Lecture 20: Wess-Zumino-Witten CFT and Fusion Algebras

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Overview

We have so far emphasized the BPZ minimal models, in which c takes particular values < 1. These are special in that the Hilbert space $\mathcal{H}$ decomposes into a finite number of $\mathbf{Vir} \oplus \mathbf{Vir}$ modules.

For $c > 1$, in order to obtain such a finite decomposition it is necessary to enlarge $\mathbf{Vir}$. If this can be done, we speak of a rational CFT: see Lecture 19 for a more careful definition of this concept, due to Moore and Seiberg.

One natural such extension enlarges $\mathbf{Vir}$ to include an affine Lie algebra. This leads to the Wess-Zumino-Witten (WZW) models. A further generalization of this theory, the coset models of Goddard-Kent-Olive ([GKO]) contains both the minimal models and the WZW models.

Lecture 6 review: A note on notation

References: [DMS] Chapter 14
Kac, Infinite-Dimensional Lie algebras, Chapters 6,7,12,13
Kac and Peterson, Adv. in Math. 53 (1984).

We begin by reviewing the construction of affine Lie algebras from Lecture 6, changing the notation slightly.

Let $\mathfrak{g}$ be a finite-dimensional semisimple complex Lie algebra with Cartan subalgebra $\mathfrak{h}$. We will define the untwisted affine Lie algebra $\widehat{\mathfrak{g}}$ by first making a central extension, then adjoining a derivation d. Finally we will make a semidirect product of $\widehat{\mathfrak{g}}$ with $\mathbf{Vir}$ in which d is identified with $-L_0$.

Review: central extensions

Let $\mathfrak{g}$ be a Lie algebra and $\mathfrak{a}$ an abelian Lie algebra. A bilinear map $\sigma : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{a}$ is called a 2-cocycle if it is skew-symmetric and satisfies

$$\sigma([X,Y],Z) + \sigma([X,Y],Z) + \sigma([X,Y],Z) = 0, \qquad X,Y,Z \in \mathfrak{a}.$$

In this case we may define a Lie algebra structure on $\mathfrak{g} \oplus \mathfrak{a}$ by

$$[(X,a),(Y,b)] = ([X,Y], \sigma(X,Y)).$$

Denoting this Lie algebra $\mathfrak{g}'$ we have a central extension

$$0 \longrightarrow \mathfrak{a} \longrightarrow \mathfrak{g}' \longrightarrow \mathfrak{g} \longrightarrow 0.$$

Review: Adding a derivation

Suppose we have a derivation d of a Lie algebra $\mathfrak{g}$. This means

$$d([x,y]) = [dx,y] + [x,dy].$$

We may then construct a Lie algebra $\mathfrak{g}' = \mathfrak{g} \oplus \mathbb{C}d$ in which $[d,x] = d(x)$ for $x \in \mathfrak{g}$.

This is a special case of a more general construction, the semidirect product.

Review: Affine Lie algebras as Central Extensions

Although the affine Lie algebra $\widehat{\mathfrak{g}}$ may be constructed from its Cartan matrix, another construction described in Chapter 7 of Kac begins with the finite-dimensional simple Lie algebra $\mathfrak{g}$ of rank ℓ. Tensoring with the Laurent polynomial ring gives the loop Lie algebra $\mathfrak{g} \otimes \mathbb{C}[t, t^{-1}]$. Then one may make a central extension:

$$0 \to \mathbb{C} \cdot K \to \widehat{\mathfrak{g}}' \to \mathbb{C}[t, t^{-1}] \otimes \mathfrak{g} \to 0.$$

After that, one usually adjoins another basis element, which acts on $\widehat{\mathfrak{g}}'$ as a derivation d. This gives the full affine Lie algebra $\widehat{\mathfrak{g}}$.

The dual Coxeter number

The dual Coxeter number, denoted $h^\vee$ will appear frequently in formulas. We omit the definition, for which see Kac, Infinite-dimensional Lie algebras Section 6.1. But here are the values as a function of the Cartan type of $\mathfrak{g}$.

Cartan Type	$h^\vee$
A_ℓ	$l+1$
B_ℓ	$2l-1$
C_ℓ	$l+1$
D_ℓ	$2l-2$
E_6	12
E_7	18
E_8	30
F_4	9
G_2	4

The root system and Weyl group

The root system is the set of nonzero elements of $\widehat{\mathfrak{h}}^*$ that occur in the adjoint representation of $\widehat{\mathfrak{h}}$ on $\mathfrak{g}$. Particular roots are as follows.

There exist $\alpha_i^\vee \in \widehat{\mathfrak{h}}$ and $\alpha_i \in \widehat{\mathfrak{h}}^*$ called simple coroots and simple roots respectively. For $i = 0, \cdots, r$ (with r the rank of $\mathfrak{g}$ there is simple reflection acting on $\widehat{\mathfrak{h}}^*$ defined by

$$s_i(\lambda) = \lambda - \lambda(\alpha_i^\vee)\alpha_i$$

and the group generated by the s_i is the affine Weyl group W.

Real and imaginary roots

As usual, roots may be partitioned into positive and negative.

A root is called real if it is in the W-orbit of a simple root. A root is imaginary if it is not real. There is a minimal positive imaginary root δ. The roots of $\mathfrak{g}$ are all roots of $\widehat{\mathfrak{g}}$. Let Δ be the finite root system of $\mathfrak{g}$. The real roots of $\mathfrak{g}$ are

$$\{\alpha + n\delta | \alpha \in \Delta, n \in \mathbb{Z}\}.$$

The imaginary roots are $n\delta$ with $0 \neq n \in \mathbb{Z}$.

Other data

The Cartan subalgebra $\widehat{\mathfrak{h}}$ of $\widehat{\mathfrak{g}}$ is $\mathfrak{h}$ with the central element K and the derivation d adjoined.

Other data that are described in Kac, Infinite-Dimensional Lie Algebras, Chapters 6 and 12. There are coroots $\alpha_i^\vee \in \widehat{\mathfrak{h}}$ for $i = 0, 1, \cdots, r$, where r is the rank of $\mathfrak{g}$. There are positive integers $a_i^\vee$ dual to the a_i that we will call comarks such that

$$K = \sum_{i=0}^{r} a_i^\vee \alpha_i^\vee, \qquad h^\vee = \sum_{i=0}^{r} a_i$$

The numbers $a_i^\vee$ (sometimes called comarks) are all 1 if $\mathfrak{g} = \mathfrak{sl}_{r+1}$, and for all $\mathfrak{g}$ at least $a_0^\vee = 1$.

Enlargement by $\mathbf{Vir}$

Instead of (or more precisely, in addition to) enlarging $\mathfrak{g}'$ by the derivation d we may form the semidirect product of $\mathfrak{g}'$ by the entire Virasoro algebra

$$[L_i, L_j] = (i-j)L_{i+j} + \frac{1}{12}(i^3 - i)\delta_{i,-j} \cdot C$$

where L_i acs as the derivation $-t^{i+1}d/dt$. This semidirect product contains the full affine Lie algebra $\mathfrak{g} = \mathfrak{g}' \oplus \mathbb{C}d$ with $d = -L_0$.

It is also possible to embed $\mathbf{Vir}$ in the universal enveloping algebra of $\widehat{\mathfrak{g}}$. This is the Sugawara construction which we will make precise later.

Integrable highest weight representations

Let $\Lambda_i \in \widehat{\mathfrak{h}}^*$ be the fundamental weights characterized by $\Lambda_i(\alpha_j^\vee) = \delta_{ij}$ and $\Lambda_i(d) = 0$. The lattice P spanned by the Λ_i is called the weight lattice. If $\lambda = \sum_i c_i \Lambda_i$ with c_i nonnegative integers, then λ is called dominant.

Given $\lambda \in \widehat{\mathfrak{h}}$ it is possible to construct a Verma module $M(\lambda)$ that is highest weight representation of $\widehat{\mathfrak{g}}$. It has a unique irreducible quotient $L(\lambda)$.

Let P^+ be the cone of dominant weights. If $\lambda \in P^+$, then $L(\lambda)$ is integrable, namely exponentiates to a representation of the loop group. The character is therefore invariant under the affine Weyl group.

String functions

Kac and Peterson proved that the characters of integrable highest weight representations are modular forms. Let λ be a dominant weight and let μ be a weight. Let $m_\lambda(\mu)$ be the multiplicity of μ in $L(\lambda)$.

The function $m(\mu - n\delta)$ is monotone, and zero outside the cone

$$(\lambda+\rho|\lambda+\rho)-(\mu+\rho,\mu+\rho)\geqslant 0.$$

Therefore we may find a smallest n such that $m(\mu - n\delta)$ is nonzero. We call $\mu - n\delta$ a maximal weight. Define the string function

$$b_\mu^\lambda = \sum_k m(\mu - k\delta)e^{-k\delta}.$$

replacing μ by $\mu - n\delta$ just multiplies the series by $e^{k\delta}$ so there is no loss of generality in assuming that μ is maximal.

String functions (continued)

The character

$$\chi_\lambda = \sum_{\mu \text{ maximal}} b^\lambda_\mu .$$

Moreover using the fact that $\delta = w(\delta)$ for all $w \in W$ we may have $w(b^\lambda_\mu) = b^\lambda_{w\mu}$ and therefore we may rewrite b^λ_μ by finding a Weyl group element such that $w\mu$ is dominant. There are only a finite number of dominant maximal weights. Then

$$\chi_\lambda = \sum_{\mu \text{ maximal, dominant}} \sum_{w \in W/W_\mu} e^{w\mu} b^\lambda_\mu .$$

(To avoid overcounting W_μ is the stabilizer of the dominant maximal weight μ.)

Modular characteristics and the character

Define the modular characteristic (modular anamoly)

$$s_\lambda = \frac{|\lambda+\rho|^2}{2(k+h^\vee)} - \frac{|\rho|^2}{2k},$$

We are using the notation $(\ |\ )$ for an invariant inner product on Λ. Here $h^\vee$ is the dual Coxeter number and $k = (\delta|\lambda)$. Also

$$s_\lambda(\mu) = s_\lambda - \frac{|\mu|^2}{2k}.$$

Now define

$$c^\lambda_\mu = e^{-s_\lambda(\mu)\delta} b^\lambda_\mu = e^{-s_\lambda(\mu)\delta} \sum_{n\geqslant 0} m(\lambda - n\delta) e^{-n\delta}.$$

These modified string functions are theta functions which are modular forms, as was proved by Kac and Peterson. Because of this, the character χ_λ, which is the sum of all string functions, is a modular form.

The level k alcove

The level of a weight λ is

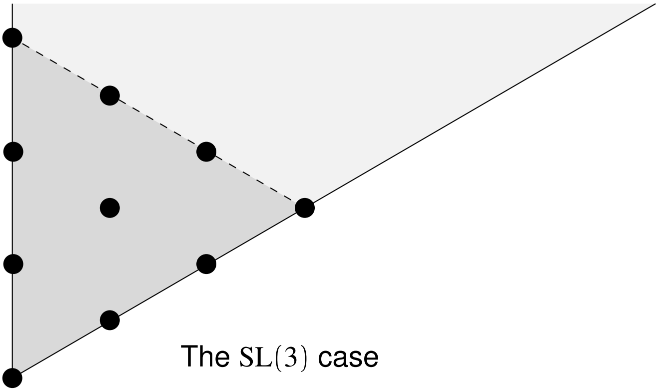
$$k = \langle \lambda, K \rangle = \sum a_i^\vee \lambda(\alpha_i^\vee).$$

There are a finite number of dominant weights of level k, and these may be visualized as follows.

We may identify the fundamental weights $\Lambda_1, \cdots, \Lambda_r$ with the weights of the finite–dimensional Lie algebra $\mathfrak{g}$. Then the weights of level k are identified with weights of $\mathfrak{g}$ by projecting on the span of $\Lambda_1, \cdots, \Lambda_r$. The positive Weyl chamber for $\mathfrak{g}$ is the set of weights λ such that $\lambda(\alpha_i^\vee) \geqslant 0$ for $1 \leqslant i \leqslant r$. The level k alcove then adds one inequality, $\langle \lambda, K \rangle \leqslant k$, where θ is the highest root of $\mathfrak{g}$.

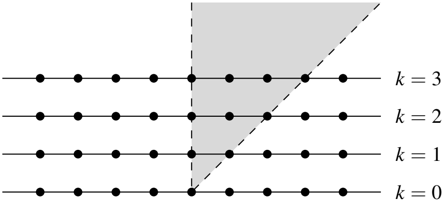
The level 3 alcove for $\widehat{\mathfrak{sl}}_3$

For example, if $\mathfrak{g} = \mathfrak{sl}_3$, here is the level 3 alcove inside the positive Weyl chamber, showing that $\hat{\mathfrak{g}}$ has 10 positive weights of level 3:



The $\mathrm{SL}(3)$ case

The affine Weyl group (continued)



The affine Weyl group fixes the origin. But parallel to the level 0 plane, it induces affine linear maps on each level (except level 0). The fundamental alcove on the k level set is the intersection with the shaded area, consisting of dominant weights.

The Hilbert space

To define a WZW CFT we pick a finite-dimensional Lie algebra $\mathfrak{g}$ with corresponding affine Lie algebra $\mathfrak{g}$ and a level k. In place of the Virasoro algebra we have the semidirect product $\mathcal{A} = \hat{\mathfrak{g}} \oplus \mathbf{Vir}$. This acts on $L(\lambda)$ for any dominant weight λ, due to the Sugawara construction. The Hilbert space of the model is

$$\bigoplus_{\lambda \text{ of level } k} L(\lambda) \otimes L(\lambda).$$

As usual, any field can be decomposed into a sum of fields $\phi(z) \otimes \overline{\phi}(\bar{z})$ where ϕ is holomorphic and $\overline{\phi}$ is antiholomorphic. The holomorphic fields are called chiral.

Affine primary fields and currents

Let Φ_λ be the highest weight element of $L(\lambda)$, identified with a chiral field. The field $\Phi_\lambda(z)\Phi_\lambda(\bar{z})$ in $L(\lambda) \otimes L(\lambda)$ is called an affine primary field. We will concentrate on the chiral field $\Phi_\lambda(z)$, which we will also call affine primary fields.

Currents

I addition to the primary fields and their descendents, there are other fields called currents. Let $J \in \mathfrak{g}$. The current $J(z)$ is the formal expression

$$J(z) = \sum_{n=-\infty}^{\infty} z^{-n-1} J_n, \qquad J_n = J \otimes t^n \in \hat{\mathfrak{g}}.$$

Currents had their origin in particle physics where they were introduced by Gell-Mann in 1964.

The Sugawara construction

Another field is the energy-momentum tensor, which is made explicit through the Sugawara construction. Indeed the Sugawara construction which embeds $\mathbf{Vir}$ in the universal enveloping algebra of $\hat{\mathfrak{g}}$ can be written succinctly as

$$T(z) = \sum_{n\in\mathbb{Z}} z^{-n-2} L_n = \frac{1}{2(k+h^\vee)} \sum_a : J^a J^a : (z)$$

where the sum is over an orthonormal basis of $\mathfrak{g}$ with respect to the inner product given by the Killing form divided by $2h^\vee$.

Summary

This gives a conformal field theory with central charge

$$c = \frac{k \dim(\mathfrak{g})}{k + h^\vee}.$$

Through the OPE expansion, the primary fields acquire a composition $\Phi_\lambda \star \Phi_\mu$ making a Fusion ring or Verlinde algebra. This is discussed further in two lectures from the previous course on quantum groups:

- Lecture 15 The Racah-Speiser algorithm. The Kac-Walton formula.
- Lecture 17 Affine Lie algebras. The fusion ring.

Modularity considerations are closely associated with the fusion rules, as we mention at the end of Lecture 17.