Lecture 6: The Casimir operator

Daniel Bump

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The invariant inner product and the Casimir operator

The text for today's lecture is Kac, Chapter 2. We will construct an invariant bilinear form $(\ |\ )$ on a Kac-Moody Lie algebra, then apply it to construct the important Casimir operator on representations of Category $\mathcal{O}$, and prove facts about it that lead to a proof of the Kac-Weyl character formula for characters of integrable highest weight representations.

For example, the Casimir operator acts on irreducible highest weight representations as a scalar, which we can compute. This will be of use in locating primitive vectors, among other things.

The center of $U(\mathfrak{g})$

Let us start with a finite-dimensional semisimple Lie algebra $\mathfrak{g}$ of rank r. The center of $U(\mathfrak{g})$ acts by scalars in any irreducible module. Harish-Chandra proved that $Z(U(\mathfrak{g}))$ is a polynomial ring in r variables. The generator of lowest degree is the Casimir element, of degree 2.

If $\mathfrak{g}$ is the Lie algebra of a Lie group G, the Lie algebra $\mathfrak{g}$ can be understood as vector fields on G. Since we can differentiate along a vector field, they are linear differential operators. So $U(\mathfrak{g})$ can be understood as the ring of all differential operators that are invariant under left translation. Its center $Z(U(\mathfrak{g}))$ is the ring of differential operators invariant under both left and right translation.

The Casimir operator

Let $(\mid)$ be an invariant bilinear form on $\mathfrak{g}$, which must be the Killing form up to normalization. Let X_i be a basis of $\mathfrak{g}$ and let Y_i be the dual basis. Then we may define

$$\Omega = \sum_{i=1}^{\dim(\mathfrak{g})} X_i Y_i \in U(\mathfrak{g}).$$

Proposition

The definition of Ω is independent of the choice of basis X_i, and $\Omega \in Z(U(\mathfrak{g}))$.

Proof

We omit the verification that Ω is unchanged if we change basis. To check that Ω is central, suppose that $T \in \mathfrak{g}$ and write

$$[T, X_i] = \sum_j c_{ij} X_j, \qquad [T, Y_j] = \sum_i d_{ij} Y_i$$

We have

$$([T, X_i]|Y_j) = \sum_k c_{ik}(X_k|Y_j) = c_{ij},$$

and similarly

$$(X_i|[T, Y_j]) = \sum d_{ij}.$$

Since the form $(\ |\ )$ is invariant, $([T, X_i]|Y_j) = -(X_i|[T, Y_j])$. Thus $c_{ij} = -d_{ij}$. Now

$$[T, \Omega] = \sum_i [T, X_i] Y_i + \sum X_i [T, Y_i] = \sum c_{ij} X_j Y_i + d_{ji} X_i Y_j = 0.$$

Weight spaces dually paired by $(\ |\ )$

Proposition

Let $\mathfrak{g}$ be a Lie algebra with a maximal abelian subgroup $\mathfrak{h}$. Assume that $\mathfrak{g}$ has a weight space decomposition

$$\mathfrak{g} = \bigoplus_{\alpha \in \mathfrak{h}^*} \mathfrak{g}_\alpha .$$

Assume that $(\ |\ )$ is an invariant bilinear form. Then if $X \in \mathfrak{g}_\alpha$ and $Y = \mathfrak{g}_\beta$ we have $(X|Y) = 0$ unless $\alpha = -\beta$.

Indeed, if $H \in \mathfrak{h}$ we have

$$\alpha(H)(X|Y) = ([H,X]|Y) = -(X|[H,Y]) = -\beta(H)(X|Y),$$

so if $(X|Y) \neq 0$ we must have $\alpha(H) = -\beta(H)$.

The isomorphism $\nu : \mathfrak{h} \to \mathfrak{h}^*$

It will be useful to describe H_α (for all positive roots α) another way. There is a homomorphism $\nu : \mathfrak{h} \longrightarrow \mathfrak{h}^*$ associated to the inner pairing $(\ |\ )$ by

$$\langle H, \nu(H') \rangle = (H|H'), \qquad \langle H, \lambda \rangle = \lambda(H) = (H|\nu^{-1}(\lambda)).$$

We may then define an inner product on $\mathfrak{h}^*$ by requiring ν to be an isometry. That is,

$$(\nu(H)|\nu(H)) = (H|H').$$

The scalar value of Ω

Our goal is to prove:

Theorem

Let V be an irreducible finite-dimensional module of the finite-dimensional simple Lie algebra $\mathfrak{g}$ with highest weight $\lambda \in \mathfrak{h}^*$. Then Ω acts on V by the scalar $(\lambda|\lambda + 2\rho)$.

To prove this we will first describe dual bases of $\mathfrak{g}$. When α is a root, $\mathfrak{X}_\alpha$ $(= \mathfrak{g}_\alpha)$ is one dimensional, and if the basis $\{X_i\}$ of $\mathfrak{g}$ is chosen so that each vector X_i lies in a root space, there will be one vector in each $\mathfrak{X}_\alpha$ and r vectors in $\mathfrak{g}_0 = \mathfrak{h}$. Let us denote the vectors in $\mathfrak{X}_\alpha$ by X_α, and the vectors in $\mathfrak{h}$ by H_i.

Normalizations

It will be convenient to adjust the $\mathfrak{X}_\alpha$ so that $(X_\alpha|X_{-\alpha}) = 1$. Also, the form $(\ |\ )$ restricts to a nondegenerate pairing of $\mathfrak{h}$, so let H^i be the dual basis of $\mathfrak{h}$ such that $(H_i|H^j) = \delta_{ij}$.
Now our dual bases X_i and Y_i of $\mathfrak{g}$ may be chosen to be $\{X_{-\alpha}, H_i\}$ and $\{X_\alpha, H^i\}$. Thus the Casimir operator is

$$\Omega = \sum_{\alpha\in\Phi} X_{-\alpha}X_\alpha + \sum_{i=1}^{r} H_i H^i.$$

We will also denote $H_\alpha = [X_\alpha, X_{-\alpha}]$.

Computation of H_α

Lemma

We have $H_\alpha = \nu^{-1}(\alpha)$.

To check this, we use the fact that the inner product $(\ |\ )$ is invariant. We have

$$(H|[X_\alpha, X_{-\alpha}]) = ([H, X_\alpha]|X_{-\alpha}) = \alpha(H)(X_\alpha|X_{-a}) = \alpha(H).$$

Now by the definition of ν, this proves that $H_\alpha = [X_\alpha, X_{-\alpha}] = \nu^{-1}(\alpha)$.

Another version of Ω

Now we may rewrite

$$\Omega = \sum_{\alpha\in\Phi} X_{-\alpha}\cdot X_\alpha + \sum_{i=1}^{r} H_i H^i$$

It is useful to write this in another form using

$$[X_\alpha, X_{-\alpha}] = H_\alpha = \nu^{-1}(\alpha)$$

by the Lemma, and applying this to those α that are negative roots:

$$\Omega = \sum_{\alpha\in\Phi^+} 2X_{-\alpha}\cdot X_\alpha + \sum_{i=1}^{r} H_i H^i + \sum_{\alpha\in\Phi^+} [X_\alpha, X_{-\alpha}]$$

or:

$$\boxed{\Omega = \sum_{\alpha\in\Phi^+} 2X_{-\alpha}\cdot X_\alpha + \sum_{i=1}^{r} H_i H^i + 2\nu^{-1}(\rho).}$$

Proof of the Theorem

This form is useful since it allows us to prove the Theorem, by computing the eigenvalue of Ω in an irreducible representation V_λ of highest weight λ.

Let $v_\lambda \in V_\lambda$ be the highest weight vector. Then $X_\alpha v_\lambda = 0$ for every positive root, while if $X \in \mathfrak{h}$ we have $Hv_\lambda = \lambda(H)v_\lambda$. So

$$\Omega v_\lambda = cv_\lambda, \qquad c = \sum_{i=1}^{r} \lambda(H_i)\lambda(H^i) + 2\langle \nu^{-1}(\rho), \lambda \rangle$$

Noting $2\langle \nu^{-1}(\rho)|\lambda \rangle = 2(\lambda|\rho)$, the theorem will follow from:

Lemma

If $\lambda, \mu \in \mathfrak{h}^*$ *then*

$$\sum_{i=1}^{r} \lambda(H_i)\mu(H^i) = (\lambda|\mu).$$

Proof (continued)

To prove this start with the formula (for $H, H' \in \mathfrak{h}$):

$$(H|H') = \sum_i (H|H_i)(H'|H^i).$$

Indeed we have the orthogonal expansion

$$H = \sum (H|H_i)H^i.$$

Then we take the inner product of this expression of H'. Now we have

$$(\lambda|\mu) = (\nu^{-1}(\lambda)|\nu^{-1}(\mu))$$

$$= \sum_i (\nu^{-1}(\lambda)|H_i)(\nu^{-1}(\mu)|H^\iota) = \sum_i \lambda(H_i)\mu(H^i),$$

proving the Lemma.

Symmetrizability

Kac proved the existence of a nondegenerate invariant bilinear form on $\mathfrak{g}$. We start by constructing the pairing on $\mathfrak{h}$, then show that it may be extended to $\mathfrak{g}$. Recall that symmetrizable means that if A is the Cartan matrix is of the form DB where D is diagonal and B is symmetric.

The symmetrizable case is no harder than the symmetric case. However we will assume in today's lecture that A is symmetric just to eliminate some minor bookkeeping. See Kac Chapter 2 for the general case.

The inner product on $\mathfrak{h}$

Recall that $\mathfrak{g} = \mathfrak{g}(A)$ has generators consisting of all of $\mathfrak{h}$ and e_i, f_i that satisfy

$$[e_i, f_j] = \delta_{ij}\alpha_i^\vee, \qquad [h, e_i] = \langle h, \alpha_i\rangle e_i, \qquad [h, f_i] = -\langle h, \alpha_i\rangle f_i.$$

We may define a symmetric bilinear form $(\,|\,)$ on $\mathfrak{h}$ by

$$(\alpha_i^\vee | h) = \langle h, \alpha_i\rangle.$$

Note that

$$(\alpha_i^\vee | \alpha_j^\vee) = a_{ji}$$

is symmetric. This does not completely determine the form ...

Finishing the inner product on $\mathfrak{h}$

Let $\mathfrak{h}' = \sum \mathbb{C}\alpha_i^\vee$ and let $\mathfrak{h}''$ be a complementary subspace to $\mathfrak{h}'$ in $\mathfrak{h}$. Then we may complete the characterization of $(\ |\ )$ by requiring that $(\mathfrak{h}''|\mathfrak{h}'') = 0$.

It is easy to check (since the α_i and $\alpha_i^\vee$ are both linearly independent) that $(\ |\ )$ is nondegenerate on $\mathfrak{h}$.

The inner product on $\mathfrak{g}$

We have a vector space isomorphism $\nu : \mathfrak{h} \longrightarrow \mathfrak{h}'$ determined by the formula

$$\langle h, \nu(h') \rangle = (h|h').$$

Since $(h|\alpha_i^\vee) = \langle h, \alpha_i \rangle$, this implies that

$$\nu(\alpha_i^\vee) = \alpha_i.$$

Theorem (Theorem 2.2 in Kac)

We may extend $(\ |\ )$ to an invariant bilinear form on $\mathfrak{g}$. The form is nondegnerate and satisfies

$$[x, y] = (x|y)\nu^{-1}(\alpha)$$

if $x \in \mathfrak{X}_\alpha$, $y \in \mathfrak{X}_{-\alpha}$.

The $\mathbb{Z}$-grading

We make use of the principal $\mathbb{Z}$-grading

$$\mathfrak{g} = \bigoplus_{j \in \mathbb{Z}} \mathfrak{g}_j$$

where $\mathfrak{g}_0 = \mathfrak{h}$, $\mathfrak{g}_1$ is the span of the e_i, and $\mathfrak{g}_{-1}$ is the span of the f_i, and the remaining $\mathfrak{g}_k$ are determined by the requirement that

$$[\mathfrak{g}_i, \mathfrak{g}_j] \subseteq \mathfrak{g}_{i+j}.$$

Thus

$$\mathfrak{h} = \mathfrak{g}_0, \qquad \mathfrak{n}_+ = \bigoplus_{i=1}^{\infty} \mathfrak{g}_i, \quad \mathfrak{n}_- = \bigoplus_{i=1}^{\infty} \mathfrak{g}_{-i}.$$

Now let

$$\mathfrak{g}(N) = \bigoplus_{j=-N}^{N} \mathfrak{g}_j.$$

Thus $\mathfrak{g}$ is generated by $\mathfrak{g}(1)$.

The grading on generators

We will first define $(\ |\ )$ on $\mathfrak{g}(1)$. Note that $(\ |\ )$ is uniquely determined on $\mathfrak{g}(1)$ by its restriction to $\mathfrak{h}$, and the required properties that:

- $[x, y] = (x|y)\nu^{-1}(\alpha)$.
- $(x|y) = 0$ for $x \in \mathfrak{g}_\alpha$ and $y \in \mathfrak{g}_\beta$ unless $\beta = -\alpha$.

The first property boils down to:

$$(e_i|f_i) = 1$$

because $\alpha_i^\vee = [e_i, f_i] = (e_i|f_i)\alpha_i^\vee$. Also note

$$(\alpha_i^\vee|\alpha_j^\vee) = a_{ij}.$$

A partial invariance property

Lemma

If $x, y, z, [x,y], [y,z] \in \mathfrak{g}(1)$ *then*

$$([x,y]|z) = (x|[y,z]).$$

There are only a couple of cases to consider if $x \in \mathfrak{g}_\alpha$, $y \in \mathfrak{g}_\beta$, $z \in \mathfrak{g}_\gamma$ where α, β, γ are either simple roots, negatives of simple roots, or 0.

- $([e_i, f_i]|h) = (e_i|[f_i, h])$: both sides equal $(\alpha_i^\vee|h) = \langle h, \alpha_i \rangle$.
- $([e_i, h]|f_i) = (e_i|[h, f_i])$: both sides equal $-\langle h, \alpha_i \rangle$.
- $([f_i, h]|e_i) = (f_i|[h, e_i])$: both sides equal $\langle h, \alpha_i \rangle$.

The remaining case with $x, y, z \in \mathfrak{g}(1)$ are zero.

Expanding the definition

Now we have constructed the invariant inner product on $\mathfrak{g}(1) = \mathfrak{h} \oplus \bigoplus \mathbb{C}e_i \oplus \bigoplus \mathbb{C}f_i$. This is the base case for an induction.

Lemma

Suppose there exists an inner product $(\ |\ )$ defined on $\mathfrak{g}(N-1)$ subject to the condition that

$$x, y, z, [x, y], [y, z] \in \mathfrak{g}(N-1) \qquad \Rightarrow \qquad ([x, y]|z) = (x|[y, z]).$$

Then we may extend it to $\mathfrak{g}(N)$ with the same property.

The recursive definition

It is necessary to define $(x|y)$ when $x \in \mathfrak{g}_{(N)}$ and $y \in \mathfrak{g}_{(-N)}$. We assume that x, y are homogenous, that is, lie in weight spaces. We may write $y = \sum_i [u_i, v_i]$ where u_i and v_i are in $\mathfrak{g}_{(N-1)}$ and are homogeneous. Then we define

$$(x|y) = \sum_i ([x, u_i]|v_i).$$

Note that $[x, u_i]$ is homogeneous since x, u_i are. It is understood that since $v_i \in \mathfrak{g}(N-1)$, the term $([x, u_i]|v_i)$ is interpreted as zero if $[x, u_i]$ is not in $\mathfrak{g}(N-1)$.

It must be checked that this expression is well-defined, independent of the decomposition $y = \sum [u_i, v_i]$.

An alternate expression proves (:) well-defined

We make an alternative expansion $x = \sum_j [s_j, t_j]$. We will prove

$$([[s_j, t_j], u_i]|v_j) = (s_j|[t_j, [u_i, v_i]]).$$

Using the Jacobi identity and the induction hypothesis (which is applicable since $s_j, t_j, u_i, v_i \in \mathfrak{g}(N-1)$) we

$$([[s_j, t_j], u_i]|v_i) = ([[s_j, u_i], t_j]|v_i) + ([s_j, [t_j, u_i]]|v_i)$$

$$= ([s_j, u_i]|[t_j, v_i]) + (s_j|[[t_j, i_i], v_i])$$

$$(s_j|[u_i, [t_j, v_i]]) + (s_j|[[t_j, i_i], v_i]) = (s_j|[t_j, [u_i, v_i]]).$$

Thus

$$\sum_i ([x, u_i]|v_i) = \sum_j (s_j|[t_j, y]).$$

The independence of the definition $(x|y) = \sum_i ([x, u_i]|v_i)$ on the decomposition $y = \sum_i [u_i, v_i]$ follows from this expression.

The end of the proof

One may check that the induction hypothesis is satisfied for this extension, that is:

$$x, y, z, [x, y], [y, z] \in \mathfrak{g}(N) \qquad \Rightarrow \qquad ([x, y]|z) = (x|[y, z]).$$

Formulaire (symmetric case)

$$[e_i, f_j] = \delta_{ij}\alpha_i^\vee, \qquad [h, e_i] = \langle h, \alpha_i\rangle e_i, \qquad [h, f_i] = -\langle h, \alpha_i\rangle f_i.$$

$$[x, y] = (x|y)\nu^{-1}(\alpha), \qquad x \in \mathfrak{g}_\alpha, y \in \mathfrak{g}_{-\alpha}$$

$$\nu(\alpha_i^\vee) = \alpha_i, \qquad \langle h, \nu(h')\rangle = (h|h'), \qquad (\alpha_i^\vee|h) = \langle h, \alpha_i\rangle.$$

Formulaire (symmetrizable case)

$A = DB$, $D = \mathrm{diag}(\varepsilon_1, \cdots, \varepsilon_n)$, $B = (b_{ij})$ symmetric

$$[e_i, f_j] = \delta_{ij}\alpha_i^\vee, \qquad [h, e_i] = \langle h, \alpha_i\rangle e_i, \qquad [h, f_i] = -\langle h, \alpha_i\rangle f_i.$$

$$[x, y] = (x|y)\nu^{-1}(\alpha), \qquad x \in \mathfrak{g}_\alpha, y \in \mathfrak{g}_{-\alpha}$$

$$\nu(\alpha_i^\vee) = \varepsilon_i\alpha_i, \quad \langle h, \nu(h')\rangle = (h|h'), \qquad (\alpha_i^\vee|h) = \varepsilon_i\langle h, \alpha_i\rangle$$

First definition

Now let us develop the Casimir element in the Kac-Moody case. It is not to be an element of $U(\mathfrak{g})$, though it can be defined for a completion of $U(\mathfrak{g})$. Instead, as in Chapter 2 of Kac' book, the Casimir element will be an operator that can be defined for representations of Category $\mathcal{O}$.

In the Kac-Moody case some spaces $\mathfrak{g}_\alpha = \mathfrak{X}_\alpha$ may have dimension $\geqslant 1$ so we should choose a basis $X_\alpha^{(t)}$ such that that $X_{-\alpha}^{(t)}$ is to be the dual basis and formally write:

$$\Omega = \sum_{\alpha \in \Phi} \sum_{t=1}^{\dim(\mathfrak{X}_\alpha)} X_\alpha^{(t)} \cdot X_{-\alpha}^{(t)} + \sum_{i=1}^{r} H_i H^i .$$

(For real roots $\mathfrak{X}_\alpha$ will be one-dimensional, but for imaginary roots the dimension may be $\geqslant 1$ as in the affine case.)

Modified definition

This form is not useful but the modified version

$$\Omega = \sum_{\alpha \in \Phi^+} \sum_{t=1}^{\dim(\mathfrak{X}_\alpha)} X_{-\alpha}^{(t)} \cdot X_{\alpha}^{(t)} + \sum_{i=1}^{r} H_i H^i + \nu^{-1}(\rho)$$

works well.

So let us write

$$\Omega = \Omega_0 + \sum_{i=1}^{r} H_i H^i + \nu^{-1}(\rho), \qquad \Omega_0 = \sum_{\alpha \in \Phi^+} \sum_{t=1}^{\dim(\mathfrak{X}_\alpha)} X_{=\alpha}^{(t)} \cdot X_{\alpha}^{(t)}.$$

Why it works

Lemma

Let V be a representation in Category $\mathcal{O}$ and let $v \in V$. Then $X_\alpha^{(t)} v = 0$ for all but finitely many α.

By definition of Category $\mathcal{O}$ there are a finite number of $\lambda_1, \cdots, \lambda_N \in \mathfrak{h}^*$ such that if $V_\mu \neq 0$ then $\mu \preccurlyeq \lambda_i$ for some i. For example if V is a highest weight representation with highest weight λ we may take $N = 1$ and $\lambda_1 = \lambda$.

Now let $v \in V$ have weight μ. Then $X_\alpha v$ has weight $\mu + \alpha$. There are only a finite number of positive roots α such that $\mu + \alpha \preccurlyeq \lambda_i$, so $X_\alpha v = 0$ for all but finitely many α. Thus the sum $\sum X_{-\alpha} X_\alpha v$ is actually finite.

Normal ordering

In physics one encounters the notion of "normal ordering" which may be summed up with the admonish to perform annihilation operators before creation operators. In physics the operators are applied to a Hilbert space representing the state of a physical system and there are operators that create or annihilate particles. Normal ordering is the procedure of always applying annihilation operators before creation operators. This allows one to work with series that would otherwise be divergent. Normal ordering is also a source of the central extensions of infinite-dimensional Lie algebras that often pop up. In the context of representations in Category $\mathcal{O}$, the operators X_α with $\alpha \in \Phi^+$ are annihilation operators, and the operators $X_{-\alpha}$ are creation operators, writing Ω this way is a perfect example of normal ordering.

Normal ordering (continued)

The operator Ω_0 makes sense applied to any vector in a representation V in Category $\mathcal{O}$, since the sum $\sum X_{-\alpha}^{(t)} X_{\alpha}^{(t)} v$ has only finitely many nonzero terms. Hence the Casimir operator is defined.

In Category $\mathcal{O}$ we may think of X_α as an annihilation operator if α is a positive root, and as a creation operator when α is a negative root. So when we write

$$\Omega_0 = \sum_{\alpha \in \Phi^+} \sum_{i=1}^{\dim(\mathfrak{X}_\alpha)} 2X_{-\alpha}^{(i)} \cdot X_{\alpha}^{(i)},$$

this is an example of normal ordering.

s_i permutes $\Phi^+ - \{\alpha_i\}$

Lemma

Let s_i be a simple reflection. If α is a positive root, then $s_i(\alpha) = -\alpha$ if $\alpha = \alpha_i$; otherwise $s_i(\alpha)$ is positive.

To prove this we note that the positive roots are the weight spaces in $\mathfrak{n}_+$. It is spanned by elements of the form

$$[e_{i_1}[e_{i_2},[e_{i_3}, \cdots, [e_{i_{k-1}}, e_{i_k}]]]].$$

From this description, the positive roots have the form

$$\alpha = \sum_{j=1}^{r} k_j \alpha_j$$

where $(k_1, \cdots, k_r)$ is a tuple of nonnegative integers.

Proof (continued)

The weight $k\alpha_i$ is only a root if $k = 1$, because $[\alpha_i, \alpha_i] = 0$, so if the above expression has all $i_1, i_2, \cdots = i$ we must have $k = 1$. Now apply s_i to $\sum k_i \alpha_i$ to obtain

$$s_i(\alpha) = \sum k_j \alpha_j - \langle \alpha_i^\vee, \alpha \rangle \alpha_i = \sum_{j=1}^{r} k_j' \alpha_j$$

where $k_j' = k_j$ if $j \neq i$. If $\alpha \neq \alpha_i$ at least one $k_j = k_j'$ is positive, so $s_i(\alpha)$ cannot be a negative root.

The Weyl vector

Now let us consider the Weyl vector $\rho = \frac{1}{2}\sum_{\alpha\in\Phi^+}\alpha$, which we treat as a divergent sum to be renormalized. Since s_i sends α_i to $-\alpha_i$ and permutes the remaining positive roots, we have (formally) $s_i(\rho) = \rho - \alpha_i$. Remembering that

$$s_\alpha(x) = x - \langle\alpha^\vee, x\rangle\alpha$$

we need ρ to be an element of the weight lattice P such that

$$\langle\alpha_i^\vee, \rho\rangle = 1 \qquad \text{for simple roots } \alpha_i.$$

This does not quite fully characterize ρ since the $\alpha^\vee$ do not span $\mathfrak{h}$ in the Kac-Moody case. However this is the only property that we need ρ to have. We choose $\rho \in \mathfrak{h}^*$ to be a fixed element with this property.

What we will prove

We may now state our goals. For representations of Category $\mathcal{O}$:

- We will prove that Ω commutes with the action of the e_i and f_i.
- Therefore it acts as a scalar on irreducible representations.
- For highest weight representations we may compute this scalar.
- This will contain enough information to help prove the Kac-Weyl character formula.

Invariance of each weight space

Lemma

If V is a representation of Category $\mathcal{O}$, then $\Omega(V_\mu) \subseteq V_\mu$ for every $\mu \in \mathfrak{h}^*$.

Indeed each of the three terms Ω_0, $\sum H_i H^i$ and $2\nu^{-1}(\rho)$ has this property. For $\Omega_0 = 2\sum X_{-\alpha}X_\alpha$ note that X_α maps V_μ into $V_{\mu+\alpha}$, then $X_{-\alpha}$ maps $V_{\mu+\alpha}$ into V_μ.

Ω commutes with the action of e_i, f_i

We defined $(|)$ as an inner product on $\mathfrak{h}$ before extending it to $\mathfrak{g}$. We also defined an isomorphism $\nu : \mathfrak{h} \to \mathfrak{h}^*$ by $\langle H', \nu(H) \rangle = (H'|H)$. We then defined an inner product on $\mathfrak{h}^*$ by $(\lambda|\mu) = (\nu^{-1}(\lambda)|\nu^{-1}(\mu))$.

Theorem

Let V be a $\mathfrak{g}$-module in Category $\mathcal{O}$. Then the action of Ω commutes with the action of e_i and f_i.

We will give a proof that is different from that in Kac' book. Let S be a subset of $\Phi \cup \{0\}$ such that if $\alpha \in S$ and $\alpha + \alpha_i \in \Phi \cup \{0\}$ then $\alpha + \alpha_i \in S$. We assume that all but finitely many elements of S are positive roots. Let

$$\mathfrak{g}^S = \bigoplus_{\alpha \in S} \mathfrak{g}_\alpha, \qquad \mathfrak{g}_S = \bigoplus_{\alpha \in S} \mathfrak{g}_{-\alpha}.$$

Proof

The spaces

$$\mathfrak{g}^S = \bigoplus_{\alpha\in S} \mathfrak{g}_\alpha, \qquad \mathfrak{g}_S = \bigoplus_{\alpha\in S} \mathfrak{g}_{-\alpha}.$$

are dually paired under $(\ |\ )$ so let X^t be a basis of $\mathfrak{g}^S$ and X_t be the dual basis of $\mathfrak{g}_S$. Define

$$\Omega_S = \sum_t X_t X^t.$$

We will prove that e_i commutes with Ω_S. Since all but finitely many of the X^t are in $\mathfrak{n}_+$, this makes sense as an operator on V.

Note that $[e_i, \mathfrak{g}^S] \subseteq \mathfrak{g}^S$ and $[e_i, \mathfrak{g}_S] \subseteq \mathfrak{g}_S$. We write

$$[e_i, X^t] = \sum_u c_{tu} X^u, \qquad [e_i, X_t] = \sum_u d_{tu} X_u.$$

Proof (continued)

Using the fact that X_u is the dual basis of X^t

$$c_{tu} = ([e_i, X^t]|X_u), \qquad d_{tu} = (X_u|[e_i, X_t]).$$

Since $(\ |\ )$ is invariant, we have $c_{tu} = -d_{ut}$. Now

$$\left[e_i, \sum_t X_t X^t\right] = \sum_t [e_i, X_t]X^t + \sum_t X_t[e_i, X^t].$$

This equals

$$\sum_{t,u} c_{tu} X_u X^t + \sum_{t,u} d_{tu} X_t X^u = 0.$$

We have proved that $[e_i, \Omega_S] = 0$.

We will apply this twice.

Proof (continued)

Let $S_1 := \Phi^+ - \{\alpha_i\}$ and $S_2 := \{\alpha_i, 0, -\alpha_i\}$. We see that e_i commutes with

$$\Omega_{S_1} = \sum_{\substack{\alpha\in\Phi^+ \\ \alpha\neq\alpha_i}} \sum_j X^j_{-\alpha} X^j_{\alpha}, \qquad \Omega_{S_2} = X_{-\alpha_i} X_{\alpha_i} + X_{\alpha_i} X^{\alpha_i} + \sum_j H_j H^j.$$

Since $X_{\alpha_i} X_{-\alpha_i} = X_{-\alpha_i} X_{\alpha_i} + \alpha_i^\vee$ we have

$$\Omega = \Omega_{S_1} + \Omega_{S_2} + (2\nu^{-1}(\rho) - \alpha_i^\vee).$$

So we have only to check that e_i commutes with $2\nu^{-1}(\alpha) - \alpha_i^\vee$.

Proof (concluded)

Indeed

$$[2\nu^{-1}(\rho)-\alpha_i^\vee, e_i] = \langle 2\nu^{-1}(\rho)-\alpha_i^\vee, \alpha_i\rangle e_i = 0$$

since $\langle \alpha_i^\vee, \alpha_i\rangle = a_{ii} = 2$ while $\langle \nu^{-1}(\rho), \alpha_i\rangle = 1$ because using the properties in the Formulaire

$$1 = \langle \alpha_i^\vee, \rho\rangle = \langle \alpha_i^\vee, \nu(\nu^{-1}\rho)\rangle = (\alpha_i^\vee|\nu^{-1}(\rho))$$

$$= (\nu^{-1}(\rho)|\alpha_i^\vee) = (\nu^{-1}(\rho)|\nu(\alpha_i)) = \langle \nu^{-1}(\rho), \alpha_i\rangle$$

We have proved that Ω commutes with e_i. A similar argument will show that it commutes with the f_i.

The main theorem

Theorem

Suppose V is a highest weight representation with highest weight λ. Then Ω acts as a scalar on V, with value $(\lambda|\lambda+2\rho)$.

Let v_λ be the highest weight vector. Since $X_\alpha^{(t)}$ annihilates v_λ we have to compute the eigenvalue of

$$\sum_i (H_i H^i) + 2\nu^{-1}(\rho)$$

on v_λ. The calculations that show this is $(\lambda|\lambda) + 2(\lambda|\rho)$ are identical to the finite-dimensional simple case.