Lecture 19: Modularity and the Partition Function

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Review: theories with $c < 1$

We saw in Lectures 16,17 and 18 that there exist Virasoro modules with $c < 1$ and particular values $h = h_{r,s}$ that contain singular vectors. For these, [BPZ] showed that the OPE is truncated, i.e. contain fewer primary fields than expected. Iif

$$c = 1 - 6\frac{(p-p')^2}{pp'}$$

where p, p' are coprime integers we can construct a minimal CFT $\mathcal{M}(p, p')$ out of these. Only if $|p - p'| = 1$ is the Virasoro representation unitary, so only $\mathcal{M}(m+1, m)$ can have direct physical significance.

Nonunitary minimal models

Though nonphysical, the nonunitary fields $\mathcal{M}(p,p')$ with $p > p'+1$ can arise indirectly. For example $\mathcal{M}(5,2)$ can arise as the Yang-Lee edge singularity of the Ising model with a magnetic field. This system is "unphysical" since $\mathcal{M}(5,2)$ arises when the magnetic field strength is analytically continued to an imaginary value.

Consider the Ising model in a magnetic field of strength h and temperature t. The energy of a state $\mathfrak{s}$ is

$$\varepsilon_{\mathfrak{s}} = -J \sum_{i,j \text{ adjacent}} \sigma_i \sigma_j - h \sum_i \sigma_i .$$

As in Lecture 18 the Partition function is

$$Z(t,h) = \sum_i e^{-\varepsilon_{\mathfrak{s}}/t}.$$

Non-unitary example: the Lee-Yang singularity

Yang and Lee (1952) considered the analytic continuation of partition functions in a general setting that contains the following particular example. Consider the Ising model in dimension d at temperature t in a constant magnetic field of strength h. There is a critical temperature t_c and for $t > t_c$ Yang and Lee showed that the zeros in h lie on the purely imaginary axis, a kind of Riemann hypothesis. There is also a singularity on the imaginary axis at a value $h = ih_c$ which may be regarded as a critical point though if h is not real the system has no physical meaning. There are no zeros between $-ih_c$ and ih_c so this is called the gap singularity.

Cardy (1985) showed that if $d = 2$ then the system is described by the non-unitary model $\mathcal{M}(5, 2)$ with $c = -\frac{22}{5}$. So non-unitary minimal models can have some meaning.

Theories with $c > 1$

Interesting CFT occur if $c > 1$. Now the Verma module $M(c, h)$ is always irreducible and unitary so as a $\mathbf{Vir} \oplus \mathbf{Vir}$ module

$$\mathcal{H} = \bigoplus_i M(c, h_i) \otimes M(c, \overline{h}_i).$$

There are no singular vectors so the truncation of the OPE that helped us in the minimal case is no longer applicable. Hence there are now infinitely many terms.

It is desirable that that the sum here be finite. If it is not (as when $c > 1$) it may be possible to combine the $\mathbf{Vir}$ modules into a finite number of groups by enlarging the Virasoro algebra. The candidates for such enlargement include affine Lie algebras and W-algebras.

Rational CFT

Reference: Moore and Seiberg Lectures on RCFT in Physics, Geometry and Topology, H.C. Lee (ed.), (1990).

Even when $\mathcal{H}$ does not decompose with finite multiplicity under $\mathbf{Vir} \otimes \mathbf{Vir}$, it may be possible to enlarge the Virasoro algebra to make the decomposition finite. Moore and Seiberg first informally define a CFT to be rational if there is a finite decomposition

$$\mathcal{H} = \bigoplus_{i=1}^{N} \mathcal{H}_i \otimes \overline{\mathcal{H}_i}$$

where if $\phi_i \otimes \overline{\phi_i} \in \mathcal{H}_i \otimes \overline{\mathcal{H}_i}$ then in any correlation function

$$\langle \cdots \phi_i(z) \otimes \overline{\phi}_i(\bar{z}) \cdots \rangle$$

is holomorphic in z and antiholomorphic in $\bar{z}$.

Rational CFT, continued

Moore and Seiberg also give a more formal equivalent definition of a RCFT. They require that there exist holomorphic and anti-holomorphic monodromy-free subalgebras $\mathcal{A}$ and $\overline{\mathcal{A}}$ of the operator product algebra such that the physical Hilbert space $\mathcal{H}$ decomposes as a finite sum of $\mathcal{A} \otimes \overline{\mathcal{A}}$ modules

$$\mathcal{H} = \bigoplus_{i=1}^{N} \mathcal{H}_i \otimes \overline{\mathcal{H}_i}.$$

There are different candidates for the algebra $\mathcal{A}$. For the minimal models, we may take just $\mathbf{Vir}$, or its enveloping algebra.

An important class of RCFT with $c > 1$ are the Wess-Zumino-Witten (WZW) theories that we will discuss in Lecture 20.

The partition function

There is a partition function in QFT as well as statistical mechanics. For CFT it is the trace of the operator

$$q^{L_0 - c/24}\,\overline{q}^{\overline{L}_0 - c/24},$$

where $q = e^{2\pi i\tau}$ with τ in the upper half-plane $\mathfrak{H}$, so that $|q| < 1$. Here $\overline{q}$ is the complex conjugate, and L_0, $\overline{L}_0$ are the energy operators acting on the holomorphic and antiholomorphic fields.

As in the case of statistical mechanical partition function, the QFT partition function is a generating function from which quantities such as the correlation functions may be extracted.

- Wikipedia link Partition Function (Statistical Mechanics)
- Wikipedia link Partition Function (QFT)

Virasoro characters

If V is a Virasoro module, its character $\chi_V(q)$ is

$$\mathrm{tr}(q^{L_0 - c/24}|V).$$

The correction $c/24$ turns out to make formulas nicer.

So the partition function is essentially character of the $\mathbf{Vir} \oplus \mathbf{Vir}$ module $\mathcal{H}$. If

$$\mathcal{H} = \oplus_i V_i \otimes \overline{V}_i$$

then the partition function is

$$\sum_i \chi_{V_i}(q)\chi_{\overline{V}_i}(\overline{q}).$$

In the diagonal case $V_i \cong \overline{V}_i$ this is

$$\sum_i |\chi_{V_i}(q)|^2.$$

Automorphicity of the partition function

Cardy (1986) argued that the partition function should be invariant under the action of $SL(2,\mathbb{Z})$ on $\mathfrak{H}$. This is a significant constraint on the theory. In this lecture we will consider how this works for the minimal models.

In addition to the partition function, many things in CFT have modularity properties for $SL(2,\mathbb{Z})$. For example, Kac and Peterson proved that the fields of a WZW theory, which are characters and string functions for affine Lie algebras are theta functions, hence modular forms.

Theta functions and quadratic forms

Theta functions are associated with quadratic forms. Let Q be a quadratic form in d variables, with integer coefficients. Thus there exists a $d \times d$ symmetric integer matrix A_{ij} such that

$$Q(z) = \sum_{i,j} A_{ij} z_i z_j, \qquad \mathbf{z} = (z_1, \cdots, z_d) \in \mathbb{Z}^d$$

Then writing $q = e^{2\pi i \tau}$

$$\theta_Q(\tau) = \sum_{\mathbf{a} \in \mathbb{Z}^n} q^{Q(\mathbf{a})/2}.$$

This theta function is modular for a congruence subgroup $\Gamma_0(N)$ of $SL(2,\mathbb{Z})$ of weight $d/2$, where N depends on Q and

$$\Gamma_0(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{Z}) | c \equiv 0 \text{ mod N} \right\}.$$

Full modular group

The group $SL(2,\mathbb{Z})$ has generators

$$T = \begin{pmatrix} 1 & 1 \\ & 1 \end{pmatrix}, \qquad S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

In some cases a transformation property with respect to S, whicha is the map $\tau \mapsto -1/\tau$ may be deduced from the Poisson summation formula. To give an example, consider the Jacobi theta function

$$\theta(\tau) = \sum_{n=-\infty}^{\infty} q^{n^2/2}.$$

This is modular not with respect to S but with respect to the similar transformation $\tau \to -1/4\tau$. We will prove this.

Poisson summation

The Poisson summation formula asserts in its simplest form

$$\sum_{n=-\infty}^{\infty} f(n) = \sum_{n=-\infty}^{\infty} \hat{f}(n),$$

where the Fourier transform

$$\hat{f}(x) = \int_{-\infty}^{\infty} f(y) e^{2\pi ixy}\, dy.$$

If

$$f(x) = e^{\pi x^2 \tau}, \qquad \hat{f}(x) = \frac{1}{\sqrt{-i\tau}} e^{-ix^2/\tau}$$

so Poisson summation implies the identity

$$\theta(\tau) = \frac{1}{\sqrt{-2i\tau}} \theta\left(-\frac{1}{4\tau}\right),$$

used by Riemann to prove the functional equation of ζ.

Automorphicity in general

All modularity properties of theta functions may be deduced from the Poisson summation formula with more work. For example $\theta(\tau)$ is modular for the congruence subgroup

$$\Gamma_0(4) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{Z}) | c \equiv 0 \text{ mod } 4 \right\}.$$

More precisely

$$\theta\left(\frac{a\tau+b}{c\tau+d}\right) = \varepsilon_d^{-1}\left(\frac{c}{d}\right)(c\tau+d)^{1/2}\theta(\tau), \qquad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(4)$$

where

$$\varepsilon_d = \begin{cases} 1 & d \equiv 1 \text{ mod } 4 \\ i & d \equiv 3 \text{ mod } 4 \end{cases}$$

and $\left(\frac{c}{d}\right)$ is a quadratic symbol variant defined by Shimura.

The Dedekind η function

Let

$$\eta(\tau) = q^{1/24} \prod_{n=1}^{\infty} (1 - q^n).$$

It is known that

$$\eta\left(\frac{a\tau + b}{c\tau + d}\right) = (*)(c\tau + d)^{1/2}\eta(\tau), \qquad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbb{Z}),$$

where $(*)$ is a 24-th root of unity that we will not make precise. The Dedekind eta function is related to the number-theoretic partition function $p(n)$ since

$$\sum_{n=0}^{\infty} p(n) q^n = \prod_{n=1}^{\infty} (1 - q^n)^{-1} = q^{1/24}\eta(\tau)^{-1}.$$

The eta function is a theta function

The Dedekind η function is also a kind of theta function due to the formula

$$\eta(\tau) = \sum_{n=-\infty}^{\infty} \chi(n) q^{n^2/24}$$

where χ is the Dirichlet character

$$\chi(n) = \begin{cases} 1 & \text{if } n \equiv \pm 1 \text{ mod } 12, \\ -1 & \text{if } n \equiv \pm 0 \text{ mod } 12, \\ 0 & \text{otherwise.} \end{cases}$$

Note the terms in this series are sparse, so there is much cancellation in the product

$$\eta(\tau) = q^{1/24} \prod_{n=1}^{\infty} (1 - q^n).$$

Minimal Models

We will follow [DMS] Sections 8.1 and 10.6 to show how the theory of theta functions leads to the automorphicity of the partition function

$$\mathrm{tr}\left(q^{L_0-c/24}\cdot\bar{q}^{\bar{L}_0-c/24}\right)$$

for the minimal models. (Recall that this modularity is a desideratum for general CFT.)

To analyze the partition function, if V is a Virasoro module in Category $\mathcal{O}$, we define its character by:

$$\chi_V(q)=\mathrm{tr}(q^{L_0-c/24}|V)$$

For the minimal models

$$\mathcal{H}=\bigoplus_{\substack{1\leqslant r<m\\1\leqslant s<p}} L(c,h_{r,s})\otimes L(c,h_{r,s}).$$

Generators of $SL(2,\mathbb{Z})$

For computation of the partition function, we have $h_{r,s} = h_{p'-r,p-s}$ so we can omit the condition $r/s < p'/p$ and

$$\mathrm{tr}\left(q^{L_0-c/24}\cdot \bar{q}^{\bar{L}_0-c/24}\right) = \frac{1}{2}\sum_{\substack{1\leqslant r<m\\ 1\leqslant s<p}} |\chi_{r,s}(q)|^2$$

where the character

$$\chi_{r,s}(q) = \chi_{L(c,h_{r,s})} = \mathrm{tr}(q^{L_0-c/24}|L(c,h_{r,s})).$$

Now $\chi_{r,s}(q)$ is invariant under the action of $SL(2,\mathbb{Z})$ on τ, where $q = e^{2\pi i\tau}$. Since $SL(2,\mathbb{Z})$ is generated by

$$S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \qquad T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

it is enough to check invariance by these elements.

The setup

Now $T : \tau \longrightarrow \tau + 1$ which does not affect q so it is enough to check invariance under $S : \tau \longrightarrow -1/\tau$. This will bring us into contact with topics that have profound significance in CFT beyond the special case of the minimal models, such as the S-matrix and Verlinde formula.

Recall that with p, p' coprime integers

$$h_{r,s} = \frac{(pr - p's)^2 - (p - p')^2}{4pp'}, \qquad c = 1 - 6\frac{(p - p')^2}{pp'}.$$

We assume $1 \leqslant r < p'$, $1 \leqslant s < p$ since these are the $L(c, h_{r,s})$ that appear in the minimal models. We will encounter $h_{a,b}$ where a, b are not both in this range. Note that $h_{a,b} = h_{a+p',b+p}$. So we may assume a and b are nonnegative and as small as possible, i.e. either $0 \leqslant a < p'$ or $0 \leqslant b < p$ but not necessarily both.

Characters of Verma modules

Note that

$$h - c/24 = \frac{(pr - p's)^2}{4pp'} - \frac{1}{24}.$$

Recalling that

$$\sum_{n=0}^{\infty} p(n) q^n = \prod_{n=1}^{\infty} (1 - q^n)^{-1} = q^{1/24} \eta(\tau)^{-1},$$

we have

$$\mathrm{tr}(q^{L_0 - c/24} | M(c, h_{r,s})) = q^{(pr - p's)^2/4pp'} \eta(\tau)^{-1}.$$

To compute the character of $L(c, h_{r,s})$ we will construct a resolution by Verma modules.

Reference: [DMS] Sections 8.1 and 10.5.

Submodules of the degenerate Verma module

Proposition

$M(c, h_{r,s})$ contains singular vectors of levels rs and $(p' - r)(p - s)$ generating submodules isomorphic to $M(c, h_{p'+r,p-s})$ and $M(c, h_{2p'-r,s})$.

Indeed, by the Kac determinant formula, $M(c, h_{r,s})$ has a singular vector of level rs. Since $h_{r,s} + rs = h_{p'+r,p-s}$ this vector generates a submodule isomorphic to $M(c, h_{p'+r,p-s})$. On the other hand $h_{r,s} = h_{p'-r,p-s}$ and by the Kac determinant formula it has a singular vector of level

$$h_{p'-r,p-s} + (p' - r)(p - s) = h_{p'+r,p-s},$$

and hence a submodule isomorphic to $M(c, h_{p'+r,p-s})$.

Submodules of the degenerate Verma module

The same method shows that the submodules isomorphic to $M(c, h_{p'+r,p-s})$ and $M(c, h_{2p'-r,s})$ both contain submodules isomorphic to $M(c, h_{2p'+r,s})$ and $M(c, h_{r,2p+s})$. These however are equal. Thus we have a lattice of subgroups:

$$\begin{tikzcd}
& M(c, h_{r,s}) \arrow[dl, dash] \arrow[dr, dash] & \\
M(c, h_{p'+r,p-s}) \arrow[d, dash] \arrow[drr, dash] & & M(c, h_{r,2p-s}) \arrow[d, dash] \arrow[dll, dash] \\
M(c, h_{2p'+r,s}) \arrow[d, dash] \arrow[drr, dash] & & M(c, h_{r,2p+s}) \arrow[d, dash] \arrow[dll, dash] \\
\vdots & & \vdots
\end{tikzcd}$$

Verma resolution

The chain continues, and at the k-th level with $k \geqslant 1$ we have two submodules

$$\begin{cases} M(c, h_{kp'+r,p-s}), & M(c, h_{r,(k+1)p-s}) \quad \text{if } k \text{ is odd,} \\ M(c, h_{kp'+r,s}), & M(c, h_{r,kp+s}) \quad \text{if } k \text{ is even.} \end{cases}$$

Hence we have a resolution:

$$\cdots \longrightarrow M_2 \longrightarrow M_1 \longrightarrow M_0 \longrightarrow L(c, h_{r,s}) \longrightarrow 0$$

where $M_0 = M(c, h_{r,s})$ and if $k \geqslant 1$

$$M_k = \begin{cases} M(c, h_{kp'+r,p-s}) \oplus M(c, h_{r,(k+1)p-s}) & \text{if } k \text{ is odd,} \\ M(c, h_{kp'+r,s}) \oplus M(c, h_{r,kp+s}) & \text{if } k \text{ is even.} \end{cases}$$

The character in terms of theta functions

We have

$$\mathrm{tr}(q^{L_0-c/24}|L(c,h_{r,s})) = \sum_{k=0}^{\infty}(-1)^k\,\mathrm{tr}(q^{L_0-c/24}|M_k).$$

Now remembering

$$\mathrm{tr}(q^{L_0-c/24}|M(c,h_{r,s})) = q^{(pr-p's)^2/4pp'}\eta(\tau)^{-1},$$

and rearranging we have can express the Virasoro character

$$\chi_{r,s}(q) = \mathrm{tr}(q^{L_0-c/24}|L(c,h_{r,s}))$$

as

$$\chi_{r,s}(q) = \frac{\Theta_{r,s}(\tau)-\Theta_{r,-s}(\tau)}{\eta(\tau)}$$

in terms of a theta function

$$\Theta_{r,s}(\tau) = \sum_{n\in\mathbb{Z}} q^{(2pp'n+pr-p's)^2/4pp'}.$$

Functional equations

We can write

$$\Theta_{r,s}(\tau) = \Theta_{\lambda_{r,s}}(\tau)$$

where, denoting $\lambda_{r,s} = pr - p's$ and $N = 2pp'$

$$\Theta_\lambda(\tau) = \sum_{n\in\mathbb{Z}} q^{(Nn+\lambda)^2/2N}.$$

With $N = 2pp'$ we have

$$\eta\left(-\frac{1}{\tau}\right) = \sqrt{-i\tau}\,\eta(\tau), \qquad \Theta_\lambda\left(-\frac{1}{\tau}\right) = \sqrt{-i\tau}\sum_{\mu=0}^{N} e^{2\pi i\lambda\mu/N}\Theta_\mu(\tau).$$

Both formulas may be proved using the Poisson summation formula.

Functional equations

We have enough information to check that the partition function

$$|\operatorname{tr}(q^{L_0-c/24}|L(c,h_{r,s}))|^2 = \frac{1}{2}\sum_{\substack{1\leqslant r<m \\ 1\leqslant s<p}} |\chi_{r,s}(q)|^2$$

is invariant under $\tau \mapsto -\frac{1}{\tau}$ (using finite Plancherel formula).