^*))\,du.\]

This is a Schwartz function on $W$. Therefore $P = \Pi(\phi_{\theta})$ is also defined.
Since $p$ is self adjoint and $p^2 = p$, the function $\phi_\theta$ satisfies $\phi_\theta(h^{-1}) = \overline{\phi_\theta(h)}$ and $\phi_\theta \ast \phi_\theta = \phi_\theta$. This implies that $P : \mathfrak{H} \rightarrow \mathfrak{H}$ is self adjoint and $P^2 = P$. It is therefore the projection onto a closed subspace $\mathfrak{H}_0$ of $\mathfrak{H}$.

Now pick a vector $x_0 \in \mathfrak{H}_0$ of length 1. Then we may define a map

$$
t_{x_0} : L^2(V) \rightarrow \mathfrak{H}_0
$$

as follows. First we define $t_{x_0}$ for functions of the form $\pi(\phi)\theta$. These are dense because $\pi$ is irreducible. We define

$$
t_{x_0}(\pi(\phi)\theta) = \Pi(\phi)x_0.
$$

We check that this is well-defined. If $\pi(\phi)\theta = \pi(\phi')\theta$ then $\pi(\phi)\pi(\phi_0) = \pi(\phi)p = \pi(\phi')\pi(\phi_0)$ so $\phi \ast \phi_0 = \phi' \ast \phi_0$ and so $\Pi(\phi)P = \Pi(\phi')P$. Since $P x_0 = x_0$ this implies that $\Pi(\phi)x_0 = \Pi(\phi')x_0$.

The map $t_{x_0}$ is clearly $H(W)$-equivariant. It extends to all of $L^2(V)$ by continuity. This shows that $\mathfrak{H}$ contains a submodule isomorphic to $L^2(V)$, and a Zorn’s Lemma argument then implies that $\mathfrak{H}$ decomposes into a direct sum of copies of $L^2(V)$. \hfill \Box

5 Weil continued: the dual pair $(SL(2), O(V))$

Let $F$ be a local field of characteristic not equal to 2. (If $F$ is nonarchimedean, the residue characteristic may equal 2.) Let and $V$ a quadratic space over $F$, that is a space equipped with a nondegenerate symmetric bilinear form $B : V \times V \rightarrow F$. We will associate to a representation of $O(V, F)$ a projective representation of $SL(2, F)$; and we will show that if $V$ is even-dimensional then the corresponding central extension of $SL(2, F)$ is trivial. The results are due to Weil; we will also use an approach from Jacquet and Langlands (1970) (see their Lemma 1.3). the exposition follows Bump (1997) Section 4.8.

Let $\psi : F \rightarrow \mathbb{C}$ be a nontrivial additive character. Let $V$ be a vector space over $F$ of finite dimension $d$, and let $B : V \times V \rightarrow \mathbb{C}$ be a nondegenerate symmetric bilinear form. $O(V)$ will denote the orthogonal group of endomorphisms of $V$ preserving $B$. Let $H$ be the Heisenberg group which as a set is $V \times V \times F$, with group law

$$
(v_1^*, v_1, x_1)(v_2^*, v_2, x_2) = (v_1^* + v_2^*, v_1 + v_2, x_1 + x_2 + B(v_1^*, v_2) - B(v_1, v_2^*)).
$$

(24)

for $v_1, v_2, v_1^*, v_2^* \in V, x_1, x_2 \in F$. We may identify $V$ with its dual by means of the pairing $\langle v, v^* \rangle \mapsto \langle v, v^* \rangle = \psi(-2B(v, v^*))$. Then the group $A(V)$ is as a set $V \times V \times \mathbb{T}$ with multiplication $(v_1^*, v_1, t_1)(v_2^*, v_2, t_2) = (v_1^* + v_2^*, v_1 + v_2, t_1 t_2 \psi(-2B(v_1, v_2^*)).)$ We have a homomorphism $\tau : H \rightarrow A(V)$ by $\tau(v^*, v, x) = (v^*, v, \psi(x) \psi(-B(v, v^*))$. As in the previous Theorem, there exists a unitary representation $\rho$ of $A(V)$ given by (8.6). Let $\pi = \rho \circ \tau$ be the corresponding representation of $H$ on $L^2(V)$. Then it is easy to check that $\pi$ is given by the same formulas (1.29) and (1.30) as in the case of a finite field.

We have actions of $SL(2, F)$ and $O(V)$ on $H$ as follows: if $g = \binom{a b}{c d} \in SL(2, F)$, we let $g(v_1, v_2, x) = (av_1 + bv_2, cv_1 + dv_2, x)$, and if $k \in O(V)$, we let $k(v_1, v_2, x) = (k(v_1), k(v_2), x)$. If $\Phi \in L^2(V)$, we define the Fourier transform

$$
\hat{\Phi}(v) = \int_{V} \Phi(u) \psi(2B(u, v)) \, du,
$$

(25)
where \( f_V \, du \) is the Haar measure on \( V \) which is self-dual with respect to the pairing \((v^*, v) \mapsto \psi(-2B(v, v^*))\).

**Theorem 5.1.** There exists a unitary projective representation \( \omega_1 \) of \( \text{SL}(2, F) \) on \( L^2(V) \) such that for \( g \in \text{SL}(2, F) \) and \( h \in H \), we have \( \omega_1(g) \, \pi(h) \, \omega_1(g)^{-1} = \pi(gh) \). There exists a (true) representation \( \omega_2 \) of \( \text{O}(V) \) on \( L^2(V) \) such that \( \omega_2(k) \, \pi(h) \, \omega_2(k)^{-1} = \pi(kh) \). The projective representation \( \omega_1 \) and the representation \( \omega_2 \) commute with each other. The Schwartz space \( S(V) \) is invariant under both \( \omega_1 \) and \( \omega_2 \). We have

\[
\left( \omega_1 \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \right) \Phi(v) = \psi(x B(v, v)) \Phi(v),
\]

\[
\left( \omega_1 \begin{pmatrix} a & \cdot \\ \cdot & a^{-1} \end{pmatrix} \right) \Phi(v) = |a|^{d/2} \Phi(av),
\]

and

\[
\omega_1(w_1) \Phi = \Phi, \quad w_1 = \Phi \begin{pmatrix} -1 & 1 \\ \cdot & \cdot \end{pmatrix}
\]

and

\[
(\omega_2(k) \Phi)(v) = \Phi(k^{-1}v).
\]

Here, of course, \( \omega_1(g) \) is only determined up to a complex number of absolute value one. Later, if \( V \) is even-dimensional, we will prescribe \( \omega_1 \) more precisely so as to obtain a true representation. We recall that if \( F \) is Archimedean the Schwartz space \( S(V) \) consists of functions which (together with all derivatives) are of faster than polynomial decay, while if \( F \) is nonarchimedean, then \( S(V) = C_c^\infty(V) \).

**Proof.** There is an evident action of \( \text{SL}(2, F) \) on \( A(V) \) such that \( \tau(\pi(h)) = \tau(\pi gh) \) for \( g \in \text{SL}(2, F), \, h \in H \). Thus \( g \) gives rise to an element of \( B_0(V) \), and it follows from Theorem 3.1 that there exists a unitary transformation \( \omega(g) \), determined up to a complex number of absolute value one such that

\[
\omega(g) \, \pi(h) \, \omega(g)^{-1} = \pi(gh).
\]

It follows from the uniqueness of \( \omega(g) \) that \( \omega(g_1g_2) \) differs from \( \omega(g_1) \, \omega(g_2) \) by a complex constant, which must have absolute value one since the operators \( \omega(g) \) are unitary. Hence we have a projective representation. The formulas (26–28) are verified by checking (30) directly when \( g \) is one of the three types of matrices in question, and it is sufficient to do this when \( h \) is of the form \((v^*, 0, 0)\) or \((0, v, 0)\) with \( v^* \) or \( v \) \( \in V \), since these generate \( H \); and we leave this elementary check to the reader. Note that the factor \(|a|^{d/2}\) in (27) is needed for unitarity.

The group \( \text{O}(V) \) also embeds in \( B_0(V) \), and the existence of a projective representation follows similarly. However it is unnecessary to make use of Theorem 3.1 here, since it is evident that \( \omega_2 \) as defined by (29) is a unitary representation of \( \text{O}(V) \) commuting with \( \omega_1 \).

It is clear that the Schwartz space is preserved by the operations (26–29), and so it is invariant under \( \omega_1 \) and \( \omega_2 \).
Our aim is to show that if the dimension $d$ of $V$ is even, then the cohomology class in $H^2(\text{SL}(2,F), \mathbb{T})$ attached to the representation $\omega_1$ is trivial. (If $d$ is odd, it may be shown that unless $F = \mathbb{C}$ the cocycle is nontrivial, and the cocycle defines an important central extension of $\text{SL}(2,F)$, the metaplectic group.) We will make use of quaternion algebras and Hilbert symbols to prove this. A quaternion algebra over a field $F$ is a two-dimensional central simple algebra. We will need only basic properties of quaternion algebras which we need can be found in Weil (1974). Let us only point out that this topic is closely connected with a central issue in local class field theory, namely the computation of the Brauer group of a local field. References for this theory are Weil (1974), Chapters 9 and 10, Serre (1979) Chapters 10, 12 and 13 and O’Meara (1971), Sections 52 and 57.

A quaternion algebra over $F$ is either isomorphic to $\text{Mat}_2(F)$ or else it is a division algebra. It is known that if $F$ is a local field (except $\mathbb{C}$) there exists exactly one isomorphism class of quaternion division algebras over $F$. We do not need this fact, however. Let $a, b \in F$. We define a quaternion algebra $\text{Quat}(a,b)$ as follows: as a vector space, $\text{Quat}(a,b)$ has a basis \{1, $i, j, k$\}, where $i, j$ and $k$ are subject to the following laws: $i^2 = a$, $j^2 = b$, $k^2 = -ab$, $ij = -ji = k$, $jk = -kj = -bi$ and $ki = -ik = -aj$. This algebra has an antiautomorphism $\xi \mapsto \overline{\xi}$ defined by $\overline{7(x + yi + zj + wk)} = x - yi - zj - wk$. The reduced norm and reduced trace are given by $N(\xi) = \xi \overline{\xi}$ and $\text{tr}(\xi) = \xi + \overline{\xi}$. We define the Hilbert symbol $(a, b) = 1$ if $\text{Quat}(a,b) \cong \text{Mat}_2(F)$, and $(a, b) = -1$ if $\text{Quat}(a,b)$ is a division ring.

We recall from local class-field theory that if $F$ is a local field, and $E$ is a quadratic extension, then the multiplicative group of the norms from $E^\times$ is a subgroup of index two in $F^\times$. The unique nontrivial character of $F^\times$ which is trivial on this subgroup is called the quadratic character of $F^\times$ attached to $E$.

**Proposition 5.2.** Let $a, b \in F^\times$. A necessary and sufficient condition for $(a, b) = 1$ is that $x^2 - ay^2 - bz^2 + abw^2 = 0$ have a solution with $x, y, z$ and $w$ not all zero, or equivalently, that $a$ is a norm from $F(\sqrt{b})$. The Hilbert symbol has the following further properties:

\[
(aa', b) = (a,b)(a', b); \quad (a, bb') = (a,b)(a, b');
\]

\[
(a, b) = (b,a);
\]

\[
(a, 1-a) = (a, -a) = 1,
\]

when $a \neq 0$ and (for $(a, 1-a)$) $a \neq 1$. The Hilbert symbol $(a, b)$ only depends on the classes of $a$ and $b$ modulo squares.

**Proof.** A necessary and sufficient condition that $(a, b) = 1$ is that $N(\xi) = 0$ for some nonzero $\xi \in \text{Quat}(a,b)$. Indeed $N(\xi) = 0$ if and only if $\xi$ is a zero divisor, since if $N(\xi) \neq 0$, then $N(\xi)^{-1}\overline{\xi}$ is an inverse to $\xi$, which is thus a unit; so $N(\xi) = 0$ is solvable with nonzero $\xi$ if and only of $\text{Quat}(a,b)$ fails to be a division ring.

Thus, writing $\xi = x + yi + zj + wk$, a necessary and sufficient condition for $(a, b) = 1$ is that $\nu(\xi) = x^2 - ay^2 - bz^2 + abw^2 = 0$ have a solution with $x, y, z$ and $w$ not all zero. We may write this identity $a = N((x + z\sqrt{b})/(y + w\sqrt{b})$, so this condition is also equivalent to $a$ being a norm from $F(\sqrt{b})$. From the second necessary and sufficient condition, it is clear
that the function \( a \mapsto (a, b) \) is trivial if \( b \) is a square, or is the quadratic character of \( F^\times \) attached to the quadratic extension \( F(\sqrt{b})/F \) if \( b \) is not a square. In either case, \( a \mapsto (a, b) \) is a character, whence (31).

It is clear from the first necessary and sufficient condition that \( (a, b) \) is symmetrical in \( a \) and \( b \), whence (32), and if \( b = -a \) or \( b = 1 - a \) we have specific solutions \( (x, y, z, w) = (0, 1, 1, 0) \) or \( (1, 1, 1, 0) \) to \( x^2 - ay^2 - bz^2 + abw^2 = 0 \), whence (33). Finally, we have \( (ar^2, b) = (a, b)^2 = (a, b) \) since \( (r, b) = \pm 1 \), so the Hilbert symbol only depends on square classes of \( a \) and \( b \).

Now we turn to Weil’s analytic interpretation of the Hilbert symbol. This is based on his observation that the function \( F_B(v) = \psi(B(v, v)) \) on \( V \) has a formal Fourier transform.

Since the function \( F_B \) has absolute value one for all \( v \), it is of course not \( L^2 \), but it has a property which compensates for this defect, namely, it is rapidly oscillating. That is, if \( |v| \) is large, \( F_B \) changes sign very rapidly, and so in some respects it behaves as if it were of rapid decay. We will show below that \( F_B \) has a Fourier transform in a natural sense.

Let \( S(V) \) denote the Schwartz space of \( V \). If \( \Phi \) is an element of the Schwartz space \( S(V) \), we will sometimes denote the Fourier transform \( \hat{\Phi} \), defined by (25), by \( \mathcal{F} \Phi \). If \( \Phi_1 * \Phi_2 \) denotes the convolution of two elements \( \Phi_1 \) and \( \Phi_2 \) of \( S(V) \), defined by

\[
(\Phi_1 * \Phi_2)(v) = \int_V \Phi_1(u) \Phi_2(v - u) \, du
\]

then it is well known and easy to check that

\[
\mathcal{F}(\Phi_1 * \Phi_2) = \mathcal{F} \Phi_1 \cdot \mathcal{F} \Phi_2.
\]

**Proposition 5.3** (Weil (1964)). If \( \Phi \in S(V) \), then the convolution \( \Phi * F_B \), defined by (34), lies in \( S(V) \). There exists a complex number \( \gamma(B) \) of absolute value one such that for all \( \Phi \in S(V) \),

\[
\mathcal{F}(\Phi * F_B) = \gamma(B) \mathcal{F} \Phi \cdot F_{-B}.
\]

More generally, if \( a \in F^\times \) and \( d = \dim(V) \),

\[
\mathcal{F}(\Phi * F_{aB}) = |a|^{-d/2} \gamma(aB) \mathcal{F} \Phi \cdot F_{-a^{-1}B}.
\]

In view of (35), this means that we should regard \( |a|^{-1/2} \gamma(aB) F_{-a^{-1}B} \) as a Fourier transform of \( F_B \), at least formally.

**Proof.** It is easy to check that the convolution

\[
(\Phi * F_B)(v) = \int_V \Phi(u) \psi(B(v - u, v - u)) \, du
\]
equals \( F_B(v) \mathcal{F}(\Phi * F_B)(-v) \), and \( \Phi \cdot F_B \) is an element of \( S(V) \); since the Schwartz space is stable under the Fourier transform, this proves the first assertion. We will deduce the second assertion from Theorem 5.1.
We make use of the matrix identity
\[ w_1 \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} w_1 \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} -1 \\ -1 \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} w_1 \]
with \( w_1 \) as in (28). Because \( \omega_1 \) is a unitary projective representation,
\[ \omega_1(w_1) \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \omega_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \gamma(B) \omega_1 \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \omega_1(w_1) \]
for some constant \( \gamma(B) \) of absolute value 1. Now
\[ \omega_1(w_1) \begin{pmatrix} 1 \\ 1 \end{pmatrix} \omega_1 \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \omega_1 \begin{pmatrix} -1 \\ -1 \end{pmatrix} = \hat{\Phi}_1, \]
where \( \Phi_1 = \omega_1 \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \omega_1 \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \omega_1 \begin{pmatrix} -1 \\ -1 \end{pmatrix} \Phi. \)
It is easy to compute that
\[ \Phi_1(v) = \int_V \Phi(u) \psi(\beta(v - u, v - u)) \, du, \]
so \( \Phi_1 = \Phi * F_B. \) On the other hand, we have
\[ \left( \omega_1 \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \omega_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} \Phi \right)(v) = \psi(-B(v,v)) \hat{\Phi}(v), \]
and comparing these two expressions we obtain (36).

We recall that in (36) the Fourier transform is with respect to the Haar measure \( dx \) on \( V \) which is self-dual with respect to the pairing \( \psi(2B(u,v)) \) on \( V \). The measure which is self-dual with respect to the pairing \( \psi(2aB(u,v)) \) is \( |a|^{d/2} \, dx \), and the Fourier transform of \( \Phi \) with respect to this measure and this pairing is \( \mathcal{F} \Phi(v) = |a|^{d/2} \mathcal{F} \Phi(au) \). Also, multiplying the measure by \( |a|^{1/2} \) multiplies the convolution on the left side of (36) by \( |a|^{d/2} \). We have therefore \( |a|^{d/2} \mathcal{F} \Phi(aB,F_aB) = \gamma(aB) \mathcal{F} \Phi \cdot F_{-aB} \). Evaluating this identity at \( a^{-1}v \) gives (37) evaluated at \( v \).

If \( \beta(v) = B(v,v) \) is the quadratic form associated with a symmetric bilinear form \( B \), then we note that (since we are assuming the characteristic of \( F \) is not two) \( B \) can be reconstructed from \( \beta \) by
\[ B(u,v) = \frac{1}{2} (\beta(u + v) - \beta(u) - \beta(v)). \]

A quadratic space is a pair \((V, \beta)\) consisting of vector space \( V \) and a quadratic form \( \beta \) on \( V \) such that the associated symmetric bilinear form \( B \) is nondegenerate. If \( X = (V, \beta) \) is a quadratic space, we will sometimes write \( \gamma(\beta) \) or \( \gamma(X) \) for \( \gamma(B) \).
If \((V_1, \beta_1)\) and \((V_2, \beta_2)\) are quadratic spaces, then we can form the direct sum \((V_1 \oplus V_2, \beta_1 \oplus \beta_2)\), where the quadratic form \((\beta_1 \oplus \beta_2)(v_1, v_2) = \beta_1(v_1) + \beta_2(v_2)\). If \(a_1, \ldots, a_n \in F^\times\), we can form the quadratic space \(\text{QF}(a_1, \ldots, a_n) = (F^n, \beta), \) where \(\beta(x_1, \ldots, x_n) = \sum_i a_i x_i^2\).

Since the characteristic of \(F\) is not two, every quadratic form can be diagonalized, so every quadratic space has the form \(\text{QF}(a_1, \ldots, a_n)\) for some \(a_i\). We call a quadratic space split if it is isomorphic a direct sum of copies of \(\text{QF}(1, -1)\). Thus the dimension of a split quadratic space is even by definition. The isomorphism classes of quadratic spaces form a commutative monoid with respect to direct sum, and the quotient by the submonoid of split quadratic spaces is a group, known as the Witt group. See Lang (2002) Theorem XV.11.1 on p. 594.

**Proposition 5.4.** (i) Let \((V_1, \beta_1)\) and \((V_2, \beta_2)\) be quadratic spaces. Then \(\gamma(\beta_1 \oplus \beta_2) = \gamma(\beta_1) \gamma(\beta_2)\). (ii) Let \((V, \beta)\) be a quadratic space. Then \(\gamma(-\beta) = \gamma(\beta)^{-1}\), where \(-\beta\) is the quadratic form on \(V\) defined by \((-\beta)(v) = -\beta(v)\). (iii) If \((V, \beta)\) is a split quadratic space, then \(\gamma(\beta) = 1\). (iv) The function \(\gamma\) is a character of the Witt group.

**Proof.** (i) is easily verified by taking a test function in \(\Phi \in S(V_1 \oplus V_2)\) of the form \(\Phi(v_1, v_2) = \Phi_1(v_1) \Phi_2(v_2)\) with \(\Phi_i \in S(V_i)\). For (ii), one simply takes complex conjugation in the definition (36) of \(\gamma\), bearing in mind that the Fourier transform \(\mathcal{F}\) is with respect to the nondegenerate pairing \(B\) on \(V\) associated with \(\beta\). For (iii), it is necessary to show that \(\gamma(\beta) = 1\) when \((V, \beta) = \text{QF}(1, -1) = \text{QF}(1) \oplus \text{QF}(-1)\), and this follows immediately from (i) and (ii). Part (iv) follows from (i) and (iii).

If \(F\) is nonarchimedean, and \(V\) is a finite-dimensional vector space over \(F\), we recall that a lattice in \(V\) is a compact open subgroup.

We make use of the matrix identity

\[
\begin{pmatrix}
1 & 1 \\
1 & 1 \\
\end{pmatrix}
\begin{pmatrix}
1 & -1 \\
-1 & 1 \\
\end{pmatrix} =
\begin{pmatrix}
1 & -1 \\
1 & 1 \\
\end{pmatrix}
\begin{pmatrix}
1 & -1 \\
1 & 1 \\
\end{pmatrix}
\begin{pmatrix}
1 & -1 \\
-1 & 1 \\
\end{pmatrix} =
\gamma(B)
\begin{pmatrix}
1 & -1 \\
1 & 1 \\
\end{pmatrix}
\begin{pmatrix}
1 & -1 \\
1 & 1 \\
\end{pmatrix}.
\]

**Proposition 5.5.** Let \((V, \beta)\) be a quadratic space over the local field \(F\), and let \(B\) be the symmetric bilinear form associated with \(\beta\). Let \(\int_V dv\) denote the Haar measure on \(V\) which is self-dual with respect to the pairing \((u, v) \mapsto \psi(2B(u, v))\). (i) If \(\Phi \in S(V)\), then

\[
\int_V (\Phi \ast F_B)(v) dv = \gamma(B) \int_V \Phi(v) dv. \tag{39}
\]

(ii) If \(F\) is nonarchimedean, then
\[ \gamma(B) = \int_L F_B(v) \, dv \quad (40) \]

for any sufficiently large lattice \( L \).

**Proof.** (39) results from evaluating both sides of (36) at zero. As for (ii), let \( L' \) be the dual lattice to \( L \), that is, \( L' = \{ u \in V | \psi(2B(u,v)) = 1 \text{ for all } v \in L \} \). If \( L \) is sufficiently large, then \( L' \) is small enough that \( F_B(u) = 1 \) for all \( u \in L' \). Now let \( \Phi \) be the characteristic function of \( L' \). Then

\[ (\Phi \ast F_B)(v) = \int_{L'} F_B(v-u) \, du = \psi(B(v,v)) \int_{L'} \psi(2B(u,v)) \, du, \]

so \( \Phi \ast F_B \) is \( \text{vol}(L') F_B \) times the characteristic function of \( L \). Thus the left side of (40) is \( \text{vol}(L') \) times \( \int_L F_B(v) \, dv \); and the right side is \( \gamma(B) \) times \( \text{vol}(L') \). Comparing, we obtain (40). \( \square \)

If \( A \) is a quaternion algebra over \( F \), and if \( \nu : A \to F \) is the reduced norm, then \( (A,\nu) \) is a quadratic space. Specifically, if \( A = \text{Quat}(a,b) \), then \( (A,V) \cong \text{QF}(1,-a,-b,ab) \), since \( \nu(x+yi+zj+wk) = x^2 - ay^2 - bz^2 + abw^2 \).

**Theorem 5.6** (Weil). Let \( A \) be a quaternion algebra over \( F \), and let \( \nu : A \to F \) be the reduced norm. Then \( \gamma(\nu) = 1 \) if \( A \) is a matrix algebra, and \( \gamma(\nu) = -1 \) if \( A \) is a division ring.

Applying this to the quaternion algebra \( \text{Quat}(a,b) \cong \text{QF}(1,-a,-b,ab) \), we obtain

\[ (a,b) = \gamma(\text{QF}(1,-a,-b,ab)). \quad (41) \]

**Proof.** If \( A \) is a matrix ring, the reduced norm coincides with the determinant. Since \( \det(\begin{pmatrix} x & y \\ z & w \end{pmatrix}) = xw - yz \), and since the quadratic forms \( (x,w) \mapsto xw \) and \( (y,z) \mapsto -yz \) are easily seen to be split (they are diagonalized by a simple change of variables), \( \gamma(\nu) = 1 \) by Proposition 5.4 (iii).

For the remainder of the proof, we assume that \( A \) is a division ring. If \( F \) is Archimedean, the existence of a quaternion division algebra implies that \( F \) is real, and \( A \) is the ring of Hamiltonian quaternions. Its norm form is \( \text{QF}(1,1,1,1) \), and so \( \gamma(\nu) = \gamma(\text{QF}(1))^4 \). Since our main interest is in the nonarchimedean case, we will leave it to the reader to show that \( \gamma(\text{QF}(1)) \) is a primitive 8-th root of unity. We only remark that to evaluate \( \gamma(B) \), it is convenient to take \( \Phi(x) = e^{-\pi x^2} \) in (36).

We assume for the remainder of the proof that \( F \) is nonarchimedean and that \( A \) is a quaternion division algebra. Since \( |\gamma(\nu)| = 1 \), it is sufficient to prove that the right side of (40) is real and negative, for some conveniently chosen additive Haar measure. We will denote by \( dz \) an additive Haar measure on \( A \). Then \( d^x z = \nu(z)^{-2} \, dz \) is a multiplicative Haar measure. We wish to evaluate the sign of

\[ \int_L \psi(\nu(z)) |\nu(z)|^2 \, d^x z \]

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Lemma 5.8. and Langlands (1970). The quadratic spaces $QF(1)$ are aβ equal simply (41), and $\psi$ is a lattice, where $o$ is the ring of integers in $F$, and $N$ is suitably large integer. Then we note that the integral factors through the homomorphism $\nu: A^\times \to F^\times$, and we are left to evaluate of the sign of

$$\int_{\mathfrak{o}^{-N}} \psi(x) |x|^2 d^x x = \int_{\mathfrak{o}^{-N}} \psi(x) |x| dx,$$

(42)

where for our Haar measures $d^x x$ and $dx$ on $F^\times$ and $F$, we take $d^x x = |x|^{-1} dx$, with $dx$ the measure with respect to which $o$ has volume one. Suppose that $\mathfrak{o}^r o$ is the conductor of $\psi$, that is, the largest fractional ideal on which $\psi$ is trivial. We note that

$$\int_{|x|=q^{-s}} \psi(x) dx = \begin{cases} q^{-s}(1 - q^{-1}) & \text{if } s \geq r; \\ -q^{-r} & \text{if } s = r - 1; \\ 0 & \text{if } s < r - 1. \end{cases}$$

Thus, if $N$ is large, the integral (42) equals

$$-q^{1-2r} + \sum_{s=r}^{\infty} q^{-2s}(1 - q^{-1}) = -q^{1-2r}(1 + q^{-1})^{-1}. $$

This is negative, as required. \hfill \Box

**Proposition 5.7.** Suppose that $(V, \beta)$ is a quadratic space of even dimension $d$. Write $(V, \beta) = QF(r_1, r_2, \cdots, r_d)$ with $r_i \in F^\times$, and let $\Delta = (-1)^{d/2} r_1 r_2 \cdots r_d$. Let $a \in F^\times$. Then $\gamma(a\beta) = (\Delta, a) \gamma(\beta)$.

**Proof.** Since $QF(r_1, r_2, \cdots, r_d)$ decomposes into the direct sum of $d/2$ two-dimensional quadratic spaces, this reduces to the case $d$ by Propositions 5.2 and 5.3. We begin by noting that by (41),

$$\gamma(QF(r_1, r_2)) = (r_1, r_2) \gamma(QF(1, r_1 r_2)).$$

(43)

This implies that with $(V, \beta) = QF(r_1, r_2)$, we have

$$\gamma(a\beta) \gamma(\beta)^{-1} = (ar_1, ar_2) \gamma(QF(1, a^2 r_1 r_2)) (r_1, r_2)^{-1} \gamma(QF(1, r_1 r_2))^{-1}. $$

The quadratic spaces $QF(1, a^2 r_1 r_2)$ and $QF(1, r_1 r_2)$ are equivalent, so this last expression equals simply $(ar_1, ar_2) (r_1, r_2)^{-1}$ which, using Proposition 5.2, simplifies to $(a, -r_1 r_2)$. \hfill \Box

We digress to introduce generators and relations for $SL(2)$. Such relations are due (in much greater generality) to Chevalley; their use in the Weil representation dates to Jacquet and Langlands (1970).

**Lemma 5.8.** Let $F$ be a field. Let $S$ be the group generated by elements $t(y)$, $n(z)$ and $w_1$ with $y \in F^\times$, $z \in F$, subject to the following relations:

$$t(y_1) t(y_2) = t(y_1 y_2); \quad n(z_1) n(z_2) = n(z_1 + z_2); $$

(44)

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\[ t(y) n(z) t(y)^{-1} = n(y^2 z); \quad w_1 t(y) w_1 = t(-y^{-1}); \quad (45) \]

and
\[ w_1 n(z) w_1 = t(-z^{-1}) n(-z) w_1 n(-z^{-1}), \quad (z \neq 0). \quad (46) \]

Then \( S \) is isomorphic to \( \text{SL}(2, F) \); in this isomorphism
\[ t(y) \mapsto \begin{pmatrix} y & 1 \\ y & 1 \end{pmatrix}, \quad n(z) \mapsto \begin{pmatrix} 1 & z \\ 1 & 1 \end{pmatrix}, \quad w_1 \mapsto \begin{pmatrix} x & 1 \\ 1 & -1 \end{pmatrix}. \]

Proof. Let
\[ t'(y) = \begin{pmatrix} y & 1 \\ y & 1 \end{pmatrix}, \quad n'(z) = \begin{pmatrix} 1 & z \\ 1 & 1 \end{pmatrix}, \quad w'_1 = \begin{pmatrix} x & 1 \\ 1 & -1 \end{pmatrix} \in \text{SL}(2, F). \]

These elements satisfy the relations \((1.12-14)\), so by the universal property of \( S \), there exists a homomorphism \( \phi : S \to \text{SL}(2, F) \) such that \( \phi(t(y)) = t'(y) \), etc., and what we must show is that \( \phi \) is an isomorphism. We will accomplish this by constructing an inverse map \( \psi : \text{SL}(2, F) \to S \). We define
\[ \psi \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{cases} n(a/c) t(-c^{-1}) w_1 n(d/c) & \text{if } c \neq 0; \\ t(a) n(b/a) & \text{if } c = 0. \end{cases} \]

We check that \( \psi \) is a homomorphism, that is, if
\[ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}, \]

we must check that
\[ \psi \begin{pmatrix} a & b \\ c & d \end{pmatrix} \psi \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \psi \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}. \quad (47) \]

There are several cases; we will check this if \( c, C \) and \( \gamma \) are all nonzero, and leave the remaining cases to the reader. The left side equals
\[ n(a/c) t(-c^{-1}) w_1 n(d/c) n(A/C) t(-C^{-1}) w_1 n(D/C) \]
\[ = n(a/c) t(-c^{-1}) w_1 n(d/c) n(A/C) w_1 t(-C) n(D/C) \]
\[ = n(a/c) t(-c^{-1}) w_1 n(\gamma c^{-1} C^{-1}) w_1 t(-C) n(D/C). \]

Using \((46)\), this equals
\[ n(a/c) t(-c^{-1}) t(-cC/\gamma) n(-\gamma/cC) w_1 n(-cC/\gamma) t(-C) n(D/C). \]

Making use of \((45)\) and the identities
\[ a - \frac{C}{\gamma} = \frac{a(cA + dC) - C(ad - bc)}{\gamma} = \frac{ca}{\gamma}, \]

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\[ D - \frac{c}{\gamma} = D(cA + dC) - c(AD - BC) = \frac{C\delta}{\gamma}, \]

this may be simplified to obtain (47).

Now let us note that \( \psi \circ \phi \) is the identity map on \( S \); indeed, since \( \phi \) and \( \psi \) are homomorphisms, it is sufficient to check this on generators, and this is easily done. It follows that \( \phi \) is injective. It is easy to see that \( \text{SL}(2, F) \) is generated by the image of \( \phi \), and so \( \phi \) is an isomorphism.

**Theorem 5.9.** Suppose that \((V, \beta)\) is a quadratic space of even dimension \(d\). Then the cohomology class in \(H^2(\text{SL}(2, F), \mathbb{T})\) determined by the projective representation \(\omega_1\) is trivial. More specifically, let \(\Delta\) be as in Proposition 5.7, and let \(\chi : F^\times \to \{\pm 1\}\) be the quadratic character \(\chi(a) = (\Delta, a)\). Then there exists a true representation \(\omega_0\) of \(\text{SL}(2, F)\) on \(L^2(V)\), having the Schwartz space \(S(V)\) as an invariant subspace, such that

\[
\omega_0 \left( \begin{array}{cc} 1 & x \\ 0 & 1 \end{array} \right) \Phi(v) = \psi(x B(v, v)) \Phi(v), \tag{48}
\]

\[
\omega_0 \left( \begin{array}{cc} a & -1 \\ 1 & a \end{array} \right) \Phi(v) = |a|^{d/2} \chi(a) \Phi(av), \tag{49}
\]

and

\[
\omega_0(w_1) = \gamma(B) \hat{\Phi}. \tag{50}
\]

**Proof.** It is clear that the cohomology class associated with a true representation is trivial. Hence it is sufficient to show that there (48-50) describe a true representation of \(\text{SL}(2, F)\).

We will deduce this by showing that the operators \(\omega_0(g)\) defined by (48-50) satisfy the relations (44), (45) and (46). The first two relations are simple to check, except that the second identity in (45) requires us to know that

\[
\gamma(B)^2 = \chi(-1). \tag{51}
\]

This reduces easily to the case where \(d = 2\), so what we must prove is that \((-1, -r_1 r_2) = \gamma(QF(r_1, r_2))^2.\) By (41), we have

\[
(-1, -r_1 r_2) = \gamma(QF(1, 1, r_1 r_2, r_1 r_2)) = \gamma(QF(1, r_1 r_2))^2,
\]

and squaring (43), this equals \(\gamma(QF(r_1, r_2))^2\), as required.

We are left with the problem of verifying (46); given (44) and (45), which are already verified, this is equivalent to showing for \(\Phi \in L^2(V)\),

\[
\omega_0(w_1) \omega_0 \left( \begin{array}{cc} a^{-1} & -a \\ 1 & 1 \end{array} \right) \omega_0 \left( 1, -a \right) \omega_0(w_1) \omega_0 \left( 1, -a^{-1} \right) \Phi = \omega_0 \left( 1, a \right) \omega_0(w_1) \Phi. \tag{52}
\]

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The left side equals $\gamma(B) F(\Phi_1)$, where

$$\Phi_1 = \omega_0 \begin{pmatrix} a^{-1} & 0 \\ a & 1 \end{pmatrix} \omega_0 \begin{pmatrix} 1 & -a \\ a & 1 \end{pmatrix} \omega_0 \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \Phi.$$ 

Unraveling the definitions, we find that $\Phi_1 = \gamma(B) |a|^{-d/2} \chi(a) F_a B$. Thus by (51) the left side of (52) equals $|a|^{-d/2} \chi(a) F(\Phi \ast F_a B)$ and by (37) and Proposition 5.7, this equals $\gamma(B) F \Phi \ast F_a B$. This equals the right side of (52), so (52) is proved.

\[
\square
\]

\textbf{References}


