Lecture 7: The Kac-Weyl character formula

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Review: s_i permutes $\Phi - \{\alpha_i\}$

The roots Φ may be classified as real and imaginary. A root α is real if $w(\alpha) = \alpha_i$ is a simple root for some $w \in W$. Otherwise, it is imaginary. In Chapter 5 Kac proves a number of important facts about the root system. We will omit these proofs since they can be see by inspection in affine Lie algebras, the case we will ultimately focus on. A real root has norm $|\alpha|^2 = (\alpha|\alpha) > 0$, while an imaginary root has norm $\leqslant 0$. In the affine case, the imaginary roots are $n\delta$ with $0 \neq n \in \mathbb{Z}$, and these are isotropic: $(\delta|\delta) = 0$.

Lemma (Proved in Lecture 6)

If s_i is the simple reflection corresponding to the simple root α_i and if α is a positive root then either $s_i(\alpha)$ is positive or $\alpha = \alpha_i$ and $s_i(\alpha) = -\alpha_i$.

Imaginary roots

This was proved in Lecture 6. Now we deduce some implications.

Proposition

If α is a positive imaginary root so is $w(\alpha)$ for any $\alpha \in W$.

For affine root systems, if α is an imaginary root $n\delta$ then $w(\alpha) = \alpha$ for all $w \in W$. This is not true for other Kac-Moody root systems, but at least we know that the positive imaginary roots are permuted by the Weyl group. This follows from the Lemma, since α_i is not an imaginary root.

The length function on W

We now define the length function on the Weyl group. If $w \in W$ let

$$\ell(w) = |\{\alpha \in \Phi^+ | w^{-1}(\alpha) \in \Phi^-\}|.$$

Note that this also equals $|\{\alpha \in \Phi^+ | w(\alpha) \in \Phi^-\}|$ since $\alpha \mapsto -w^{-1}(\alpha)$ is a bijection

$$\{\alpha \in \Phi^+ | w^{-1}(\alpha) \in \Phi^-\} \quad \longrightarrow \quad \{\alpha \in \Phi^+ | w(\alpha) \in \Phi^-\}.$$

Thus $\ell(w) = \ell(w^{-1})$.

Proposition

We have

$$\ell(s_i w) = \begin{cases} \ell(w) + 1 & \textit{if } w^{-1}(\alpha_i) \in \Phi^+, \\ \ell(w) - 1 & \textit{if } w^{-1}(\alpha_i) \in \Phi^-. \end{cases}$$

Proof

To prove this, suppose that $w^{-1}(\alpha_i) \in \Phi^+$. Then it is easy to show that

$$\{\alpha \in \Phi^+ | s_i w(\alpha) \in \Phi^-\} = \{\alpha \in \Phi^+ | w(\alpha) \in \Phi^-\} \cup \{w^{-1}(\alpha_i)\}$$

and this union is disjoint. For example, to show that the left-hand side is contained in the right, assume that $s_i w(\alpha) \in \Phi^-$ but $w(\alpha) \notin \Phi^-$. Then since α_i is the only element of Φ^+ mapped to Φ^- by s_i we have $w(\alpha) = \alpha_i$ and so α is indeed contained in the right-hand side. We leave the other inclusion to the reader.

This shows that if $w^{-1}(\alpha_i) \in \Phi^+$ then $\ell(s_i w) = \ell(w) + 1$. To prove the other inclusion, interchange the roles of w and $s_i w$, and apply the case just proved to $s_i w$.

Functorialty of reflections

Lemma

Let α_i and α_j be simple roots, $w \in W$. If $w(\alpha_i) = \alpha_j$ then $w s_i w^{-1} = s_j$.

We make use of the definition $s_i(x) = x - \langle \alpha_i^\vee, x \rangle \alpha_i$. So

$$w s_i w^{-1}(x) = w(w^{-1}(x) - \langle \alpha_i^\vee, w^{-1}(x) \rangle \alpha_i)$$

$$= x - \langle w(\alpha_i^\vee), x \rangle w(\alpha_i) = x - \langle \alpha_j^\vee, x \rangle \alpha_j = s_j(x).$$

Discarding redundant elements of a decomposition

It is known that Kac-Moody Weyl groups are Coxeter groups. We won't prove that but the following Proposition is a well-known property of Coxeter groups

Proposition

Suppose $w = s_{i_1} \cdots s_{i_k}$ *and* $\ell(w) < k$*. Then there exists* $1 \leqslant m < n \leqslant k$ *such that*

$$w = s_{i_1} \cdots \hat{s}_{i_m} \cdots \hat{s}_{i_n} \cdots s_{i_k},$$

where the hats mean the two elements are omitted.

Let m be the largest integer such that $\ell(s_{i_{m+1}} \cdots s_{i_k}) = k - m$. Then $\ell(s_{i_m} \cdots s_{i_k}) \neq k - m + 1$, so by the Proposition $\ell(s_{i_m} \cdots s_{i_k}) = k - m - 1$ and $(s_{i_{m+1}} \cdots s_{i_k})^{-1}(\alpha_{i_m}) \in \Phi^-$.

Proof (continued)

Now let n be the smallest integer such that $(s_{i_{m+1}} \cdots s_{i_n})^{-1}(\alpha_{i_m}) \in \Phi^-$. Then $(s_{i_{m+1}} \cdots s_{i_{n-1}})^{-1}(\alpha_{i_m}) \in \Phi^+$. Then $\beta = (s_{i_{m+1}} \cdots s_{i_{n-1}})^{-1}(\alpha_{i_m})$ is a positive root such that $s_{i_n}(\beta)$ is in Φ^-. There is a unique such positive root, and so $\beta = \alpha_{i_n}$. Now by the Lemma

$$(s_{i_{m+1}} \cdots s_{i_{n-1}})^{-1} s_{i_m} (s_{i_{m+1}} \cdots s_{i_{n-1}}) = s_{i_n}.$$

This implies that

$$s_{i_1} \cdots s_{i_k} = s_{i_1} \cdots \hat{s}_{i_m} \cdots \hat{s}_{i_n} \cdots s_{i_k}.$$

Reduced decompositions

If $w \in W$ we may write it as a product of simple reflections:

$$w = s_{i_1} \cdots s_{i_k}.$$

If this decomposition is as short as possible, it is reduced.

Proposition

The length $\ell(w)$ is the number of elements in a reduced expression.

Let $w = s_{i_1} \cdots s_{i_k}$ be a reduced decomposition of w. Since $\ell(s_i w) = \ell(w) \pm 1 \leqslant \ell(w) + 1$ it is clear that $\ell(w) \leqslant k$. We must prove the converse. Let $w = s_{j_1} \cdots s_{j_{k'}}$ be any expression for w, reduced or not. If $k' > \ell(w)$ the last Proposition shows that we may discard elements 2 at a time, until we arrive at an expression of with $\leqslant \ell(w)$.

Primitive vectors

We recall the definition from Lecture 5.

Definition

Let V be a module with a weight space decomposition. If μ is a weight of V, a nonzero vector $v \in V(\mu)$ is called a primitive vector if there exists a submodule U of V such that $v \notin U$ but $e_i v \in U$ for all i.

Particularly, if $e_i(v) = 0$ for all i, then v is a primitive vector.

Functoriality properties of primitive vectors

Lemma

Let

$$0 \longrightarrow V' \xrightarrow{i} V \xrightarrow{p} V'' \longrightarrow 0$$

be a short exact sequence in Category-$\mathcal{O}$.
(i) If v' is a primitive vector in V' then $i(v')$ is a primitive vector in V.
(ii) If v'' is a primitive vector in V'' and $v \in V$ is a preimage of v'' then v is a primitive vector in V.
(iii) If v is a primitive vector in V then either $v = i(v')$ where v' is a primitive vector in V', as in (i), or $p(v) = v''$ is a primitive vector in V''.

The proof is straightforward from the definition of a primitive vector.

A module is generated by primitive vectors

Proposition

Let V be a module in Category $\mathcal{O}$. Then V is generated by its primitive vectors.

Let U be the submodule of V generated by its primitive vectors and let $W = V/U$. If W is nonzero, let μ be a weight that is maximal with respect to $\succcurlyeq$ such that $W_\mu \neq 0$. If u is a nonzero vector in W_μ, then obviously u is a primitive vector in W so if v is a preimage in V, then v is a primitive vector that is not in U, which is a contradiction.

Input from the Casimir operator

Proposition

Let V be a highest weight module with highest weight λ. Then the Casimir element Ω acts by the scalar $(\lambda + 2\rho|\lambda) = \|\lambda + \rho\|^2 - \|\rho^2\|$ on V.

This was proved in Lecture 6.

Subquotients

If U and V are modules in Category $\mathcal{O}$ we say U is a subquotient of V if V has submodule $0 \subseteq V_1 \subseteq V_2 \subseteq V$ such that $U \cong V_2/V_1$.

Proposition

Let V and U be highest weight modules with highest weights λ and μ, respectively. Suppose U is a subquotient of V. Then $\|\lambda + \rho\|^2 = \|\mu + \rho^2\|$ and $\mu \preccurlyeq \lambda$.

Since Ω acts by the scalar $\|\lambda + \rho\|^2 - \|\rho\|^2$ on V, it acts by the same scalar on any subquotient. Thus $\|\lambda + \rho\|^2 - \|\rho\|^2 = \|\mu + \rho\|^2 - \|\rho\|^2$. Moreover writing $U \cong V_2/V_1$ for $V_1 \subset V_2 \subseteq V$, the module V_2 has a vector of weight μ and so $V_\mu \neq 0$, which implies $\mu \preccurlyeq \lambda$.

A completion

Recall that $\lambda \in \mathfrak{h}^*$ is an integral weight if all $\langle \alpha_i^\vee, \lambda \rangle \in \mathbb{Z}$. The integral weights form the weight lattice P. If furthermore all $\langle \alpha_i^\vee, \lambda \rangle \geqslant 0$, then λ is called a dominant weight. The dominant weights form a cone $P^+ \subset P$.

Let $\mathcal{E}$ be the free abelian group on P; if $\lambda \in P$ we will denote e^λ the corresponding basis element. We wish to consider a completion $\hat{\mathcal{E}}$ that will contain the characters of representations of Category $\mathcal{O}$. If $f = \sum a_\mu e^\mu \in \mathcal{E}$ let $\mathrm{supp}(f) = \{\mu | a_\mu \neq 0\}$. We will say a sequence $\{f_n\}$ with $f_n \in \mathcal{E}$ converges to 0 if for any $\lambda \in P$ there is an N such that if $n > N$ and $\mu \succcurlyeq \lambda$ then $\mu \notin \mathrm{supp}(f_n)$.

The character

With this topology, the completion $\hat{\mathcal{E}}$ contains (for example)

$$\sum_{n=0}^{\infty} e^{-n\alpha}$$

if α is a positive root and

$$\sum_{n=0}^{\infty} e^{-n\alpha} = (1 - e^{-\alpha}).$$

If V is a module in Category $\mathcal{O}$, then we may define the character

$$\text{ch } V = \sum_{\mu} \dim(V_\mu) e^{\mu}$$

and this series is convergent in $\hat{\mathcal{E}}$.

The character of an irreducible

We proved that if $\lambda \in P^+$ then $L(\lambda)$ is an integrable representation, so its character

$$\mathrm{ch}\ L(\lambda) = \sum_{\mu \in \mathfrak{h}^*} \dim(L(\lambda)_\mu) e^\mu$$

is invariant under the Weyl group W.

The character of a Verma module

The other thing we know is the character of Verma module:

$$\mathrm{ch}\ M(\lambda) = e^{\lambda} \prod_{\alpha \in \Phi^+} (1 - e^{-\alpha})^{-1}.$$

Formally, this is because if $v_\lambda \in M(\lambda)$ is a highest weight vector, then the map $U(\mathfrak{n}_-) \longrightarrow M(\lambda)$ in which $\xi \in U(\mathfrak{n}_-)$ maps to $\xi \cdot v_\lambda$ is a vector space isomorphism, and the character formula

$$\mathrm{ch}\ U(\mathfrak{n}_-) = \prod_{\alpha} \sum_{k_\alpha = 0}^{\infty} e^{-k_\alpha \alpha} = \prod_{\alpha \in \Phi^+} (1 - e^{-\alpha})^{-1}$$

follows from the PBW theorem.

The Kac-Weyl Character formula

A proof of the Weyl character formula for a finite-dimensional simple Lie algebra $\mathfrak{g}$ was given by Bernstein, Gelfand and Gelfand using Category $\mathcal{O}$ methods. Kac adapted their method to the symmetrizable Kac-Moody case. Critical input comes from the center of the universal enveloping algebra. While BGG used the entire center, Kac showed that for the generalizationof the Weyl character formula to the infinite-dimensional case, only the Casimir element is really needed.

Web link:

BGG: Category of $\mathfrak{g}$-modules from Joseph Bernstein's home page

Restriction to finite-dimensional Lie algebras

In this section we will assume that $\mathfrak{g}$ is a finite-dimensional simple Lie algebra. In this case, the Cartan matrix A is positive definite, and it follows that the inner product $(\,|\,)$ on $\mathfrak{h}$, or equivalently on $\mathfrak{h}^*$ is positive definite.

Proposition

Let $\mathfrak{g}$ be a finite-dimensional simple Lie algebra and V a highest weight module with highest weight λ. Then V has finite length. That is, it has a composition series

$$0 = V_0 \subset V_1 \subset \cdots \subset V_n = V$$

where the quotients V_i/V_{i+1} are irreducible. Moreover each $V_i/V_{i+1} \cong L(\mu_i)$ with $\mu_i \preccurlyeq \lambda$ and

$$\|\mu_i + \rho\| = \|\lambda_i + \rho\|.$$

Proof

If V is a nonzdero module in Category $\mathcal{O}$ then V has a maximal proper submodule. We proved this in Lecture 1 if V is a highest weight module, and it is not to hard to generalize the argument.

Let $V^0 = V$ and let V^1 be a maximal proper submodule of V; let V^2 be a maximal proper submodule of V^1, and so forth. Thus we have a chain

$$V = V^0 \supset V^1 \supset V^2 \supset \cdots$$

with quotients V^i/V^{i+1} irreducible, and we need to know that it terminates. The subquotients V^i/V^{i+1} are irreducible so they are of the form $L(\mu_i)$, and from what we have already proved, $\|\mu_i + \rho\|^2 = \|\lambda + \rho\|^2$. Now since $\mathfrak{g}$ is a finite-dimensional simple Lie algebra the form $(\ |\ )$ is positive definite there are only finitely many possible μ_i.

Proof (concluded)

Furthermore V_{μ_i} is finite-dimensional, so the number of subquotients is bounded by

$$\sum_{\substack{\mu_i \in \mathfrak{h}^* \\ \|\mu_i+\rho\|^2=\|\lambda+\rho\|^2}} \dim(V_{\mu_i}) < \infty.$$

The transition matrix

We apply this to Verma modules and write

$$\mathrm{ch}\ M(\lambda) = \sum_{\substack{\mu \preccurlyeq \lambda \\ \|\mu+\rho\|^2 = \|\lambda+\rho\|^2}} d_{\lambda,\mu}\ \mathrm{ch}\ L(\mu).$$

Here $d_{\lambda,\mu}$ is the number of times $L(\mu)$ occurs among the composition factors of $M(\lambda)$. The matrix $d_{\lambda,\mu}$ of nonnegative integers is triangular in the sense that

$$d_{\lambda,\lambda} = 1, \qquad d_{\lambda,\mu} = 0 \text{ unless } \mu \preccurlyeq \lambda.$$

This implies that it is invertible.

Expansion in Verma characters

Inverting the transition matrix, we may write

$$\mathrm{ch}\ L(\lambda) = \sum_{\substack{\mu \preccurlyeq \lambda \\ \|\mu+\rho\|^2 = \|\lambda+\rho\|^2}} c_{\lambda,\mu}\, \mathrm{ch}\ M(\mu)$$

for $c_{\lambda,\mu} \in \mathbb{Z}$, with

$$c_{\lambda,\lambda} = 1, \qquad c_{\lambda,\mu} = 0 \text{ unless } \mu \preccurlyeq \lambda.$$

Then remember that the map $U(\mathfrak{n}_-) \longrightarrow M(\lambda)$ in which $\xi \mapsto \xi \cdot v_\lambda$ is an isomorphism (with v_λ the highest weight vector). Using the PBW theorem this gives

$$\mathrm{ch}\ M(\mu) = e^{\mu} \prod_{\alpha \in \Phi+} (1 - e^{-\alpha})^{-1}.$$

The Weyl denominator

Now let us introduce the Weyl denominator

$$\Delta = e^{\rho} \prod_{\alpha \in \Phi^+} (1 - e^{-\alpha}).$$

Lemma

If $w \in W$ then $w(\Delta) = (-1)^{\ell(w)} \Delta$.

It is enough to prove this for simple reflections so assume $w = s_i$. Write

$$\Delta = e^{\rho}(1 - e^{-\alpha_i}) \prod_{\substack{\alpha \in \Phi^+ \\ \alpha \neq \alpha_i}} (1 - e^{-\alpha}).$$

Proof (continued)

Since s_i permutes the factors in the product,

$$s_i(\Delta) = e^{s_i\rho}(1-e^{-s_i\alpha_i}) \prod_{\substack{\alpha\in\Phi^+ \\ \alpha\neq\alpha_i}} (1-e^{-\alpha}) = e^{\rho-\alpha_i}(1-e^{\alpha_i}) \prod_{\substack{\alpha\in\Phi^+ \\ \alpha\neq\alpha_i}} (1-e^{-\alpha})$$

which equals $-\Delta$.

Back to the character

Now we may write:

$$\begin{aligned}
\operatorname{ch} L(\lambda) &= \sum_{\substack{\mu \preccurlyeq \lambda \\ \|\mu+\rho\|^2 = \|\lambda+\rho\|^2}} c_{\lambda,\mu} \operatorname{ch} M(\mu) \\
&= \sum_{\substack{\mu \preccurlyeq \lambda \\ \|\mu+\rho\|^2 = \|\lambda+\rho\|^2}} c_{\lambda,\mu} e^{\mu} \prod_{\alpha \in \Phi^+} (1 - e^{-\alpha})^{-1} \\
&= \Delta^{-1} \sum_{\substack{\mu \preccurlyeq \lambda \\ \|\mu+\rho\|^2 = \|\lambda+\rho\|^2}} c_{\lambda,\mu} e^{\mu+\rho}.
\end{aligned}$$

We may learn something about the $c_{\lambda,\mu}$ from the W-invariance of $L(\lambda)$ and anti-invariance of Δ

Using the W-invariance of $L(\lambda)$

Since $\lambda \in P^+$, $\mathrm{ch}\ L(\lambda)$ is invariant under W and Δ is anti-invariant:

$$\Delta^{-1} \sum_{\substack{\mu \preccurlyeq \lambda \\ \|\mu+\rho\|^2 = \|\lambda+\rho\|^2}} c_{\lambda,\mu} e^{\mu+\rho}$$

$$= (-1)^{\ell(w)} \Delta^{-1} \sum_{\substack{\mu \preccurlyeq \lambda \\ \|\mu+\rho\|^2 = \|\lambda+\rho\|^2}} c_{\lambda,\mu} e^{w(\mu+\rho)}.$$

Making the variable change $\mu \mapsto w(\mu+\rho) - \rho = w \cdot \mu$

$$c_{\lambda,\mu} = (-1)^{\ell(w)} c_{\lambda, w\cdot\mu}.$$

Using information from the Casimir element

Proposition

If $c_{\lambda,\mu} \neq 0$ *then* $w \cdot \mu = \lambda$ *for some* $w \in W$. *We have*

$$c_{\lambda, w\cdot\lambda} = (-1)^{\ell(w)}.$$

Find w so that $w(\mu+\rho) \in P^+$. Let $\mu' = w \cdot \mu = w(\mu+\rho) - \rho$. Then $c_{\lambda,\mu'} = \pm c_{\lambda,\mu} \neq 0$ so $\mu' \preccurlyeq \lambda$ and

$$\|\mu'+\rho\|^2 = \|w(\mu+\rho)\|^2 = \|\mu+\rho\|^2 = \|\lambda+\rho\|^2.$$

From the identity $\|a\|^2 - \|b\|^2 = (a+b|a-b)$ we have

$$\|\lambda+\rho\|^2 - \|\mu'+\rho\|^2 = (\lambda + w(\mu+\rho) + \rho | \lambda - \mu').$$

Proof (continued)

Now λ and $w(\mu+\rho)$ are both dominant, so $\lambda + w(\mu+\rho)+\rho$ is strongly dominant in the sense that

$$(\lambda + w(\mu+\rho)+\rho|\alpha_i) > 0$$

for all simple roots α_i. Since $\lambda - \mu' \succcurlyeq 0$ we may write $\lambda - \mu' = \sum k_i\alpha_i$ where $k_i \geqslant 0$. Thus the identity

$$\|\lambda+\rho\|^2 - \|\mu'+\rho\|^2 = 0$$

implies all $k_i = 0$ and hence $\mu' = \lambda$.

The Weyl character formula

Theorem (Weyl Character Formula)

Suppose that $\mathfrak{g}$ is a finite-dimensional simple Lie algebra and λ is a dominant weight. Then

$$\mathrm{ch}\ L(\lambda) = \Delta^{-1} \sum_{w \in W} (-1)^{\ell(w)} e^{w(\lambda+\rho)}.$$

Indeed, we already proved

$$\mathrm{ch}\ L(\lambda) = \Delta^{-1} \sum_{\substack{\mu \preccurlyeq \lambda \\ \|\mu+\rho\|^2 = \|\lambda+\rho\|^2}} c_{\lambda,\mu} e^{\mu+\rho}.$$

However we also proved that the only μ that appear in the summation are those of the form $\mu = w \cdot \lambda$ so $\mu + \rho = w(\lambda + \rho)$, and for these $c_{\lambda,\mu} = (-1)^{\ell(w)}$

An approximate composition series

Now we turn to the Kac-Moody case. The arguments must be modified, but only slightly.

Proposition (Lemma 9.6 in Kac)

Let V be in Category $\mathcal{O}$ and let $\nu \in \mathfrak{h}^*$. Then V admits a partial composition series with respect to ν, meaning a filtration

$$0 = V_0 \subset V_1 \subset \cdots \subset V_n = V$$

such that each quotient V_i/V_{i-1} is either $\cong L(\mu_i)$ for some $\mu_i \succcurlyeq \nu$, or else $(V_i/V_{i-1})_\mu = 0$ for all $\mu \succcurlyeq \nu$.

Proof (continued)

To prove this, we argue by induction on

$$a(V,\nu) = \sum_{\mu \succcurlyeq \nu} \dim V_\mu.$$

If $a(V,\nu) \neq 0$ then we may choose a maximal weight $\mu \in \mathrm{supp}(V)$ such that $V_\mu \neq 0$. Then $e_i\mu = 0$ for all i so a nonzero $v \in V_\mu$ generates a highest weight module U. Then U has a maximal proper submodule U' (Lecture 1) and $U/U' \cong L(\mu)$. Thus we have

$$0 \subset U' \subset U \subset V$$

and $a(U',\nu)$, $a(V/U,\nu)$ are strictly smaller than $a(V,\nu)$. By induction these modules have partial composition series with respect to ν, and patching these together we obtain the result.

The Kac-Weyl character formula

Generalizing the Weyl character formula:

Theorem (Kac (see Chapter 10))

Suppose that $\mathfrak{g}$ is a finite-dimensional simple Lie algebra and λ is a dominant weight. Then

$$\text{ch } L(\lambda) = \Delta^{-1} \sum_{w \in W} (-1)^{\ell(w)} e^{w(\lambda+\rho)}.$$

We recall how we argued for the finite-dimensional case. The first step was to note that the composition series for $M(\lambda)$ all the irreducible subquotients were of the form $L(\mu)$ where $\mu \preccurlyeq \lambda$ and $\|\mu + \rho\|^2 = \|\lambda + \rho\|^2$, so

$$\text{ch } M(\lambda) = \sum_{\substack{\mu \preccurlyeq \lambda \\ \|\mu+\rho\|^2 = \|\lambda+\rho\|^2}} d_{\lambda,\mu} \text{ ch } L(\mu).$$

Adapting the argument

In the Kac-Moody case, this identity is also true, except that the sum may be infinite. To prove it, define $[M(\lambda) : L(\mu)]$ to be the number of times $L(\mu)$ appears as a subquotient in the partial composition series for any $\nu \preccurlyeq \mu$. This is easily seen to be independent of ν. Since $L(\mu)$ is a subquotient of $M(\lambda)$ we have $\mu \preccurlyeq \lambda$ and $\|\mu + \rho\|^2 = \|\lambda + \rho\|^2$ and

$$\mathrm{ch}\ M(\lambda) = \sum_{\substack{\mu \preccurlyeq \lambda \\ \|\mu+\rho\|^2 = \|\lambda+\rho\|^2}} [M(\lambda) : L(\mu)]\ \mathrm{ch}\ L(\mu).$$

The remainder of the proof is the same as in the finite-dimensional case.

Primitive vectors in $M(0)$

In this section we will prove:

Proposition

Let $\mathfrak{g}$ be a Kac-Moody Lie algebra with Weyl group W. For any $w \in W$, $M(0)$ has a primitive vector of weight $w(\rho) - \rho$.

This is Exercise 10.3 in Kac. If W is infinite, the result shows that the $M(0)$ has an infinite number of primitive vectors. So it is not a module of finite length.

An expression for $\rho - w(\rho)$

.

Lemma

We have

$$\rho - w(\rho) = \sum_{\substack{\alpha\in\Phi^+ \\ w^{-1}(\alpha)\in\Phi^-}} \alpha.$$

This can be proved by induction on $\ell(w)$. If $w = 1$ both sides are 0. If the identity is true for w then we will prove that it is true for $s_i w$, where $s_i w > w$. We have

$$\rho - s_i w\rho = (\rho - s_i\rho) + s_i(\rho - w\rho) = \alpha_i + s_i(\rho - w\rho)$$

and since $\langle \alpha_i^\vee, \rho\rangle = 1$ we have $s_i\rho = \rho - \langle\alpha_i^\vee, \rho\rangle\alpha_i = \rho - \alpha_i$.

Proof (continued)

So

$$\alpha_i + s_i \left(\sum_{\substack{\alpha \in \Phi^+ \\ w^{-1}(\alpha) \in \Phi^-}} \alpha \right)$$

It is not hard to check that

$$\{\alpha \in \Phi^+ | (s_i w)^{-1} \alpha \in \Phi^-\} = \{\alpha_i\} \cup \{s_i \alpha | w^{-1} \alpha \in \Phi^-\}$$

and that the union is disjoint, so have proved the formula.

Now we may prove the proposition. We will find primitive vectors $v_{w\rho-\rho}$ each of weight $w\rho - \rho$ that are primitive in the strong sense that $e_i(v_{w\rho-\rho}) = 0$ for all i. If $w = 1$ we may take v_0 to be the highest weight vector, so suppose that $v_{w\rho-\rho}$ is constructed and suppose $\ell(s_i w) > \ell(w)$. We will construct $v_{s_i w\rho-\rho}$.

Proof (continued)

We need to know that

$$\langle \alpha_i^\vee, w\rho - \rho \rangle \geqslant 0.$$

Indeed this equals

$$\langle \alpha_i^\vee, w\rho \rangle - \langle \alpha_i^\vee, \rho \rangle = \langle \alpha_i^\vee, w\rho \rangle - 1.$$

Since $w\rho \in P$ (the weight lattice) $\langle \alpha_i^\vee, w\rho \rangle \in \mathbb{Z}$ and we need to show that it is positive. This equals $\langle w^{-1}(\alpha_i^\vee), \rho \rangle$ and $w^{-1}(\alpha_i)$ is a positive root since $\ell(s_i w) > \ell(w)$. So $w^{-1}(\alpha_i^\vee)$ is the coroot associated with a positive root and its inner product with ρ is positive.

Use of $\mathrm{SL}(2)$ theory

Let $k = \langle \alpha_i^\vee, w\rho - \rho \rangle \geqslant 0$ and let $v_{s_i w\rho - \rho} = f_i^{k+1} v_{w\rho-\rho}$.

We will show that $v_{s_i w\rho-\rho}$ is a primitive vector. By the $\mathrm{SL}(2)$ theory $e_i v_{s_i\rho-\rho} = e_i f_i^{k+1} v_{w\rho-\rho} = 0$. Also $e_j f_i^{k+1} v_{s_i\rho-\rho} = 0$ since $e_j f_i = 0$. Hence $v_{s_i w\rho-\rho}$ is a primitive vector of weight $s_i w\rho - \rho$.

Its weight is

$$w\rho - \rho - \alpha_i^{k+1} = w\rho - \rho - \langle \alpha_i^\vee, w\rho - \rho \rangle - \alpha_i$$

$$s_i(w\rho - \rho) - (\rho - s_i\rho) = s_i w\rho - \rho,$$

as required.