Lecture 18: Minimal Models II

Daniel Bump

January 1, 2020

Degenerate Virasoro modules

In Lecture 17 we discussed the fusion rules for Virasoro modules when one of the primary fields is degenerate. This means that the Verma module $M(c,h)$ contains a singular vector, which by the Kac determinant formula can happen only when $h = h_{r,s}$ and $c < 1$.

There are different expressions for $h_{r,s}$. To recall one, fix $0 < c < 1$ and let $h_0 = \frac{1}{24}(c-1)$. Then if α is a real parameter, we have

$$h_{r,s} = h_0 + \frac{1}{4}(r\alpha_+ + s\alpha_-)^2, \qquad \alpha_\pm = \frac{\sqrt{1-c} \pm \sqrt{25-c}}{\sqrt{24}}.$$

Note that $\alpha_+ > 0$ while $\alpha_- < 0$.

Notations for primary fields

We have two notations for primary fields. We will denote by $\phi_{(\alpha)}$ a primary field that is not necessarily degenerate, but which has L_0-eigenvalue $h = h_0 + \frac{1}{4}\alpha^2$.

Alternatively, we will denote by $\phi_{(r,s)}$ a primary field with L_0-eigenvalue $h_{r,s}$. In other words $\phi_{(r,s)} = \phi_{(\alpha)}$ with $\alpha = r\alpha_+ + s\alpha_-$ with r, s positive integers.

From the Kac determinant formula, such a field $\phi_{(r,s)}$ is degenerate, that is, the corresponding Verma module $M(c, h_{r,s})$ contains a singular vector at level rs and the corresponding null field, which is a descendent of $\phi_{r,s}$ may be set to zero since it has no interactions with other fields (even $\phi_{(r,s)}$ itself).

Truncated fusion rule

Consider the degenerate fields with null descendent fields of level 2. In Lecture 17 we proved

$$[\phi_{(2,1)}] \times [\phi_{(\alpha)}] = [\phi_{(\alpha-\alpha_+)}] + [\phi_{(\alpha+\alpha_+)}],$$

$$[\phi_{(1,2)}] \times [\phi_{(\alpha)}] = [\phi_{(\alpha-\alpha_-)}] + [\phi_{(\alpha+\alpha_-)}].$$

This means that the three point function

$$\langle \phi_{(2,1)}(z)\phi_{(\alpha)}(w_1)\phi_{(\beta)}(w_2)\rangle$$

is zero unless $\beta = \alpha - \alpha_+$ or $\beta = \alpha + \alpha_+$. The same conclusion is true for any descendents of these primary fields. As a consequence the only fields that can occur in the OPE of $\phi_{(2,1)}(z)\phi_{(\alpha)}(w)$ are from the conformal families $[\phi_{(\beta)}(w)]$ with $\beta = \alpha \pm \alpha_+$.

Both fields degenerate

If $\phi_{(\alpha)}$ is itself a degenerate field, then the above fusion rules may be wrong. For if $\phi_{(\alpha)} = \phi_{(m,n)}$ then the above rules will state that

$$[\phi_{(2,1)}] \times [\phi_{(m,n)}] = [\phi_{(m+1,n)}] + [\phi_{(m-1,n)}],$$

$$[\phi_{(1,2)}] \times [\phi_{(m,n)}] = [\phi_{(m,n+1)}] + [\phi_{(m,n-1)}].$$

In particular

$$[\phi_{(1,2)}] \times [\phi_{(2,1)}] = [\phi_{(2,0)}] + [\phi_{(2,2)}]$$

while

$$[\phi_{(2,1)}] \times [\phi_{(1,2)}] = [\phi_{(2,2)}] + [\phi_{(0,2)}].$$

Further truncation

To repeat, we need to reconcile:

$$[\phi_{(1,2)}] \times [\phi_{(2,1)}] = [\phi_{(2,0)}] + [\phi_{(2,2)}]$$

while

$$[\phi_{(2,1)}] \times [\phi_{(1,2)}] = [\phi_{(2,2)}] + [\phi_{(0,2)}].$$

But $\phi_{(2,1)}(z)$ and $\phi_{(1,2)}(w)$ commute (by locality) and so these should be the same. The explanation is that in the case $\phi_{(\alpha)}$ is degenerate, the fusion rule is further truncated. So actually

$$[\phi_{(1,2)}] \times [\phi_{(2,1)}] = [\phi_{(2,1)}] \times [\phi_{(1,2)}] = [\phi_{(2,2)}].$$

In which $c = \frac{1}{2}$

Now let us take $c = \frac{1}{2}$. This will lead to the simplest minimal model, denoted $\mathcal{M}(4,3)$. We note that

$$\frac{\alpha_+}{\alpha_-} = \frac{\sqrt{1-c}+\sqrt{25-c}}{\sqrt{1-c}-\sqrt{25-c}} = -\frac{4}{3}.$$

Consequently

$$\alpha_+ + \alpha_- = -(2\alpha_+ + 3\alpha_-)$$

and so

$$h_{(1,1)} = h_0 + \frac{1}{4}(\alpha_+ + \alpha_-)^2 = h_{(1,1)} = h_0 + \frac{1}{4}(2\alpha_+ + 3\alpha_-)^2 = h_{(2,3)}.$$

Thus we may take $\phi_{(2,3)} = \phi_{(1,1)}$ which we recall is the constant field that is independent of z.

The fusion ring

We have seen that when $c = \frac{1}{2}$ we may identify $\phi_{(2,3)}$ with $\phi_{(1,1)}$, the constant field that is the unit in the fusion ring. Similarly

$$\phi_{(2,1)} = \phi_{(1,3)}, \qquad \phi_{(1,2)} = \phi_{(2,2)}.$$

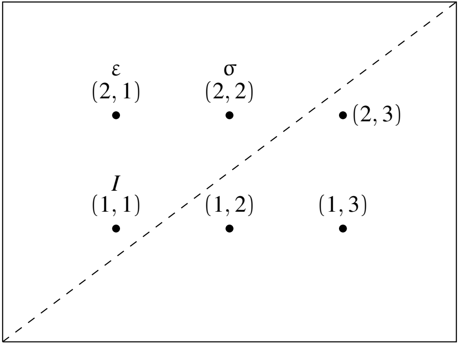
With these identifications consider the span of the fields

$$I = \phi_{(1,1)}, \qquad \varepsilon = \phi_{(2,1)}, \qquad \sigma = \phi_{(1,2)}.$$

We see that these fields are closed under the fusion product:

	I	ε	σ
I	I	ε	σ
ε	ε	I	σ
σ	σ	σ	$I + \varepsilon$

The Kac diagram for $\mathcal{M}(4,3)$



We may depict the model graphically in this Kac diagram. Since $\phi_{(1,2)} = \phi_{(2,2)}$, $\phi_{(1,3)} = \phi_{(2,1)}$ and $\phi_{(2,3)} = \phi_{(1,1)}$ the fields below the dashed line are duplicates of I, ε, σ.

Conformal weights and the Hilbert space

The conformal weights may also be made more explicit:

$$h_{(1,1)} = 0, \qquad h_{(2,1)} = \frac{1}{2}, \qquad h_{(1,2)} = \frac{1}{16}.$$

We have ignored the antiholomorphic part of the CFT. As a $\mathbf{Vir} \oplus \mathbf{Vir}$ module

$$\mathcal{H} = L\left(\tfrac{1}{2}, 0\right) \otimes L\left(\tfrac{1}{2}, 0\right) \oplus L\left(\tfrac{1}{2}, \tfrac{1}{2}\right) \otimes L\left(\tfrac{1}{2}, \tfrac{1}{2}\right) \oplus L\left(\tfrac{1}{2}, \tfrac{1}{16}\right) \otimes L\left(\tfrac{1}{2}, \tfrac{1}{16}\right).$$

This is the BPZ minimal CFT denoted $\mathcal{M}(4,3)$.

The setup

We now describe the general minimal model $\mathcal{M}(p,p')$ where p and p' are coprime integers with $p > p'$. These will be unphysical unless $p = p' + 1$ but we will only impose this condition at the very end.

We seek models in which every primary field is degenerate, and where a finite number of conformal families are closed under the fusion product. Such systems can occur for values of c such that the slope

$$-\frac{\alpha_+}{\alpha_-} = \frac{\sqrt{1-c}+\sqrt{25-c}}{-\sqrt{1-c}+\sqrt{25-c}}$$

is rational.

Rational slope

If $\frac{p}{p'}$ is this rational slope then

$$c = 1 - 6\frac{(p-p')^2}{pp'}, \qquad h_{r,s} = \frac{(pr-p's)^2 - (p-p')^2}{4pp'}.$$

We consider $1 \leqslant r < p'$ and $1 \leqslant s < p$.
Now any transformation of (r,s) that sends α to either α or $-\alpha$ preserves

$$h_{r,s} = h_0 + \frac{1}{4}\alpha^2.$$

Here

$$\alpha = r\alpha_+ + s\alpha_- = \frac{\alpha_+}{p}(rp - sp').$$

Thus

$$h_{r,s} = h_{p'-r,p-s} \ .$$

The Hilbert space

Now we may construct a theory with

$$\mathcal{H} = \bigoplus_{\substack{1 \leqslant r < p' \\ 1 \leqslant s < p \\ r/s < p'/p}} L(c, h_{r,s}) \otimes L(c, h_{r,s})$$

and as in the $M(4,3)$ case it may be shown that these conformal families are closed under fusion product.

Constraint from Unitarity

Now we recall the theorem of Friedan, Qiu and Shenkar, that the irreducible highest weight representation $L(c,h)$ is unitary if and only if

$$c = 1 - \frac{6}{m(m+1)}, \qquad h = h_{r,s}$$

for some $m \geqslant 2$ and r,s can be chosen in the range $1 \leqslant r \leqslant s \leqslant m+1$. Thus the system $\mathcal{M}(p,p')$ is unphysical unless $p - p' = 1$.

Therefore we take $p = m+1$, $p' = m$. In this case the slope p'/p is nearly 1 and the condition $r/s < m/(m+1)$ may be replaced by $r \leqslant s$, so

$$\mathcal{H} = \bigoplus_{\substack{1 \leqslant r < m \\ 1 \leqslant s < m+1 \\ r \leqslant s}} L(c, h_{r,s}) \otimes L(c, h_{r,s}).$$

Statistical Mechanics

For us statistical mechanical system $\mathfrak{S}_i$ is an ensemble containing a large number of states $\mathfrak{s}$. Each state is assigned an energy $\varepsilon_{\mathfrak{s}}$. At a temperature T, the probability that the system will be found in the state $\mathfrak{s}$ is

$$\frac{e^{-\varepsilon_{\mathfrak{s}}/T}}{Z(\mathfrak{S},T)},$$

where $Z(\mathfrak{S},T)$ is the partition function

$$Z(\mathfrak{S},T) = \sum_i e^{-\varepsilon_{\mathfrak{s}}/T}.$$

We will explain one such system, the critical Ising model, and show that it is related to $\mathcal{M}(4,3)$.

The Ising Model

The (two-dimensional ferromagnetic) Ising model takes place on a large $(N \times N)$ 2-dimensional square grid. In a state of the system a spin $\sigma_i = \pm 1$ is assigned to each vertex v_i. The energy of the state $\mathfrak{s}$ is $-J \sum \sigma_i \sigma_j$ where the sum is over adjacent sites v_i and v_j.

The two-dimensional Ising model was solved by Onsager (1944) after significant work by Krammers and Wannier who found a relationship between the partition functions at two values one above and one below a critical value T_c. (See [DMS] equation (12.7)). Among Onsager's results was a proof that T_c is indeed a phase transition point.

Conformal invariance at the critical temperature

Thus system has a phase transition at the critical temperature T_c where $\sinh(2J/T_c) = 1$. Below the critical temperature the spins and local energies at distant points are strongly correlated; above the critical temperature they are not.

In Onsager's solution, the system is equivalent to a free-fermionic quantum mechanical system. Away from T_c, the fermions are massive, but at $T = T_c$ they are massless, and the system is equivalent to a massless free-fermionic system that is described by a conformal field theory with central charge $c = \frac{1}{2}$.

Critical Exponents

Assuming that the system is at the critical temperature, and imagining the grid fine enough that the spins $\sigma(z)$ and energy density $\varepsilon(z)$ may be approximated by functions on the complex plain, we have the correlations (DMS Section 12.2.2)

$$\varepsilon(z)\varepsilon(w) \sim \frac{1}{|z-w|^2}, \qquad \sigma(z)\sigma(w) \sim \frac{1}{|z-w|^{1/4}} + C|z-w|^{3/4}\varepsilon(w).$$

The critical exponents 2 and $\frac{1}{4}$ are predicted by the L_0 eigenvalues

$$h_{(2,1)} = \frac{1}{2}, \qquad h_{(1,2)} = \frac{1}{16}.$$

The appearance of $\varepsilon(w)$ in the OPE of $\sigma(z)\sigma(w)$ is predicted by the fusion rule $[\sigma]^2 = [I] + [\varepsilon]$.

Andrews, Baxter and Forrester

Other two-dimensional solvable lattice models are associated with other BPZ minimal models. Particularly, an infinite family of such models (the RSOS models of Andrews, Baxter and Forrester 1984) shows that all minimal models are associated with statistical-mechanical systems.