Lecture 14: Virasoro Vertex Algebras

Daniel Bump

November 8, 2019
We recall from Lecture 13 that a field $\Phi_a (z, \bar{z})$ is primary if there exist positive constants $\Delta_a$ and $\bar{\Delta}_a$ such that

$$\left[ L_n, \Phi_a (z, \bar{z}) \right] = (n + 1) \Delta_a z^n \Phi_a (z, \bar{z}) + z^{n+1} \frac{\partial}{\partial z} \Phi_a (z, \bar{z}),$$

$$\left[ \bar{L}_n, \Phi_a (z, \bar{z}) \right] = (n + 1) \bar{\Delta}_a \bar{z}^n \Phi_a (z, \bar{z}) + \bar{z}^{n+1} \frac{\partial}{\partial \bar{z}} \Phi_a (z, \bar{z}).$$

Remember from Lecture 13 the two components of the energy-momentum tensor are

$$T(z) = \sum_{n \in \mathbb{Z}} L_n z^{-n-2}, \quad \bar{T}(\bar{z}) = \sum_{n \in \mathbb{Z}} \bar{L}_n \bar{z}^{-n-2}.$$

These are fields (not primary).
Normal Order

We define the normal order for two fields

\[ A(z) = \sum A_n z^{-n}, \quad B(w) = \sum B_m w^{-m} \]

to be

\[ \sum_{n,m} : A_n z^{-n} B_m w^{-m} : \]

where

\[ : A_n z^{-n} B_m w^{-m} : = \begin{cases} B_m A_n z^{-n} w^{-m} & \text{if } -n \geq 0, \\ A_n B_m z^{-n} w^{-m} & \text{otherwise.} \end{cases} \]

(Since the field might be written \( A_n z^{-n-1} \) we note that it is the exponent \( z^{-n} \) that determines the two cases, not the subscript of the operator \( A_n \).)
The operator product expansion

With $\Phi_a$ a primary field, we will prove the operator product expansion

$$T(z) \Phi_a(w, \bar{w}) = \frac{\Delta_a}{(z-w)^2} \Phi_a(w) + \frac{1}{z-w} \frac{\partial}{\partial w} \Phi_a(w) + \: T(z) \Phi_a(w, \bar{w}) : .$$

We have

$$T(z) \Phi_a(w) - \: T(z) \Phi_a(w) : = \sum_{n \leq -2} [L_n, \Phi_a(w)] z^{-n-2} =$$

$$\sum_{n \leq -2} (n + 1) \Delta_a w^n z^{-n-2} \Phi_a(w) + w^{n+1} z^{-n-2} \frac{\partial}{\partial z} \Phi_a(w) .$$

We regard this as an expansion at $z$. Let $k = -2 - n$. This equals

$$-\Delta_a \sum_{k=0}^{\infty} (k + 1) w^{-k-2} z^k + \sum_{k=0}^{\infty} w^{-1-k} z^k \frac{\partial}{\partial z} \Phi_a(w) .$$
Proof: OPE

Both series are convergent when $|z| < |w|$ this equals

$$\frac{\Delta_a}{(z-w)^2} \Phi_a(w) + \frac{1}{z-w} \frac{\partial}{\partial w} \Phi_a(w).$$

Note that although we proved this expansion by summing a power series in $z$, the operator product expansion gives information at $w$. That is, since $\mathcal{T}(z) \Phi_a (w, w)$ is analytic when $z = w$ we may write

$$\mathcal{T}(z) \Phi_a (w, \bar{w}) \sim \frac{\Delta_a}{(z - w)^2} \Phi_a(w) + \frac{1}{z-w} \frac{\partial}{\partial w} \Phi_a(w)$$

and this expands the product of the two operators in terms of local fields at $w$. 
OPE for the Virasoro field

Assume that the Virasoro generator $C$ acts by the scalar $c$ on all fields, including $T(z)$ itself. Then we say that the CFT has **central charge** $c$. In this case a slightly more difficult computation shows that

$$T(z)T(w) = \frac{c}{2} \frac{1}{(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{T'(w)}{z-w} + : T(z)T(w) : ,$$

or

$$T(z)T(w) \sim \frac{c}{2} \frac{1}{(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{T'(w)}{z-w} .$$

This can be deduced from the Virasoro commutation rules

$$[L_n, L_m] = (n-m)L_{n+m} + \delta_{n,-m} \frac{n^3 - n}{12} c ,$$

and indeed this OPE encodes this identity: see Kac **Vertex Algebras for Beginners**, Theorems 2.6 and 4.10.
Conformal vectors

This leads to the notion (due to Borcherds) of a conformal vertex algebra. In addition to Kac, see [FBZ] for this notion.

Let $V$ be a vertex algebra, $\omega \in V$ a vector. Usually we use the notation $Y(v, z) = \sum v(n)z^{-n-1}$ but shift the indices and write

$$Y(v, z) = \sum_{n \in \mathbb{Z}} L_n v^{-n-2}.$$ 

Denote $T(z) = Y(v, z)$ and assume that we have the OPE

$$T(z)T(w) \sim \frac{c}{2} \frac{1}{(z - w)^4} + \frac{2T(w)}{(z - w)^2} + \frac{T'(w)}{z - w}.$$ 

This implies that the $L_n$ generate a Virasoro algebra with central charge $c$. Assume also that $L_{-1} = T$, the translation operator, and that $L_0$ diagonalizable. Then $\omega$ is called a conformal vector and $V$ is called a conformal vertex algebra.
Alternative formulation of the OPE

For a vertex algebra, the identity

\[ T(z)T(w) = \frac{c}{2} \frac{1}{(z - w)^4} + \frac{2T(w)}{(z - w)^2} + \frac{T'(w)}{z - w} + :T(z)T(w):, \]

is equivalent to

\[ [T(z), T(w)] = \frac{c}{12} \partial_w^3 \delta(z - w) + 2T(w) \partial_w \delta(z - w) + T'(w) \delta(z - w). \]

See [FBZ] Lemma 2.5.4, Proposition 3.3.1 and (3.4.2).
Kac’s Lemma on the OPE

In the context of a vertex algebra, the equivalence of the two statements is a lemma due to Kac, that for fields $\phi, \psi$ and $\gamma_j$

$$[\phi(z), \psi(w)] = \sum_{j=0}^{N} \frac{1}{j!} \gamma_j(w) \delta^{(j)}_w \delta(z - w)$$

if and only if

$$\phi(z)\psi(w) = \sum_{j=0}^{N-1} \frac{\gamma_j(w)}{(z - w)^{j+1}} + :\psi(z)\phi(w):.$$

For the proof, see [FBZ] Proposition 3.3.1.
Return to Heisenberg

Let us return to the example of the Heisenberg vertex algebra \( \mathcal{H} \) from Lectures 8 and 9. The Heisenberg Lie algebra is spanned by elements \( b_n \ (n \in \mathbb{Z}) \) and \( \mathbb{1} \) such that

\[
[b_m, b_n] = m\delta_{m,-n}\mathbb{1}.
\]

The Bosonic Fock space is \( B = \mathbb{C}[b_{-1}, b_{-2}, \cdots] \), a subspace of the universal enveloping algebra \( U(\mathcal{H}) \). It is a \( \mathcal{H} \)-module:

If \( n > 0 \) then \( b_{-n} \) acts by multiplication and \( b_n \) acts by \( n\partial / \partial b_{-1} \). We let \( b_0 \) act by 0 and \( \mathbb{1} \) act by 1 on \( B \). The element \( 1 \in B \) (not to be confused with \( \mathbb{1} \)) is the vacuum vector.
Then $B$ has the structure of a vertex algebra, and we had partially proved this. Today we will talk a bit more about the proof. We begin by defining the translation operator $T$ by

$$T(b_{j_1} \cdots b_{j_k}) = - \sum_{i=1}^{k} j_i b_{j_1} \cdots b_{j_i-1} \cdots b_{j_k}.$$ 

We also require that $Y(b_{-k}, z) = b(z)$ where

$$b(z) = \sum_{n \in \mathbb{Z}} b_n z^{-n-1}.$$ 

More generally, we want

$$Y(b_{-k}, z) = \frac{1}{(k-1)!} \partial^{k-1} b(z).$$
Review: Locality of the Heisenberg fields

The fields $b(z), b(w)$ are local due to the operator product expansion

$$b(z)b(w) = \frac{1}{(z-w)^2} + :b(z)b(w):$$

or equivalently

$$[b(z), b(w)] = \partial_w \delta(z-w).$$

It follows that their derivatives $\partial^{k-1}b(z)$ and $\partial^{l-1}b(w)$ are local by differentiating the identity

$$(z-w)^2[b(z), b(w)] = 0$$

to obtain

$$(z-w)^{2+k+l}[\partial_z^k b(z), \partial_w^l b(w)] = 0.$$

At this point, one may complete the construction by invoking a reconstruction theorem which is Theorem 2.3.11 or Theorem 4.4.1 of [FBZ].
Hypotheses of the Reconstruction Theorem

Assume that we have a vector space $V$ with a nonzero vector $|0\rangle$ and a finite or countable collection of vectors with fields

$$a^\alpha(z) = \sum_{n \in \mathbb{Z}} a_{(n)}^\alpha z^{-n-1}$$

such that for all $\alpha$, $a^\alpha(z)|0\rangle \in \text{End}(V)[[z]]$ and

$$a^\alpha(z)|0\rangle|_{z=0} = a^\alpha.$$

It is further assumed that $[T, a^k(z)] = \partial_z a^k(z)$, that the fields $a^k(z), a^l(w)$ are mutually local. Finally we assume the index set $\{\alpha\}$ to be ordered such that $V$ has a basis of vectors

$$a_{(j_1)}^{\alpha_1} \cdots a_{(j_m)}^{\alpha_m}|0\rangle$$

with $j_1 \leq j_2 \leq \cdots \leq j_m < 0$ and such that if $j_i = j_{i+1}$ then $\alpha_i \leq \alpha_{i+1}$.
Reconstruction Theorem

Then the reconstruction theorem asserts that $V$ may be made into a vertex algebra with

$$Y(a_{\alpha_1}^{(j_1)} \cdots a_{\alpha_m}^{(j_m)}|0\rangle, z) =$$

$$\frac{1}{(-j_1 - 1)! \cdots (-j_m - 1)!} : \partial_z^{-j_1 - 1} a_{\alpha_1}(z) \cdots \partial_z^{-j_m - 1} a_{\alpha_m}(z) : .$$

For the Heisenberg Lie algebra, we need only one $\alpha$, and $a^\alpha = b_{-1}$,

$$a^\alpha(z) = b(z) = \sum_{n \in \mathbb{Z}} b_n z^{-n-1}.$$
Let $g$ be a complex Lie algebra that can be written as $g = h \oplus n_+ \oplus n_-$, where $h, n_+, n_-$ are Lie subalgebras with $h$ abelian, such that

$$[h, n_+] \subseteq n_+, \quad [h, n_-] \subseteq n_-.$$ 

We require that

$$[h, n_+] \subset n_+, \quad [h, n_0] \subset n_-.$$ 

This implies that $n_\mu \oplus h$ are Lie algebras, denoted $b$ and $b_-$. 

We assume that $n_\pm$ have weight space decompositions with respect to the adjoint representation under $h$ and that $0$ is not a weight. Moreover we assume there is a closed convex cone $D \subset h^*$ such that $D$ (resp. $-D$) contains the weights of $n_+$ ($n_-$) and that $D \cap (-D) = \{0\}$. 

Let $\Phi_-$ be the set of weights in $\mathfrak{n}_-$ which is an $\mathfrak{h}$-module under the adjoint representation. Let $Q_-$ be the set of finite sums of elements of $\Phi_-$ (with repetitions allowed). This is a discrete subset of $-D$.

The Bernstein-Gelfand-Gelfand (BGG) category $\mathcal{O}$ of modules can be defined for any Lie algebra with triangular decomposition. A module $V$ in this category is assumed to have a weight space decomposition with finite-dimensional weight spaces. Furthermore, it is assumed that there is a finite set of weights $\lambda_1, \cdots, \lambda_N$ such that the weights of $V$ lie in the set

$$\bigcup_i (\lambda_i + Q_-).$$
Review: Highest weight modules

A module $V$ is called a highest weight module with highest weight $\lambda \in \mathfrak{h}^*$ if there is a vector $v \in V(\lambda)$ such that $X \cdot v = 0$ for $X \in \mathfrak{n}_+$, and such that $V = U(\mathfrak{g}) \cdot v$. Since

$$U(\mathfrak{g}) \cong U(\mathfrak{h}) \otimes U(\mathfrak{n}_+) \otimes U(\mathfrak{n}_-)$$

this is equivalent to $V = U(\mathfrak{n}_n) \cdot v$.

Any highest weight module is in Category $\mathcal{O}$. 
Review: Verma modules

There exists a unique highest weight module $M_\lambda$ such that if $V$ is a highest weight module with highest weight $\lambda$ then $V$ is isomorphic to a quotient of $M_\lambda$. To construct $M_\lambda$, note that $\lambda$ extends to a character of $\mathfrak{b} = \mathfrak{h} \oplus \mathfrak{n}_+$ by letting $\mathfrak{n}_+$ act by zero. Let $\mathbb{C}_\lambda$ be $\mathbb{C}$ with this $\mathfrak{b}$-module structure, with generator $1_\lambda$. Define

$$M_\lambda = U(\mathfrak{g}) \otimes_{\mathfrak{b}} \mathbb{C}_\lambda.$$

In view of

$$U(\mathfrak{g}) \cong U(\mathfrak{h}) \otimes U(\mathfrak{n}_+) \otimes U(\mathfrak{n}_-),$$

the map $\xi \mapsto \xi \otimes 1_\lambda$ is a vector space isomorphism $U(\mathfrak{n}_- \rightarrow M_\lambda$. Let $v_\lambda = 1 \otimes 1_\lambda$ be the highest weight element of $M_\lambda$, unique up to scalar.
Lemma

Let $\mathfrak{h}$ be an abelian Lie algebra and let $V$ be a $\mathfrak{h}$-module. We say that $V$ has a weight space decomposition with respect to $\mathfrak{h}$ if

$$V = \bigoplus_{\lambda \in \mathfrak{h}^*} V_\lambda, \quad V_\lambda = \{ v | X \cdot v = \lambda(X)v, \quad X \in \mathfrak{h} \}$$

and the spaces $V_\lambda$ are finite-dimensional.

Lemma

If $V$ has a weight-space decomposition and $U$ is any submodule then $U$ also has a weight space decomposition.

For a proof see Kac-Raina, Corollary 1.1.
Review: Irreducibles

Let $V$ be a highest weight module, for example a Verma module, with highest weight vector $u_\lambda$.

**Lemma**

$V$ has a unique maximal proper submodule.

To prove this, note that a submodule $U$ of $V$ is proper if and only if $u_\lambda \notin V$. Indeed, if $U$ is not proper, then $u_\lambda \in U$, and conversely if $U$ is proper, then $u_\lambda$ cannot be in $U$ because $u_\lambda$ generates $V$. Now every proper submodule $U$ has a weight space decomposition

$$U = \bigoplus_{\mu \neq \lambda} U_\mu.$$

So the sum of the proper submodules has a weight space decomposition not involving $\lambda$, and is therefore proper.
Two triangular decompositions of Vir

There are two noteworthy triangular decompositions of Vir. The one we usually use is $\mathfrak{h} = \mathbb{C}C \oplus \mathbb{C}L_0$,

$$
n_+ = \bigoplus_{n>0} \mathbb{C}L_n, \quad n_- = \bigoplus_{n<0} \mathbb{C}L_n.
$$

The characters of $\mathfrak{h}$ are determined by the eigenvalues $c$ and $h$ of $C$ and $L_0$.

The other triangular decomposition has $\mathfrak{h}' = \mathbb{C}C$,

$$
n'_+ = \bigoplus_{n \geq -1} \mathbb{C}L_n, \quad n'_- = \bigoplus_{n < -1} \mathbb{C}L_n.
$$

To check that $n'_+$ is a Lie algebra, note that $[L_m, L_n] = (m - n)L_{m+n}$ because $\delta_{m,-n} \frac{m^3-m}{12} = 0$ if $m, n \geq -1$. 
Fix \( c \in \mathbb{C}^\times \). We take the Verma module \( M(c) \) for the alternative triangular decomposition with the character \( C \to c \) of \( \mathfrak{h}' \).

Following [FBZ] we want to make this a vertex algebra. The highest weight element becomes the vacuum \( |0\rangle \) and the translation operator \( T = L_{-1} \), so \( T|0\rangle = 0 \). We require

\[
Y(L_{-2}|0\rangle) = T(z) = \sum_{n \in \mathbb{Z}} L_n z^{-n-2}.
\]

Because

\[
[T(z), T(w)] = \frac{c}{12} \partial_w^3 \delta(z - w) + 2T(w) \partial_w \delta_w(z - w) + T'(w) \delta(z - w),
\]

we have \((z - w)^4[T(z), T(w)] = 0\) and therefore \( T(z) \) is local.

The construction of the vertex algebra is concluded using the reconstruction theorem.