Lecture 2

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Recommended texts:

- Majid: A Primer of Quantum Groups
- Kassel: Quantum Groups

The more advanced parts of the class will also draw from Turaev: Quantum Invariants of Knots and 3-Manifolds.
Review of bialgebras and Hopf algebras

Recall that a bialgebra over a field $K$ is a Vector space $V$ with linear maps

$$\mu : H \otimes H \to H, \quad \Delta : H \to H \otimes H,$$

$$\eta : K \to H, \quad \varepsilon : H \to K,$$

subject to the axioms. Associativity and coassociativity:

$$H \otimes H \otimes H \xrightarrow{\mu \otimes 1} H \otimes H \xrightarrow{1_H \otimes \mu} H \otimes H \xrightarrow{\mu} H$$

$$H \xrightarrow{\Delta} H \otimes H \xrightarrow{\Delta \otimes 1_H} H \otimes H \otimes H \xrightarrow{1_H \otimes \Delta} H \otimes H \otimes H \otimes H$$
Bialgebra axioms (continued)

Unit:

\[
\begin{align*}
H \otimes H & \xrightarrow{1_H \otimes \eta} H \otimes K \\
& \xleftarrow{\mu} H
\end{align*}
\]

\[
\begin{align*}
H \otimes H & \xrightarrow{\eta \otimes K_H} K \otimes H \\
& \xleftarrow{\mu} H
\end{align*}
\]

Counit:

\[
\begin{align*}
H \otimes H & \xleftarrow{1_H \otimes \varepsilon} H \otimes K \\
& \xrightarrow{\Delta} H
\end{align*}
\]

\[
\begin{align*}
H \otimes H & \xleftarrow{\varepsilon \otimes I_H} K \otimes H \\
& \xrightarrow{\Delta} H
\end{align*}
\]
Bialgebra axioms (continued)

The augmentation and coaugmentation axioms:

\[ H \otimes H \xrightarrow{\varepsilon \times \varepsilon} K \times K \]
\[ H \xrightarrow{\varepsilon} K \]
\[ K \xrightarrow{\eta} H \]
\[ K \times K \xrightarrow{\eta \times \eta} H \times H \]

These say that the counit is an algebra homomorphism, and that the unit is a coalgebra homomorphism, respectively.
Bialgebra axioms (concluded)

Let $\tau(a \otimes b) = b \otimes a$. Hopf:

\[
\begin{align*}
H \otimes H & \xrightarrow{\Delta \otimes \Delta} H \otimes H \otimes H \otimes H \\
& \xrightarrow{1_H \otimes \tau \otimes 1_H} H \otimes H \otimes H \otimes H \\
& \xrightarrow{\mu \otimes \mu} H \otimes H
\end{align*}
\]

- The associativity and unit axioms: $H$ is an algebra.
- The coassociativity and counit: $H$ is an coalgebra.
- Augmentation axiom: counit is an algebra homomorphism
- Dually, the unit is a coalgebra homomorphism
- The Hopf axiom: $\Delta$ is an algebra homomorphism
- Equivalently $\mu$ is a coalgebra homomorphism
A **Hopf algebra** is a bialgebra with a linear map $S : H \to H$ called the **antipode** satisfying

$$H \xrightarrow{\Delta} H \otimes H \xrightarrow{1_H \otimes S} H \otimes H$$

$$H \xrightarrow{\Delta} H \otimes H \xrightarrow{S \otimes 1_H} H \otimes H$$

$$\downarrow \varepsilon \phantom{1_H \otimes S} \quad \downarrow \mu$$

$$\downarrow \varepsilon \phantom{S \otimes 1_H} \quad \downarrow \mu$$

In the analogy between Hopf algebras and groups, this substitutes for $g \cdot g^{-1} = g^{-1} \cdot g = 1$. 
Sweedler Notation (I)

We will use ordinary ring notation for the multiplication and unit \( \mu \) and \( \eta \). Thus if \( a, b \in H \) let \( a \cdot b = \mu(a \otimes b) \) and let \( 1_H = \eta(1_K) \).
Indeed we may identify \( K \) with a subring of the center of \( H \) using \( \eta \).

The Sweedler notation streamlines the formula

\[
\Delta(a) = \sum_{i=1}^{N} a'_i \otimes a''_i.
\]

Instead, write

\[
\Delta(a) = a_{(1)} \otimes a_{(2)}.
\]

We are omitting the summation from the notation.
Sweedler Notation

Sweedler notation from the previous slide:

\[ \Delta(a) = a_{(1)} \otimes a_{(2)}. \] (1)

The co-associativity

\[ (\Delta \otimes 1_H)\Delta(a) = (1 \otimes \Delta)\Delta(a) \]

means

\[ a_{(1)(1)} \otimes a_{(1)(2)} \otimes a_{(2)} = a_{(1)} \otimes a_{(2)(1)} \otimes a_{(2)(2)}. \]

We will write

\[ a_{(1)} \otimes a_{(2)} \otimes a_{(3)} \]

for either of these decompositions. Note that \( a_{(2)} \) has a different meaning here than it did in (1).
Axioms in Sweedler notation (I)

The counit axiom:

\[ H \otimes H \xrightarrow{1_H \otimes \varepsilon} H \otimes I \xrightarrow{=\Delta} H \]

\[ H \otimes H \xrightarrow{\varepsilon \otimes I_H} I \otimes H \xrightarrow{=\Delta} H \]

In Sweedler notation,

\[ a = a_{(1)} \varepsilon(a_{(2)}) = \varepsilon(a_{(1)})a_{(2)}. \]
Axioms in Sweedler notation (II)

The antipode axiom:

\[ H \xrightarrow{\Delta} H \otimes H \xrightarrow{1_H \otimes S} H \otimes H \]

\[ \begin{array}{c}
\downarrow \varepsilon \\
K \\
\end{array} \quad \begin{array}{c}
\downarrow \mu \\
H \\
\end{array} \quad \begin{array}{c}
\downarrow \varepsilon \\
K \\
\end{array} \quad \begin{array}{c}
\downarrow \mu \\
H \\
\end{array} \]

\[ \varepsilon(a) = a_{(1)} S(a_{(2)}) = S(a_{(1)})a_{(2)}. \]

We should write

\[ \eta \varepsilon(a) = \cdots \]

but we identify \( \varepsilon(a) \in K \) with its image under \( \eta \).
If $A$ and $B$ are algebras, so is $A \otimes B$ with multiplication

$$(a \otimes b)(a' \otimes b') = (aa') \otimes (bb').$$

If $\mu_A$, $\mu_B$ and $\mu_{A \otimes B}$ are the multiplications in $A$, $B$ and $A \otimes B$ this means

$$\mu_{A \otimes B} = (\mu_A \otimes \mu_B) \circ (1_A \otimes \tau \otimes 1_B).$$

Hence we write the Hopf axiom

In other words, $\Delta : H \rightarrow H \otimes H$ is an algebra homomorphism.
We may also translate the Hopf axiom into Sweedler notation.

\[ H \otimes H \xrightarrow{\Delta \otimes \Delta} (H \otimes H) \otimes (H \otimes H) \]
\[ \downarrow \mu_H \quad \quad \downarrow \mu_{H \otimes H} \]
\[ H \xrightarrow{\Delta} H \otimes H \]

Applying this to \( x \otimes y \in H \otimes H \) gives

\[ (xy)_{(1)} \otimes (xy)_{(2)} = x_{(1)}y_{(1)} \otimes x_{(2)}y_{(2)}. \]
The antipode

**Proposition**

In a Hopf algebra $S(ab) = S(b)S(a)$.

The proof of this will be good practice for Sweedler notation so we will explain it carefully. This is an analog of the group property $(xy)^{-1} = y^{-1}x^{-1}$ so the problem will be to translate the proof of that fact into the Hopf algebra setting. Let us ponder the middle expression in

$$(xy)^{-1} = (xy)^{-1}xyy^{-1}x^{-1} = y^{-1}x^{-1}.$$ 

The Hopf analog is

$$S(x_1y_1)x_2y_2S(y_3)S(x_3)$$

where

$$x_1 \otimes x_2 \otimes x_3 = (\Delta \otimes 1)\Delta(x) = (1 \otimes \Delta)\Delta(x)$$
The antipode (continued)

The Hopf analog of \((xy)^{-1}xyy^{-1}x^{-1}\) is

\[
S(x(1)y(1))x(2)y(2)S(y(3))S(x(3)) \tag{2}
\]

where

\[
x(1) \otimes x(2) \otimes x(3) = (\Delta \otimes 1)\Delta(x) = (1 \otimes \Delta)\Delta(x).
\]

So imitating what we did for the group identity, we will unravel this expression two ways, proving both

\[
S(x(1)y(1))x(2)y(2)S(y(3))S(x(3)) = S(xy)
\]

and

\[
S(x(1)y(1))x(2)y(2)S(y(3))S(x(3)) = S(y)S(x).
\]

Using the antipode axiom \(y(1)S(y(2)) = \varepsilon(y)\)

\[
S(x(1)y(1))x(2)y(2)S(y(3))S(x(3)) = S(x(1)y(1))x(2)\varepsilon(y(2))S(x(3)).
\]

What just happened?
What just happened?

Here we are parsing

\[ y_{(1)} \otimes y_{(2)} \otimes y_{(3)} = (1 \otimes \Delta) \Delta(y) = y_{(1)} \otimes y_{(2)(1)} \otimes y_{(2)(2)}. \]

Now applying the map \( a \otimes b \otimes c \mapsto a \otimes bS(c) \) to both sides of this identity gives

\[ y_{(1)} \otimes y_{(2)} S(y_{(3)}) = y_{(1)} \otimes S(y_{(2)(1)})y_{(2)(2)} = y_{(1)} \otimes \varepsilon(y_{(2)}). \]

That is how we get

\[ S(x_{(1)}y_{(1)})x_{(2)}y_{(2)} S(y_{(3)})S(x_{(3)}) = S(x_{(1)}y_{(1)})x_{(2)}\varepsilon(y_{(2)})S(x_{(3)}). \]
Going back to (2), bear in mind that $\varepsilon(a)$ is a scalar and can be moved around at will in formulas. Also $\varepsilon(a)\varepsilon(b) = \varepsilon(ab)$ since the counit is a ring homomorphism by the augmentation axiom.

\[
S(x_1 y_1)x_2 y_2 S(y_3)S(x_3) = \\
S(x_1 y_1)x_2 \varepsilon(y_2)S(x_3) = \\
S(x_1 y_1)x_2 S(x_3)\varepsilon(y_2) = \\
S(x_1 y_1)\varepsilon(x_2)\varepsilon(y_2) = \\
S(x_1 y_1)\varepsilon(x_2 y_2)).
\]

Now by the Hopf axiom

\[
x_1 y_1 \otimes x_2 y_2 = (xy)_1 \otimes (xy)_2
\]

so

\[
S(x_1 y_1)x_2 y_2 S(y_3)S(x_3) = S((xy)_1 \varepsilon((xy)_2)) = S(xy).
\]
The other side

Now let us unravel (1) a different way.

\[
S(x_1)y_1)x_2)y_2)S(y_3)S(x_3) = \]
\[
S(x_1)y_1)x_2)y_2)S(y_2)S(x_2) = \quad \text{Hopf axiom}
\]
\[
S((x_1)y_1)(1))x_1)y_1)(2)S(y_2)S(x_2) = 
\]
\[
\varepsilon(x_1)y_1)S(y_2)S(x_2)
\]
\[
S(\varepsilon(y_1)y_2)S(\varepsilon(x_1)x_2)
\]

and so

\[
S(x_1)y_1)x_2)y_2)S(y_3)S(x_3) = S(y)S(x).
\]
Slogans

- Modules over a bialgebra are a monoidal category
- Finite-Dimensional Modules over a Hopf algebra are a rigid monoidal category
- Modules over a quasitriangular bialgebra are a braided monoidal category
- Finite-Dimensional Modules over a ribbon Hopf algebra are a ribbon category

For the first item, if $V$ and $W$ are modules over $H$, then $V \otimes W$ is a module over $H \otimes H$. Since $\Delta$ is an $H$-module homomorphism $H \to H \otimes H$, we may use it to transport this module structure to $H$. In Sweedler notation

$$a(v \otimes w) = a_{(1)} v \otimes a_{(2)} w.$$ 

The coassociativity means that $U \otimes (V \otimes W)$ and $(U \otimes V) \otimes W$ have the same module structure.
**Bialgebras and monoidal categories**

We have just seen that the category of modules over a bialgebra is a monoidal category, and that the key to this is the comultiplication.

There is a dual construction. Since the axioms of a bialgebra are self-dual, this statement has a dual one, that the category of comodules over a bialgebra is a monoidal category.

This fact arises in the theory of affine algebraic groups. If $G$ is an affine group scheme over a field $K$, the affine algebra $\mathcal{O}(G)$ is a commutative Hopf algebra, and if $V$ is a module for $G$, then the dual space of $V$ is a comodule for $\mathcal{O}(G)$. For a proof, see Waterhouse, *Introduction to Affine Group Schemes* Section 3.2.
Our second slogan is that modules over a Hopf algebra form a \textit{rigid category}. This is a category in which objects have duals. The archetype is the category finite-dimensional vector spaces over a field $K$.

Let $V$ be an object in a monoidal category with unit object $I$. We will abstract the properties of a (left) dual. This comes with morphisms $\text{ev}_V : V^* \otimes V \to I$ and $\text{coev}_V : I \to V \otimes V^*$ called \textit{evaluation} and \textit{coevaluation}.

The evaluation and coevaluation maps are subject to the following axioms.

$$(1_V \otimes \text{ev}_V) \circ (\text{coev}_V \otimes 1_V) = 1_V,$$

$$(\text{ev}_V \otimes 1_{V^*}) \circ (1_{V^*} \otimes \text{coev}_V) = 1_{V^*}.$$
Example: finite-dimensional vector spaces

In the category of finite-dimensional vector spaces, $l = K$
$V^* = \text{Hom}(V, K)$ and $\text{ev}_V(v^* \otimes v) = v^*(v)$, evaluating the linear functional $v^* \in V^*$ at the vector $v$.

Continuing with the example of a finite-dimensional vector space, we have to define the coevaluation, which is now to be a linear map $K \to V \otimes V^*$. We pick dual bases $v_i$ of $V$ and $v_i^*$ of $V^*$ and define $\text{coev}_V(a) = a \sum v_i \otimes v_i^*$. You may check that this is independent of $V$, and that the axioms are satisfied.
Diagrams for evaluation and coevaluation

We can represent \( \text{ev}_V : V^* \otimes V \to K \) and \( \text{coev}_V : K \to V^* \otimes V \)

The diagram is to be read from top to bottom.

At the top of the diagram is a sequence of spaces which are to be tensored together, hence \( V^* \otimes V \) in the left figure; so the two figures represent morphisms \( V^* \otimes V \to K \) and \( K \to V \otimes V^* \)
The first axiom

\[(1_V \otimes \text{ev}_V) \circ (\text{coev}_V \otimes 1_V) = 1_V\]

means that the following two diagram represents the identity map \( V \to V \).

Here \( \text{coev}_V \otimes 1_V : V = K \otimes V \to V \otimes V^* \otimes V \) and \( 1_V \otimes \text{ev}_V : V \otimes V^* \otimes V \to V \otimes K = V \).
Diagrams for rigidity axioms (II)

Identifying $K \otimes V = V \otimes K$ in the above diagram, the $K$ is superfluous. Moreover we may rely on the reader to slice the diagram horizontally at intervals to obtain the sequence of spaces $V, V \otimes V^* \otimes V$; we have only to label the segments of the diagram.

Then the axiom asserts that the above two morphisms are equal. (The second one is the identity map $V \to V$.)
Here is the dual axiom

\[(\text{ev}_V \otimes 1_{V^*}) \circ (1_{V^*} \otimes \text{coev}_V) = 1_{V^*}.\]

We see that the two axioms may be understood as asserting that such bends may be straightened.
Suppose we have another left dual $\tilde{V}^*$. Let $\tilde{ev}_V$ and $\tilde{coev}_V$ be the corresponding evaluation and coevaluation maps. We can construct morphisms $\phi : V^* \to \tilde{V}^*$ and $\psi : \tilde{V}^* \to V^*$ by

$$\phi = (ev_V \otimes I_{\tilde{V}^*})(1_{V^*} \otimes \tilde{coev}_V)$$

$$\psi = (\tilde{ev}_V \otimes I_{V^*})(1_{\tilde{V}^*} \otimes coev_V)$$
Uniqueness of the dual (continued)

We may diagram the composition $\psi \circ \phi$:

$$
\psi \circ \phi = (\text{ev}_V \otimes I_{V^*})(1_{V^*} \otimes \text{coev}_V)(\text{ev}_V \otimes I_{V^*})(1_{V^*} \otimes \text{coev}_V)
$$

$$
= (\text{ev}_V \otimes I_{V^*})(I_{V^*} \otimes I_V \otimes \text{ev}_V \otimes I_{V^*})(I_{V^*} \otimes \text{coev}_V \otimes I_V \otimes I_{V^*})(I_{V^*} \otimes \text{coev}_V)
$$
Uniqueness of the dual (continued)

After switching the order of $\tilde{\text{coev}}_V$ and $\tilde{\text{ev}}_V$ as above (just using the fact that $\otimes$ is a bifunctor) we may use the axioms twice to conclude that $\psi \circ \phi = I_{V^*}$:

Similarly $\phi \circ \psi = I_{V^*}$, so $\phi$ and $\psi$ are inverse isomorphisms. This proves the uniqueness of the dual, and also illustrates the use of diagrammatic methods in proofs.
We define an object $V$ in a monoidal category to be **rigid** if it has a left dual $V^*$. The category is called **rigid** if every object is.

Suppose that $V$ and $W$ are objects in a rigid category and $f : V \to W$ is a morphism. Define a morphism $f^* : W^* \to V^*$ by

$$f^* = (\text{ev}_W \otimes 1_{V^*})(1_{W^*} \otimes f \otimes 1_{V^*})(l_{W^*} \otimes \text{coev}_V).$$

Representing $f$ as a dot (left) then $f^*$ is as on the right.
Exercises

Exercise 1. Let \( f : V \to W \) and \( g : W \to U \) be morphisms in a rigid category. Prove that \((gf)^* = f^* g^*\).

Let \( V \) be a vector space and \( V^* \) its dual. We will write \( \langle v^*, v \rangle \) instead of \( v^*(v) \) for the dual pairing.

Exercise 2. Let \( H \) be a Hopf algebra, \( V \) a finite-dimensional module, and let \( V^* \) be its dual vector space. If \( a \in H \) and \( v^* \in V^* \) define \( av^* \) by

\[
\langle av^*, v \rangle = \langle v^*, S(a)v \rangle.
\]

Prove that this makes \( V^* \) into a module over \( H \).

Exercise 3. Prove that the category of finite-dimensional modules over a Hopf algebra is a rigid category.