Primitive Vectors

Let us consider the general case of a Lie algebra

\[ g = n_- \oplus h \oplus n_+ . \]

with triangular decomposition, let \( V \) be a highest weight module. Reference: Kac, *Infinite-dimensional Lie algebras*, Chapter 9. Let \( V \) be a \( g \)-module with a highest weight decomposition. A vector \( v \in V \) is called *primitive* if there exists a submodule \( U \) such that \( v \notin U \) but \( n_+ v \subseteq U \). An important special case is that \( U = 0 \). Then \( v \neq 0 \) but \( n_+ v = 0 \); in this case \( v \) generates a highest weight representation with highest weight \( \lambda \). If this is true, we say that \( v \) is a *singular vector*. 
The Kac Determinant Formula

Example: \( \mathfrak{sl}(2, \mathbb{C}) \)

A necessary and sufficient condition for a module in Category \( \mathcal{O} \) to be irreducible is that it has a unique (up to scalar) primitive vector. This vector will be a highest weight vector. If \( v \in V_\lambda \) then \( V \cong L(\lambda) \).

Let us illustrate these examples with the example \( \mathfrak{g} = \mathfrak{sl}(2, \mathbb{C}) \) spanned by

\[
H = \begin{pmatrix} 1 & \cr & -1 \end{pmatrix}, \quad E = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad F = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix},
\]

\[
\]

Let \( \lambda \) be the linear functional on \( \mathfrak{h} = \mathbb{C}H \) defined by \( \lambda(H) = k \), where \( k \in \mathbb{C} \). Let \( v_\lambda \) be the highest weight vector, so \( Hv_\lambda = kv_\lambda \) and \( Ev_\lambda = 0 \).
Because the map $U(n) = \mathbb{C}[F]$ to $M(\lambda)$ sending $\xi$ to $\xi v_\lambda$ is an isomorphism, a basis of $M(\lambda)$ consists of $F^m v$ with $m = 0, 1, 2, \cdots$. An induction using $[H, F] = -2F$ shows that $H \cdot F^m v_\lambda = (k - 2m)v_\lambda$. Then another induction using $[E, F] = H$ shows that $EF^m v_\lambda = m(k - m + 1)v_\lambda$. Thus assuming this for some $m$,

$$EF^{m+1} v_\lambda = (EF - FE)F^m v_\lambda + FEF^m v_\lambda = HF^m v_\lambda + FEF^m v_\lambda$$

and by induction this equals

$$(k - 2m)v_\lambda + m(k - m + 1)v_\lambda = (m + 1)(k - m)v_\lambda.$$  

This completes the induction.
Example, concluded

Since $EF^m v_\lambda = m(k - m + 1) v_\lambda$, we see that $F^m v_\lambda$ is a singular vector if $m = k + 1$. This means that $k$ is a nonnegative integer, or equivalently, $\lambda$ is a dominant weight. The singular vector $F^m v_\lambda$ is a highest weight vector for a submodule isomorphic to $M(\lambda - (k + 1) \alpha)$. Then $L(\lambda) = M(\lambda)/M(\lambda - (k + 1) \alpha)$ is finite-dimensional.

If we regard $\mathfrak{g}$ as the complexification of $\mathfrak{su}(2)$ then $L(\lambda)$ is unitary as an $\mathfrak{su}(2)$-module in this case where $\lambda$ is dominant. For general $\lambda$ a highest weight module for $\lambda$ will contain vectors of negative norm (“ghosts”) but not for $L(\lambda)$ when $\lambda$ is dominant.
The inner product for $\text{Vir}$

Let $V$ be a highest weight representation of $\text{Vir}$ with highest weight $(c, h)$, meaning that $Cv = cv$ for all $v \in V$, and $L_0v_\lambda = hv_\lambda$ if $v_\lambda$ is a highest weight vector. We will fix a highest weight vector and denote $v_\lambda = |h\rangle$.

In a unitary representation that comes from a conformal field theory, $L_n$ must be the adjoint of $L_{-n}$. See Ginsparg, Applied CFT (arXiv:hep-th/9108028) Section 3.4 for justification of this. It is proved in Kac and Raina, Proposition 2.2 that if $V$ is a highest weight representation of $\text{Vir}$ that there is a unique Hermitian inner product on $V$ in which $L_n$ and $L_{-n}$ are adjoints. However this inner product may not be positive definite.
Determining whether this inner product on the irreducible highest weight module $L(c, h)$ is positive definite is a problem solved by the Kac determinant, which we now describe, following [FMS] Section 7.2.1 and Kac-Raina, Chapters 8 and 12.

When $c < 1$ the representations $L(c, h)$ when $M(c, h)$ contains a singular vector are used in constructing the two-dimensional minimal models of [BPZ], which important in statistical mechanics since they often model two-dimensional solvable lattice models such as the Ising model at the critical temperature.
The Verma module

The Verma module $M(c, h)$ is graded as follows. A basis consists of vectors

$$|\mathbf{k}\rangle = L_{-k_1} \cdots L_{-k_m} |h\rangle, \quad 1 \leq k_1 \leq \cdots \leq k_m.$$  

We call $\sum k_i = N$ the level of the vector. Let $\mathbf{k} = (k_1, \cdots, k_n)$ be the corresponding partition (written backwards since traditionally partitions are written in descending order). If $\mathbf{k}$ and $\mathbf{l}$ are two such partitions of the same level $l$, then the inner product $\langle \mathbf{l}|\mathbf{k}\rangle$ equals

$$\langle h|L_{k_m} \cdots L_{k_1} L_{-l_1} \cdots L_{-k_n} |h\rangle.$$  

(If $\mathbf{l}$ and $\mathbf{k}$ have different level then $|\mathbf{l}\rangle$ and $|\mathbf{k}\rangle$ are orthogonal.)
The number of partitions of level $N$ is denoted $p(N)$. The $p(N) \times p(N)$ matrix of inner products $\langle l|k \rangle$ is denoted $\det_N(c, h)$.

Let us compute some inner products. To compute $\langle h|L_1L_{-1}|h \rangle$ we use the identity $[L_1, L_{-1}] = 2L_0$ and we see that $\langle h|L_1L_{-1}|h \rangle = 2h \langle h|h \rangle = 2h$. Again, let us compute $\langle h|L_1^2L_{-2}|h \rangle$. For this we use $L_1L_{-2} = L_{-2}L_1 + 3L_{-1}$. Remembering that $L_1|h\rangle = 0$ we get

$$\langle h|L_1^2L_{-2}|h \rangle = \langle h|3L_{-1}|h \rangle = 6h.$$ 

Again, using the cocycle $\frac{k^3 - k}{12} = \frac{1}{2}$ when $k = 2$,

$$\langle h|L_2L_{-2}|h \rangle = \left\langle h|4L_0 + \frac{C}{2}|h \right\rangle = 4h + \frac{c}{2},$$

etc.
The Kac Determinant Formula

Determinants

The determinant of the $p(N) \times p(N)$ matrix of inner products is denoted $\det_N(c, h)$ and we compute

$$\det_1(c, h) = 2h,$$

and

$$\det_2(c, h) = \det \begin{pmatrix} \langle h|L_2L_{-2}|h \rangle & \langle h|L_2L_{-1}L_{-1}|h \rangle \\ \langle h|L_2L_{-1}L_{-1}|h \rangle & \langle h|L_1L_1L_{-1}L_{-1}|h \rangle \end{pmatrix} =$$

$$= \left| \begin{array}{cc} 4h + \frac{c}{2} & 6h \\ 6h & 8h^2 + 4h \end{array} \right| = 2h(16h^2 + 2hc - 10h + c).$$

These determinants must be non-negative if the module $L(c, h)$ is unitary. (We allow the inner product to be semidefinite but not indefinite.) Thus we need $h \geq 0$ and

$$0 \leq c < 1 - (4h - 1)^2/(2h + 1).$$
The numbers $h_{r,s}$

As another application, we may now see when $M(c, h)$ has a singular vector of level 2. From the above, we must have $0 = \det_2(c, h)$ and so $16h^2 + (2c - 10)h + c$. Solving the quadratic equation for $h$ we must have

$$h = \frac{1}{16} \left(c - 5 \pm \sqrt{(c - 1)(c - 25)}\right).$$

To proceed further the higher Kac determinants are needed. A formula for these was found by Kac (1978). Let

$$h_{r,s}(c) = \frac{1}{48} \left[ (13 - c)(r^2 + s^2) + \sqrt{(c - 1)(c - 25)}(r^2 - s^2) ight]$$

$$-24rs - 2 + 2c].$$
The Kac determinant formula is

$$\det_n(c, h) = K \prod_{\substack{r, s \in \mathbb{N} \\mid 1 \leq rs \leq n}} (h - h_{r,s}(c))^{p(n-rs)}$$

where $K$ is an explicit positive constant. The proof is somewhat difficult and may be found in Kac-Raina Chapters 8 and 12. A first consequence is that the Verma module $M(c, h)$ is irreducible and unitary if $c > 1$ and $h > 0$, the key step being the positivity of all the Kac determinants. If $c = 1$ then $M(c, h)$ is unitary unless $4h$ is a square in $\mathbb{Z}$, and it is always (weakly) unitary. The case $c = 1$ is relevant to some interesting conformal field theories, including the free boson. See Ginsparg Figure 14 for a survey of CFT when $c = 1$. If $c = 0$ only the trivial representation $L(0, 0)$ is unitary.
It is better to revise the notation and write $h_{r,s}$ as a function of a parameter $m$ chosen so that

$$c(m) = 1 - \frac{6}{m(m+1)}$$

and then

$$h_{r,s}(m) = \frac{((m+1)r - ms)^2 - 1}{4m(m+1)}.$$

**Theorem (Friedan, Shenkar, Qiu)**

The module $L(c, h)$ is unitary if and only if $c = c(m)$ with $m$ an integer $\geq 2$ and $h = h_{r,s}(m)$ for some $r, s$ with $1 \leq s \leq r < m$.

Proofs of this deep result were also given by Kac-Wakimoto (independently) and Langlands (later). The Kac-Wakimoto proof is described in Kac-Raina Chapter 12.