Lecture 5: integrable representations

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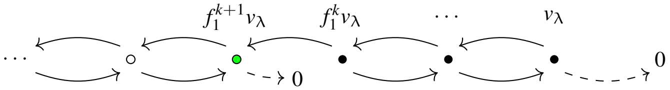


Review: $\mathfrak{sl}(2)$

We will reduce some calculations to $\mathfrak{sl}(2) = \mathbb{C}e \oplus \mathbb{C}f \oplus \mathbb{C}\alpha^\vee$. So

$$[e,f] = \alpha^\vee, \qquad [\alpha^\vee, e] = 2e, \qquad [\alpha^\vee, f] = -2f.$$

Consider a highest weight module V generated by v_λ such that

$$e(v_\lambda) = 0, \qquad \alpha^\vee v_\lambda = \lambda v_\lambda.$$

Define

$$v_{\lambda-2j} = \frac{1}{j!} f^j v_\lambda.$$

Then it is easy to prove

$$\alpha^\vee(v_{\lambda-2j}) = (\lambda - 2j)\alpha^\vee$$

$$f(v_{\lambda-2j}) = (j+1)v_{\lambda-2j-2}, \qquad e(v_{\lambda-2j}) = (\lambda - j + 1)v_{\lambda-2j+2}.$$

These identities are true in any highest weight module and can be proved by induction on j.

Review: $\mathfrak{sl}(2)$ (continued)

To repeat, in a highest weight module for $\lambda \in \mathbb{C}$:

$$\alpha^\vee(v_{\lambda-2j}) = (\lambda - 2j)\alpha^\vee$$

$$f(v_{\lambda-2j}) = (j+1)v_{\lambda-2j-2}, \qquad e(v_{\lambda-2j}) = (\lambda - j + 1)v_{\lambda-2j+2}.$$

If $\lambda = k$ is a nonnegative integer then $v_{k-2j} = 0$ for $j > k$. Then V is a basis $v_k, v_{k-2}, \cdots, v_{-k}$. Since $v_{-k-2} = 0$ we have $f^{k+1}v_k = 0$. Moreover $k = \langle \alpha^\vee, \mathrm{wt}(v) \rangle$.

There are two possibilities for the highest weight module V:

- V is the Verma module $M(\lambda)$;
- V is the irreducible quotient $L(\lambda)$.

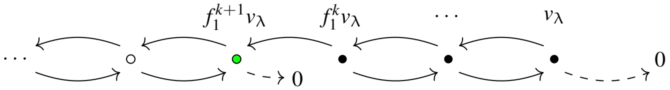
Primitive vectors

Let V be a module with a weight space decomposition. If μ is a weight of V, a nonzero vector $v \in V(\mu)$ is called a primitive vector if there exists a submodule U of V such that $v \notin U$ but $e_i v \in U$ for all i.

For example, if $e_i v = 0$ for all i, then v is a primitive vector. If V is a highest weight module, then V is irreducible if and only if its only primitive vectors are multiples of the highest weight vector.

Review: $\mathfrak{sl}(2)$: $M(\lambda)$ and $L(\lambda)$

With $k = \langle \alpha_1^\vee, \lambda \rangle$ ($= 2$ in this example) here is the Verma module $M(\lambda)$:



The dashed arrows are zero. The lighter dots mark the maximal proper submodule. Dividing by this submodule gives the unique irreducible quotient $L(\lambda)$ (black dots), which is finite-dimensional and integrable.

The green dot is the primitive vector $f_1^{k+1}v_\lambda$. It is primitive since $e_1 f_1^{k+1} v_\lambda = 0$.

Finite-dimensional modules

Let $\mathfrak{g}$ be a finite-dimensional simple Lie algebra, and let V be an irreducible finite-dimensional module. As usual, V has a weight space decomposition

$$V = \bigoplus_{\mu} V_{\mu}$$

and we will write $\mathrm{wt}(v) = \mu$ if $v \in V_{\mu}$. This means $Hv = \mu(H)v$ for $H \in \mathfrak{h}$.

Lemma

Let $v \in V$ be a vector such that $e_i v = 0$ for some i. Then

$$k = \langle \alpha_i^{\vee}, \mathrm{wt}(v) \rangle \geqslant 0$$

and $f_i^{k+1} v = 0$.

Proof

Indeed, let

$$\mathfrak{g}_{(i)} = \langle e_i, f_i \rangle = \mathbb{C}e_i \oplus \mathbb{C}f_i \oplus \mathbb{C}\alpha_i^\vee$$

be the copy of $\mathfrak{sl}(2)$ generated by e_i, f_i. Then v generates a finite-dimensional highest weight module for this $\mathfrak{sl}(2)$. Applying our knowledge of $\mathfrak{sl}(2)$ representations, $f_i^{k+1}v = 0$ where $k = \langle \alpha_i^\vee, \mathrm{wt}(v) \rangle \geqslant 0$.

The Serre relations

Proposition (Serre relations)

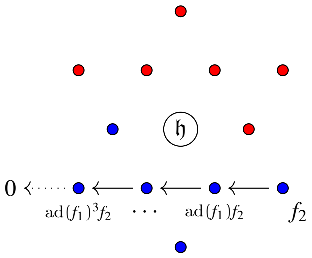
Let $\mathfrak{g}$ be a finite-dimensional simple Lie algebra with Cartan matrix a_{ij}. If $i \neq j$ then

$$\mathrm{ad}(f_i)^{1-a_{ij}} f_j = 0, \qquad \mathrm{ad}(e_i)^{1-a_{ij}} e_j.$$

To prove this we apply our observations on $\mathfrak{sl}(2)$ embedded as $\mathfrak{g}_{(i)}$ to the adjoint representation which is finite-dimensional. Note that $\mathrm{ad}(e_i)f_j = [e_i, f_j] = 0$ and $\langle \alpha_i^\vee, \mathrm{wt}(f_j) \rangle = -\langle \alpha_i^\vee, \alpha_j \rangle = -a_{ij}$. Hence the Lemma applies giving $\mathrm{ad}(f_i)^{1-a_{ij}} f_j = 0$. To obtain the other relation we may apply the Chevalley involution ω of $\mathfrak{g}$ such that $\omega(e_i) = -f_i$, $\omega(f_i) = -e_i$, $\omega(\alpha_i^\vee) = -\alpha_i^\vee$ to interchange the two identities.

Example: G_2

Here is the exceptional Lie algebra $\mathfrak{g}_2$.



The arrows show $\mathrm{ad}(f_1)$ shifting between weight spaces. The dashed line is $\mathrm{ad}(f_1) : \mathrm{ad}(f_1)^3 f_2 \to 0$, illustrating the Serre relation $\mathrm{ad}(f_1)^{k+1} f_2 = 0$ with

$$k = \langle \alpha_1, \mathrm{wt}(f_2) \rangle = \langle \alpha_1, -\alpha_2 \rangle = 3.$$

Serre relations: the Kac-Moody case

The Serre relations are true for general Kac-Moody Lie algebras, but we have to argue differently since the preceding arguments relied on finite-dimensionality.

We will prove:

Theorem

The Serre relations:

$$\mathrm{ad}(f_i)^{1-a_{ij}} f_j = 0, \qquad \mathrm{ad}(e_i)^{1-a_{ij}} e_j$$

are valid in an arbitrary Kac-Moody Lie algebra when $i \neq j$.

There are no primitive vectors in $\mathfrak{n}_-$

Proposition

Let $\mathfrak{g}$ be a Kac-Moody Lie algebra and suppose that $X \in \mathfrak{n}_-$ such that $[e_i, X] = 0$ for all i. Then $X = 0$.

Proof. Since $\mathfrak{g}$ acts on itself by the adjoint representation, we obtain an action of the associative algebra $U(\mathfrak{g})$. We make use of the fact that $U(\mathfrak{g}) = U(\mathfrak{n}_-) \otimes U(\mathfrak{h}) \otimes U(\mathfrak{n}_+)$. We have

$$U(\mathfrak{n}_+) \cdot X = (\mathbb{C} \oplus U(\mathfrak{n}_+)\mathfrak{n}_+)X = \mathbb{C}X$$

since $\mathfrak{n}_+$ annihilates X. Also $U(\mathfrak{h})X = \mathbb{C}X$. So $U(\mathfrak{g})X = U(\mathfrak{n}_-)X$ is an ideal that is contained in $\mathfrak{n}_-$. But $\mathfrak{g}$ has no nontrivial ideals that do not intersect $\mathfrak{h}$, so $U(\mathfrak{h})X = 0$. Thus $X = 0$.

Proof of the Serre relations

Let us denote $\theta_{ij} = \mathrm{ad}(f_i)^{1-a_{ij}} f_j$. The Proposition shows if we can show $\mathrm{ad}(e_k)(\theta_{i,j}) = 0$ for all k, then $\theta_{i,j} = 0$, which is one of the Serre relations.

First consider the case $k = i$. We consider the $\mathfrak{sl}(2) \cong \mathfrak{g}_{(i)}$ module generated by f_j. Since one of the generator relations for $\mathfrak{g}$ is $[e_i, f_j] = \delta_{ij} \alpha_i^\vee$ and we are assuming $i \neq j$, we have $\mathrm{ad}(e_i) f_j = 0$. So f_i is a highest weight vector for $\mathfrak{g}_{(i)}$. Therefore by our discussion of $\mathfrak{sl}(2)$ theory, $\mathrm{ad}(e_i)\,\mathrm{ad}(f_i)^{1+k} f_j = 0$ where

$$k = \langle \alpha_i^\vee, \mathrm{wt}(f_j) \rangle = -\langle \alpha_i^\vee, \alpha_j \rangle = -a_{ij}.$$

This proves $\mathrm{ad}(e_i)\theta_{ij} = 0$.

Proof (continued)

Next suppose $k = j$. So we need to know $[e_j, \mathrm{ad}(f_i)^{1-a_{ij}} f_j = 0]$. Note that e_j and f_i commute by the generating relations so

$$[e_j, \mathrm{ad}(f_i)^{1-a_{ij}} f_j] = \mathrm{ad}(f_i)^{1-a_{ij}} [e_j, f_j] = \mathrm{ad}(f_i)^{1-a_{ij}} \alpha_j^\vee.$$

First suppose that $a_{ij} = 0$. Then this equals

$$[f_i, \alpha_j^\vee] = \langle \alpha_j^\vee, \alpha_i \rangle f_i = a_{ij} f_i = 0.$$

Next suppose that $a_{ij} = -1$. Then

$$\mathrm{ad}(f_i)^{1-a_{ij}} \alpha_j^\vee = \mathrm{ad}(f_i)^2 \alpha_j^\vee = [f_i, [f_i, \alpha_j^\vee]] = \langle \alpha_j^\vee, \alpha_i \rangle [f_i, f_i] = 0,$$

and the cases $a_{ij} = -2, -3, \cdots$ are similar, depending on $[f_i, f_i]$=0.

Proof (concluded)

We also need to know that $[e_k, \theta_{ij}] = 0$ for other k. If $k \neq i, j$ then e_k commutes with both f_j and f_i by the generating relations of the Kac-Moody Lie algebra, so $[e_k, \mathrm{ad}(f_i)^{1-a_{ij}} f_j] = 0$ in this case also.

The Lemma is proved: we have shown that with $\theta_{i,j} = \mathrm{ad}\, f_i{}^{1-a_{ij}} f_j$, $[e_k, \theta_{i,j} = 0$ for all k. The Proposition then shows that $\theta_{i,j} = 0$.

The other Serre relation

$$\mathrm{ad}(e_i)^{1-a_{ij}} e_k = 0$$

follows by applying the Chevalley involution

$$e_i \to -f_i, \qquad f_i \to -e_i, \qquad \alpha_i^\vee \to -\alpha_i^\vee.$$

The idea of an integrable representation

We now turn to the notion of an integrable representation. As we explained in Lecture 3, a representation of the Lie algebra $\mathfrak{g}$ of a Lie group G is integrable if it is the differential of a representation of G.

However we would like an equivalent definition that does not make use of the Lie group G, since we want to work in the category of modules for a Kac-Moody Lie algebra, and we do not wish to construct an analog of the group G.

Local nilpotence and integrability

Let V be a $\mathfrak{g}$-module. We will say that an endomorphism $T : V \longrightarrow V$ is locally nilpotent if for every vector $v \in V$ there exists an $N > 0$ such that $T^N v = 0$.

For $\mathfrak{g} = \mathfrak{sl}(2)$, a representation is integrable if and only if it is finite-dimensional. Indeed, this implies that e_1 and f_1 are locally nilpotent. Conversely, an irreducible representation of $\mathfrak{sl}(2)$ in which e_1 and f_1 are locally nilpotent is easily seen to be integbrable. This equivalence is also true for more general (semsimple, simply-connected) Lie groups and their Lie algebras.

We will say that a representation V of a general Kac-Moody Lie algebra is integrable if it has a weight space decomposition and the e_i and f_i are locally nilpotent.

Integrability and the Weyl group

Integrability gives one major benefit of a lifting to G, without having to construct G. We recall that the Weyl group W is generated by the simple reflections $s_i : \mathfrak{h}^* \longrightarrow \mathfrak{h}^*$:

$$s_i(x) = x - \langle \alpha_i^\vee, x \rangle \alpha_i.$$

Proposition

Suppose V is integrable. If $\mu \in \mathfrak{h}^*$ let $m_V(\mu)$ be the weight multiplicity $\dim\ V_\mu$. Then m_V is constant on W-orbits.

Integrability and the Weyl group (continued)

To prove the W-invariance of weight multiplicities, we first note the following fact, important in its own.

Lemma

If V is an integrable representation, then every vector lies in a finite-dimensional representation of $\mathfrak{g}_{(i)}$. The module V is a direct sum of finite-dimensional irreducible $\mathfrak{g}_{(i)}$-modules.

Suppose that v is any nonzero vector. We will show that v lies in a finite-dimensional module. We may assume that $v \in V_\mu$ for some μ.

Proof of the Lemma

We have $e_i^N v = 0$ for sufficiently large v. Let n be maximal such that $e_i^n v \neq 0$ and let $v_\lambda = e_i^n v$ where $\lambda = \mu + n\alpha_i$. Then $v_\lambda \in V_\lambda$ and v_λ is a highest weight vector for $\mathfrak{g}_{(i)}$. Since $f_i^M v_\lambda = 0$ for sufficiently large M, from our knowledge of $\mathfrak{sl}(2)$-modules we learn that $\langle \alpha_i^\vee, \lambda \rangle$ is a nonnegative integer and $\mathfrak{g}_{(i)} v_\lambda$ is a finite-dimensional module of dimension $\langle \alpha_i^\vee, \lambda \rangle + 1$. It contains v. The fact that V is a direct sum of finite-dimensional modules now follows from a well-known property of $\mathfrak{sl}(2, \mathbb{C})$ (due to Weyl) that every finite-dimensional modules is completely reducible, plus an easy Zorn's Lemma argument.

The Lemma implies the Proposition, since the weight multiplicities of finite-dimensional $\mathfrak{sl}(2)$ modules are invariant under the simple reflection of $\mathfrak{sl}(2)$, which agrees with s_i.

Integrable highest weight representations

The remainder of this section will be devoted to the proof of the following result. Let P be the weight lattice consisting of $\lambda \in \mathfrak{h}^*$ such that $\langle \alpha_i^\vee, \lambda \rangle$ is an integer for all i. The cone P^+ of dominant weights consists of λ such that each $\langle \alpha_i^\vee, \lambda \rangle$ is a nonnegative integer.

Theorem

Let $\mathfrak{g}$ be a Kac-Moody Lie algebra and let V be an irreducible highest-weight representation with highest weight λ. Then V is integrable if and only if λ is a dominant weight.

Integrability criterion

Lemma

Let $V = L(\lambda)$ be the highest-weight irreducible representation with highest weight λ. A necessary and sufficient condition for V to be integrable is that $f_i^N v_\lambda = 0$ for sufficiently large N, where v_λ is the highest weight vector.

We have a partial order $\succcurlyeq$ on the weight lattice in which $\lambda \succcurlyeq \mu$ if $\lambda - \mu = \sum n_i \alpha_i$ where n_i are nonnegative integers. We define

$$\mathrm{supp}(V) = \{\mu \in \mathfrak{h}^* | V_\mu \neq 0\}.$$

Using the PBW theorem, we proved that $V = U(\mathfrak{n}_-)v_\lambda$. In particular

$$\mathrm{supp}(V) \subseteq \{\mu \in \mathfrak{h}^* | \mu \preccurlyeq \lambda\}.$$

For μ fixed we will have $\mu + N\alpha_i \not\preccurlyeq \lambda$ for sufficiently large N. Thus if $v \in V_\mu$ then $V_{\mu + N\alpha_i} = 0$ so $e_i^N v = 0$.

Proof

To prove the Lemma, it follows from the definition that if V is integrable then $f_i^N v_\lambda = 0$ for sufficiently large N. We assume that $f_i^N v_\lambda = 0$ for sufficiently large N and prove that V is integrable.

So if $v \in V$ we must also show that $f_i^N v = 0$. Since $V = U(\mathfrak{n}_-)v_\lambda$ it is sufficient to show that $f_i^N v = 0$ when v is of the form

$$f_{i_1} \cdots f_{i_M} v_\lambda, \qquad i_1 \leqslant i_2 \leqslant \cdots \leqslant i_M$$

We write

$$f_i^N f_{i_1} \cdots f_{i_M} v_\lambda = [f_i^N, f_{i_1} \cdots f_{i_M}] v_\lambda + f_{i_1} \cdots f_{i_M} \cdot f_i^N v_\lambda$$

and the second term vanishes by assumption if N is large.

(continued)

So it is enough to show that

$$\mathrm{ad}(f_i)^N f_{i_1} \cdots f_{i_M} = 0$$

in $U(\mathfrak{n}_-)$. If D is a derivation then by the multinomial generalization of the Leibniz rule

$$D^N(f_{i_1}^{k_1} \cdots f_{i_r}^{k_r}) = \sum_{N=\sum N_i} \frac{N!}{N_1! \cdots N_r!} D^{N_1}(f_{i_1}) \cdots D^{N_r}(f_{i_M}).$$

We apply this with $D = \mathrm{ad}(f_i)$. If N is sufficiently large, each term on the right vanishes by the Serre relations.

Proof of the theorem

Now we may prove the theorem. To reiterate:

Theorem

Let $\mathfrak{g}$ be a Kac-Moody Lie algebra and let V be an irreducible highest-weight representation with highest weight λ. Then V is integrable if and only if λ is a dominant weight.

Assume that V is the irreducible highest weight representation with highest weight λ. Suppose that $\langle \alpha_i^\vee, \lambda \rangle$ is a nonnegative integer for each i. To prove that V is integrable we must show that $f_i^N v_\lambda = 0$ for sufficiently large N. What we will show is that if $k = \langle \alpha_i^\vee, \lambda \rangle$ then $f_i^{k+1} v_\lambda = 0$. Suppose not. Let $u = f_i^{k+1} v_\lambda$. We will show that $e_j u = 0$ for each j. If $j \neq i$ then by the generating relations of the Kac-Moody Lie algebra e_j commutes with f_i so $e_j u = f_i^{k+1} e_j u = 0$. On the other hand if $i = j$ then $e_i f_i^{k+1} v_\lambda = 0$ by $\mathfrak{sl}(2)$ theory.

Proof (concluded)

We have shown that $e_j(f_i^{k+1}v_\lambda) = 0$ for all j. So $f_i^{k+1}v_\lambda$ is a primitive vector, which is a contradiction since V is irreducible.

(A reminder of how this goes)

Since $U(\mathfrak{g}) = U(\mathfrak{n}_-) \otimes U(\mathfrak{h}) \otimes U(\mathfrak{n}_+)$ it follows that $U(\mathfrak{g})f_i^{k+1}v_\lambda = U(\mathfrak{n}_-)f_i^{k+1}v_\lambda$. There is no way that $v_\lambda \in U(\mathfrak{n}_-)f_i^{k+1}v_\lambda$, so this is a proper submodule. But V is irreducible, and so $f_i^{k+1}v_\lambda = 0$.

Now V is integrable by the Lemma. We leave the converse to the reader.

Review

Our main interest is in highest weight representations of a Kac-Moody Lie algebra $\mathfrak{g}$. We have seen in Lecture 1 that for a fixed $\lambda \in \mathfrak{h}^*$:

- There is a unique universal highest weight representation $M(\lambda)$ with highest weight λ such that if V is any highest weight representation with highest weight λ, there is a surjection $M(\lambda) \longrightarrow V$.
- There is a unique irreducible representation $L(\lambda)$ that is a quotient of any highest weight representation V for λ.

The morphisms implied by these statements map the highest weight vector (with weight λ) to the highest weight vector, but other morphisms are possible: for example there may be embeddings $M(\mu) \to M(\lambda)$ for certain $\mu \preccurlyeq \lambda$.

Category $\mathcal{O}$

The BGG Category $\mathcal{O}$ is a slightly larger category that contains all highest weight modules (for every weight). We define it now. Recall that if $\lambda, \mu \in \mathfrak{h}^*$ then $\lambda \succcurlyeq \mu$ if $\lambda - \mu = \sum k_i \alpha_i$ where k_i are nonnegative integers. Let V be a $\mathfrak{g}$-module.

Definition

We say that a module V is in Category $\mathcal{O}$ if it has a weight space decomposition with finite-dimensional weight spaces, and if there are a finite number of $\lambda_1, \cdots, \lambda_N \in \mathfrak{h}^*$ such that $V_\mu = 0$ unless $\mu \preccurlyeq$ some λ_i.

Category $\mathcal{O}$ is an abelian category. For finite-dimensional semisimple Lie algebras, it is the subject of a book Representations of Semisimple Lie algebras in the BGG Category $\mathcal{O}$ by James Humphreys.

Modules may not have finite length

If $\mathfrak{g}$ is finite-dimensional, then a module V of Category $\mathcal{O}$ is finitely generated. Moreover it has finite length, namely it has a composition series:

$$0 = V_0 \subset V_1 \subset V_2 \subset \cdots \subset V_N = V$$

where V_i/V_{i+1} is irreducible, that is, $V_i \cong L(\lambda_i)$ for some $\lambda \in \mathfrak{h}^*$. If $\mathfrak{g}$ is infinite-dimensional, this is not necessarily true. For example by Exercise 10.3 in Kac, the Verma module $M(0)$ does not have finite length if W is infinite.

The dot action

In the next couple of lectures, we will show that if λ is a dominant integral weight then the primitive vectors in $M(\lambda)$ are at the values $w(\lambda+\rho)-\rho$ for $w \in W$.

This motivates the definition of the "dot action" of the Weyl group, which is the action of W on P shifted so that the fixed point is $-\rho$ instead of 0. Thus define

$$w \cdot \lambda = w(\lambda+\rho)-\rho.$$

Weyl character formula

If $\mathfrak{g}$ is a finite-dimensional semisimple Lie algebra, then we may infer this from the Weyl character formula which we write this way:

$$\mathrm{ch}\ L(\lambda) = \sum_{w\in W} (-1)^{\ell(w)} e^{w(\lambda+\rho)-\rho} \prod_{\alpha\in\Phi^+} (1-e^{-\alpha})^{-1}$$

Remember that

$$\mathrm{ch}\ M(\lambda) = e^{\lambda} \prod_{\alpha\in\Phi^+} (1-e^{-\alpha})^{-1}$$

and so

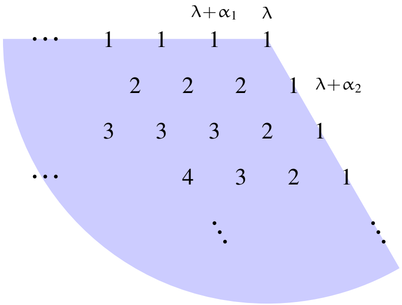
$$\mathrm{ch}\ L(\lambda) = \sum_{w\in W} (-1)^{\ell(w)}\ \mathrm{ch}\ M(w(\lambda+\rho)-\rho).$$

So

$$\mathrm{ch}\ L(\lambda) = \sum_{w\in W} (-1)^{\ell(w)}\ \mathrm{ch}\ M(w\cdot\lambda).$$

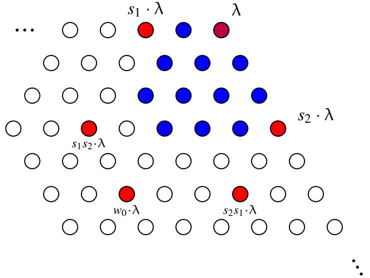
Primitive vectors in $M((3,1,0))$

Here is the Verma module $M(\lambda)$ for $\mathrm{GL}(3)$, showing the weight multiplicities. (These are the values of the Kostant partition function.)



Primitive vectors in $M((3,2,0))$

Let $\lambda = (3,2,0)$. This is a dominant integral weight, so the quotient $L(\lambda)$ of $M(\lambda)$ by its maximal proper submodule is finite-dimensional. The primitive vectors are at the red weights (except the highest weight λ which is purple).



The blue weights are the weights of $L(\lambda)$.