Lecture 4: Kac-Moody Lie algebras

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September 14, 2020

Introduction

In Chapter 1 of Kac, Infinite-dimensional Lie algebras, Kac gives an ingenious construction of the Kac-Moody Lie algebra in his Theorem 1.2. In the notes he asserts that the theorem should be attributed to Chevalley (1948).

Proofs of the construction of the Kac-Moody Lie algebras were given in 1968 independently by Moody and Kac. Both papers contain much more than this construction. The argument in the book is more similar to Moody's 1968 paper than to Kac's.

This proof relies on constructing an auxiliary Lie algebra $\tilde{\mathfrak{g}}$ of which the Kac-Moody Lie algebra is a quotient, then by arguments following Jacobson, Verma modules are constructed by by hand. This gives enough information to construct a quotient that is the desired Lie algebra.

Free Lie algebras

Today we want to describe Lie algebras that are described by generators and relations, so we begin by discussing free Lie algebras. A reference for this topic is Bourbaki, Lie groups and Lie algebras, Chapter 2. See also this article by Casselman:

Free Lie algebras by Casselman (web link)

Let X be a set, which for our purposes will be finite. A magma is a set M with a map $m : M \times M \longrightarrow M$ that we will think of as a kind of multiplication. There is an obvious notion of a homomorphism of magmas: this is a map $\phi : M \longrightarrow M'$ such that $m'(\phi(x), \phi(y)) = \phi(m(x, y))$.

A nonassociative algebra over a field F is an F-vector space A with a bilinear map $\mu : A \times A \longrightarrow A$.

Universal properties

For each of the categories of magmas, vector spaces, nonassociative algebras, Lie algebras and associative there is a notion of a free object over X. Thus the free magma on X is defined by the universal property:

Definition

A free magma on X is a magma M_X together with a map $j : X \longrightarrow M_X$ such that if $\phi : X \longrightarrow P$ is a map from X into a magma, there is a unique homomorphism $\Phi : M_X \longrightarrow P$ such that $\phi = \Phi \circ j$.

Free vector spaces, nonassociative algebras, Lie algebras and associative algebras on X are defined similarly. Two of these constructs are familiar: the free vector space is just the free module which we will denote $F[X]$, and the free associative algebra is just the tensor algebra over $F[X]$.

Free magmas

However we are interested in the free Lie algebra, so we will review its construction. We will construct the free nonassociative algebra as the free vector space over the free magma, and define the free Lie algebra as a quotient.

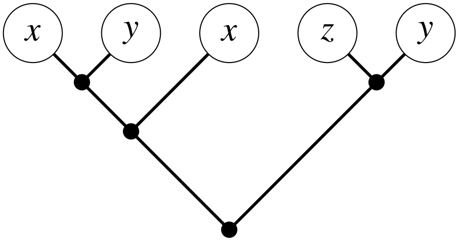
As usual, an object defined by a universal property is unique up to isomorphism. However we need to know the existence of the free object.

We will define the free magma M_X to be the set of finite rooted trees whose leaves are labeled by elements of X. To multiply two trees, we join their roots.

Rooted trees and the free magma

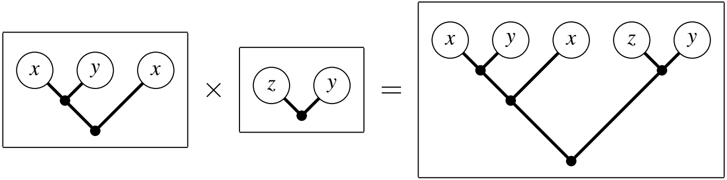
A rooted tree is a directed graph that contains no cycles, and has a initial vertex, the root. Terminal vertices are called leaves. A tree is called binary if each vertex that is not a leaf has two successor vertices, one called ‘left’ and one called ‘right’.

Let M_X be the set of all finite rooted trees with an element of X assigned to each leaf.



The free magma (continued)

To define the product of two rooted labeled binary trees we adjoin a new root. The old roots become the left and right siblings of the new root.



- Identify $x \in X$ with the singleton tree whose leaf is labeled x.
- Write $x \cdot y$ or xy as usual for the multiplication in the magma.
- Use parentheses to disambiguate expressions.

$$(xy)x \times zy = ((xy)x)(zy) \ .$$

The universal property

With X identified as a subset of M_X, the universal property has the following form.

Universal Property

Let P be any magma and $\phi : X \to P$ a mapping. Then ϕ extends uniquely to a magma homomorphism $M_X \to P$.

The universal property is almost obvious.

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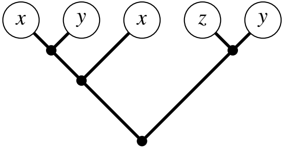


) $= \phi\big(((xy)x)(zy)\big)$

$$= ((\phi(x)\phi(y))\phi(x))(\phi(z)\phi(y)).$$

The Free Lie algebra

The free Lie algebra on X is also characterized by a universal property. We seek a Lie algebra $\mathfrak{L}_X$ together with a mapping $j : X \to \mathfrak{L}_X$ that is universal in the following sense:

Universal Property

If $\phi : X \to \mathfrak{g}$ is any mapping from X into a Lie algebra, there is a unique Lie algebra homomorphism $\Phi : \mathfrak{L}_X \to \mathfrak{g}$ such that $\phi = \Phi \circ j$.

Briefly, any homomorphism from X into a Lie algebra factors through $\mathfrak{L}_X$. As usual, an object characterized by a universal property is unique up to isomorphism, but there is an issue of existence.

Construction

By a nonassociatiative algebra (meaning, more precisely, a "not necessarily associative algebra") we mean a vector space A with a bilinear map $m : A \times A \to A$ that we will think of as a kind of multiplication. We can construct a free nonassociative algebra as the free vector space $F[M_X]$ over the free magma. Combining the universal properties of the free magma and the free vector space, we see that any map from X into a nonassociative algebra factors uniquely through $F[M_X]$.
Now let J be the (two-sided) ideal generated by elements of $F[M_X]$ of the forms:

$$x \cdot x, \qquad x \cdot y + y \cdot x, \qquad (x \cdot y) \cdot z + (y \cdot z) \cdot x + (z \cdot x) \cdot y,$$

for $x, y, z \in F[M_X]$ or equivalently (by an easy argument) for $x, y, z \in M_X$. The quotient $\mathfrak{L}_X = F[M_X]/J$ is a Lie algebra by construction.

The universal enveloping algebra of $\mathfrak{L}_X$

Let V be a vector space. The tensor algebra

$$T(V) = \bigoplus_{k=0}^{\infty} \otimes^k V$$

has the universal property that any linear map from V to an associative algebra factors through $T(V)$.

Proposition

The universal enveloping algebra of $\mathfrak{L}_X$ is isomorphic to the tensor algebra $T(F[X])$.

Both these associative algebras have the same universal property: a homomorphism from X into an associative algebra factors uniquely through A. If $A = U(\mathfrak{L}_X)$ this follows by combining the universal property of the free Lie algebra with the universal property of the enveloping algebra. If $A = T(F[X])$

Roots and coroots encode the Weyl group

If $\mathfrak{g}$ is a finite-dimensional Lie algebra, then a key aspect of its structure is the Weyl group. It acts on the Cartan subalgebra $\mathfrak{h}$ and by duality, on and its dual space $\mathfrak{h}^*$. The group is generated by simple reflections s_i which are given by

$$s_i(x) = x - \langle \alpha_i^\vee, x \rangle \alpha_i, \qquad x \in \mathfrak{h}^*.$$

Dually

$$s_i(y) = y - \langle y, \alpha_i \rangle \alpha_i^\vee, \qquad y \in \mathfrak{h}.$$

Here α_i and $\alpha_i^\vee$ are particular elements of $\mathfrak{h}$ and $\mathfrak{h}^*$ called the simple roots and simple coroots. It is easy to check that these two maps $s_i : \mathfrak{h}^* \to \mathfrak{h}^*$ and $s_i : \mathfrak{h} \to \mathfrak{h}$ have order 2 and are adjoints.

The Cartan matrix

The matrix $A = (a_{ij})$ where $a_{ij} = \langle \alpha_i^\vee, \alpha_j \rangle$ is called the Cartan matrix of $\mathfrak{g}$. A key idea is that from the Cartan matrix, we may produce a Lie algebra $\mathfrak{g}$ containing $\mathfrak{h}$ as an abelian subalgebra. This construction turns out to work nicely with very little required of A. We assume:

- The entries in A are integers.
- The diagonal entries $a_{ii} = 2$
- If $i \neq j$ then $a_{ij} \leqslant 0$.
- $a_{ij} \neq 0$ if and only if $a_{ji} \neq 0$.

For the deeper theory it is necessary to assume that A is symmetrizable. This means that there is a diagonal matrix D such that $A = DB$ where B symmetric. This implies the last condition.

Example: Cartan Type A_3

Suppose $\mathfrak{g} = \mathfrak{sl}_5$ of Cartan type A_3. This root system is simply-laced meaning that all roots have the same length so we may take $\alpha_i = \alpha_i^\vee$. We have

$$\alpha_1 = (1, -1, 0, 0), \qquad \alpha_2 = (0, 1, -1, 0), \qquad \alpha_3 = (0, 0, 1, -1)$$

and

$$A = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix}.$$

For simply laced systems the Cartan matrix is symmetric.

Example: Cartan Type B_3

Next suppose $\mathfrak{g} = \mathfrak{so}(7)$ of Cartan type B_3. We may identify $\mathfrak{h} = \mathbb{C}^3$ in such a way that the simple roots and coroots are

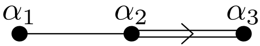
$$\alpha_1 = (1,-1,0), \qquad \alpha_2 = (0,1,-1), \qquad a_3 = (0,0,1)$$

$$\alpha_1^\vee = (1,-1,0), \qquad \alpha_2^\vee = (0,1,-1), \qquad a_3^\vee = (0,0,2).$$

The Cartan matrix is

$$A = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -2 & 2 \end{pmatrix} = DB, \qquad D = \begin{pmatrix} 1 & & \\ & 1 & \\ & & 2 \end{pmatrix}, \quad B = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{pmatrix}.$$

So this type is symmetrizable. Here's the Dynkin diagram:



Example: Cartan Type C_3

Next suppose $\mathfrak{g} = \mathfrak{sp}(6)$ of type C_3. Now we may identify $\mathfrak{h} = \mathbb{C}^3$ in such a way that

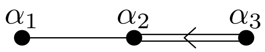
$$\alpha_1 = (1, -1, 0), \qquad \alpha_2 = (0, 1, -1), \qquad a_3 = (0, 0, 2),$$

$$\alpha_1^\vee = (1, -1, 0), \qquad \alpha_2^\vee = (0, 1, -1), \qquad a_3^\vee = (0, 0, 1).$$

Now

$$A = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -2 \\ 0 & -1 & 2 \end{pmatrix} = DB, \qquad D = \begin{pmatrix} 1 & & \\ & 1 & \\ & & 1/2 \end{pmatrix}, \quad B = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -2 \\ 0 & -2 & 2 \end{pmatrix}.$$

This type is also symmetrizable. Here's the Dynkin diagram:



Reminder: Generalized Cartan matrices

Assume that $A = (a_{ij})$ is an $n \times n$ integer matrix such that:

- $a_{ii} = 2$;
- $a_{ij} \leqslant 0$ if $i \neq j$.

Instead of assuming that A is symmetrizable, at first it is enough to assume

- $a_{ij} = 0$ if and only if $a_{ji} = 0$.

Such an A will be called a generalized Cartan matrix.

Realization

Suppose that A is $n \times n$ and that ℓ is the rank of A. Following Kac, Infinite-dimensional Lie algebras, Section 1.1, let us define a realization of A to be a pair of vector spaces $\mathfrak{h}$, $\mathfrak{h}^*$ in duality, with elements $\alpha_i \in \mathfrak{h}^*$, $\alpha_i^\vee \in \mathfrak{h}$. We require:

- Both sets $\{\alpha_i\}$ and $\{\alpha_i^\vee\}$ are linearly independent;
- $\langle \alpha_i^\vee, \alpha_j \rangle = a_{ij}$;
- $n - \ell = \dim(\mathfrak{h}) - n$.

In the case where A is the Cartan matrix of a semisimple Lie algebra $n = \ell$, so $\dim(\mathfrak{h}) = n$. In general we need $\mathfrak{h}$ to be a bit larger than n. In the affine case, we will find that $\dim(\mathfrak{h}) = n + 1$.

Existence of a realization

Proposition (Kac, Proposition 1.1)

There exists a realization of A.

Kac proves that the realization is unique up to an obvious notion of isomorphism.

Proof. Permute the indices if necessary so that the first ℓ rows of A are linearly dependent, and write

$$A = \begin{pmatrix} A_1 \\ A_2 \end{pmatrix}$$

where $\text{rank}(A_1) = \text{rank}(A) = \ell$.

Proof, continued

Consider the $n \times (2n-\ell)$ matrix

$$C = \begin{pmatrix} A_1 & \\ A_2 & I_{n-\ell} \end{pmatrix}.$$

Taking $\mathfrak{h} = \mathbb{C}^{2n-\ell}$ let $\alpha_1, \cdots, \alpha_n \in \mathfrak{h}^*$ be the coordinate functions, and let $\alpha_i^\vee$ be the rows of C. Then we have $\langle \alpha_i^\vee, \alpha_j \rangle = a_{ij}$ and both sets of vectors $\{\alpha_i\}$ and $\{\alpha_i^\vee\}$ are linearly independent.

The plan

The goal is to construct a Lie algebra $\mathfrak{g}$ with a triangular decomposition

$$\mathfrak{g} = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}_+,$$

from the Cartan matrix A and its realization. We will see that $\mathfrak{g}$ can be infinite-dimensional, though of course $\mathfrak{h}$ is finite-dimensional.

The Lie algebra $\mathfrak{g}$ will be a quotient of a larger Lie algebra

$$\tilde{\mathfrak{g}} = \tilde{\mathfrak{n}}_- \oplus \mathfrak{h} \oplus \tilde{\mathfrak{n}}_+$$

which will be more straightforward to construct. The first step is to construct $\tilde{\mathfrak{g}}$. This is done by generators and relations.

A key idea

Before launching into the construction, we isolate a key idea.

Proposition

Let $\mathfrak{h}$ be a finite-dimensional abelian Lie algebra and let V be a module with a weight space decomposition

$$V = \bigoplus_{\mu \in \mathfrak{h}^*} V_\lambda.$$

Let S be subset of the weights in this decomposition. Then V has a subalgebra U that is maximal with respect to the condition that

$$U \cap \bigoplus_{\mu \in S} V_\mu = 0.$$

Proof

This follows from the fact (proved in Lecture 2) that any submodule W of V has itself a weight decomposition, so

$$W = \bigoplus_{\mu \in \mathfrak{h}^*} W_\mu,$$

$W_\mu \subset V_\mu$. So a necessary and sufficient condition for

$$W \cap \bigoplus_{\mu \in S} V_\mu = 0$$

is that $W_\mu = 0$ for $\mu \in S$. Thus the sum of all such W also satisfies this condition.

The Proposition implies, for example that if V is highest weight module, then V has a maximal proper submodule. We just take $S = \{\lambda\}$ where λ is the highest weight.

The Lie algebra $\tilde{\mathfrak{g}}$

We reiterate that the Kac-Moody Lie algebra will be a quotient of another Lie algebra $\widetilde{\mathfrak{g}}$. The Lie algebra $\tilde{\mathfrak{g}}$ is generated by $\mathfrak{h}$ and generators e_i, f_i subject to the relations

$$[e_i, f_j] = \delta_{ij}\alpha_i^\vee \qquad (i, j = 1, \cdots, n)$$

$$[h, h'] = 0 \qquad (h, h' \in \mathfrak{h})$$

$$[h, e_i] = \langle h, \alpha_i \rangle e_i$$

$$[h, f_i] = -\langle h, \alpha_i \rangle f_i \qquad (i = 1, \cdots, n, h \in \mathfrak{h}).$$

We can take the free Lie algebra on the generators e_i, f_i and a basis of $\mathfrak{h}$, and quotient by the ideal generated by these relations.

Digression: the Serre relations

It is possible to add relations called Serre relations to these to obtain the Kac-Moody Lie algebra $\mathfrak{g}$. The Serre relations are precisely what is needed to make the adjoint representation itself integrable. Including these would give a presentation of $\mathfrak{g}$. However what is done in Kac's book is to omit the Serre relations, producing this Lie algebra $\tilde{\mathfrak{g}}$, then obtain $\mathfrak{g}$ as a quotient.

The Serre relations have the form

$$\mathrm{ad}(e_i)^{1-a_{ij}} e_j = -\,\mathrm{ad}(f_i)^{1-a_{ij}} f_j = 0$$

It is easy to see (conceptually) why these should be true and we will consider them later when we discuss integrability.

The main theorem

Theorem (Theorem 1.2 in Kac)

The Lie algebra $\tilde{\mathfrak{n}}_+$ generated by the e_i is free on these generators, and the algebra $\tilde{\mathfrak{n}}_-$ generated by the f_i is similarly free. The Lie algebra $\tilde{\mathfrak{g}}$ has a weight space decomposition with finite-dimensional weight spaces. There is a unique ideal $\mathfrak{r}$ that is maximal with respect to the condition that $\mathfrak{r} \cap \mathfrak{h} = 0$. Furthermore

$$\mathfrak{r} = (\mathfrak{r} \cap \tilde{\mathfrak{n}}_+) \oplus (\mathfrak{r} \cap \tilde{\mathfrak{n}}_-) \ .$$

This was proved independently by Kac and Moody in 1968. The proof in Moody's paper (A New Class of Lie Algebras) is similar to what Kac does in his book. The key idea is to build a Verma module by hand, and extract key information from it. This construction is essentially in Jacobson's 1962 book on Lie algebras.

Proof

Proof. Let $\lambda : \mathfrak{h} \longrightarrow \mathbb{C}$ be a linear functional. The proof depends on constructing the corresponding Verma module by hand. Since $\tilde{\mathfrak{n}}_-$ is larger than the corresponding $\mathfrak{n}_-$ in the quotient $\mathfrak{g}$, this Verma module is larger (but also simpler) than the Verma module for $\mathfrak{g}$.

Once the Verma module is in hand we will extract key information that will allow us to prove that $\mathfrak{n}_-$ and $\mathfrak{n}_+$ are free Lie algebras on the f_i and e_i, and to obtain the triangular decomposition. Finishing the proof will use "key idea" argument.

Proof (continued)

Let V be an n-dimensional vector space with basis $v_1, \cdots, v_n$. We will construct a representation of $\tilde{\mathfrak{g}}$ on the tensor algebra $T(V)$. We define

$$f(a) = v_i \otimes a, \qquad a \in T(V)$$

$$h(1) = \lambda(h)1, \qquad h \in \mathfrak{h},$$

and recursively on $\otimes^s V$:

$$h(v_j \otimes a) = \alpha_j(h) v_j \otimes a + v_j \otimes h(a), \qquad a \in \otimes^{s-1} V$$

$$e_i(1) = 0$$

and recursively on $\otimes^s V$:

$$e_i(v_j \otimes a) = \delta_{ij} \alpha_i^\vee(a) + v_j \otimes e_i(a), \qquad a \in \otimes^{s-1} V.$$

These maps are well-defined, depending only on the universal property of the tensor product. But we have to show that they satisfy the relations that we've imposed on the generators.

Proof (continued)

The relation

$$[h, h'] = 0 \qquad (h, h' \in \mathfrak{h})$$

is obvious since h acts diagonally. We have

$$(e_i f_j - f_j e_i)(a) = e_i(v_j \otimes a) - v_j \otimes e_i(a) = \delta_{ij}\alpha_i^\vee(a) + v_j \otimes e_i(a) - v_j \otimes e_i(a)$$

so

$$(e_i f_j - f_j e_i)(a) = \delta_{ij}\alpha_i^\vee(a).$$

We omit the verification of the other two properties

$$[h, e_i] = \langle h, \alpha_i \rangle e_i,$$

$$[h, f_i] = -\langle h, \alpha_i \rangle f_i \qquad (i = 1, \cdots, n, h \in \mathfrak{h}).$$

See Kac for details of these.

The Lie algebra $\mathfrak{n}_-$ is free

We will argue next that $T(V)$ is a universal enveloping algebra for $\mathfrak{n}_-$, and that $\mathfrak{n}_-$ is free on the generators f_i.

We have a representation of $\mathfrak{n}_-$ on $T(V)$ from restricting the representation of $\tilde{\mathfrak{g}}$ constructed above, hence we have a homomorphism of $U(\mathfrak{n}_-)$ to $T(V)$, which is a surjection since the f_i map to algebra generators v_i of $T(V)$. We wish to argue that this homomorphism is an isomorphism. Let $\mathfrak{f}$ be the free Lie algebra on $v_1, \cdots, v_n$. We have shown earlier in this lecture that $T(V) \cong U(\mathfrak{f})$. We have a Lie algebra homomorphism $\mathfrak{f} \longrightarrow \mathfrak{n}_-$ in which $v_i \longmapsto f_i$. The kernel of this Lie algebra homomorphism is contained in the kernel of the composition $T(V) \cong U(\mathfrak{f}) \longrightarrow U(\mathfrak{n}_-) \longrightarrow T(V)$, which is the identity map, so actually $\mathfrak{f} \cong \mathfrak{n}_-$. We have proved that $\mathfrak{n}_-$ is free on the generators f_i, and also that its enveloping algebra is $T(V)$.

Towards the triangular decomposition

Our next task is to argue that

$$\tilde{\mathfrak{g}} = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}_+.$$

First note that $\mathfrak{n}_- + \mathfrak{h} + \mathfrak{n}_+ = \tilde{\mathfrak{g}}$. Indeed $\mathfrak{n}_- + \mathfrak{h} + \mathfrak{n}_+$ contains the generators e_i, f_i and $\mathfrak{h}$ and (from the defining relations) is closed under $\mathrm{ad}(e_i)$, $\mathrm{ad}(h_i)$ and $\mathrm{ad}(h)$ for $h \in \mathfrak{h}$, so this much is clear.

However we must argue that the sum $\mathfrak{n}_- + \mathfrak{h} + \mathfrak{n}_+ = \tilde{\mathfrak{g}}$ is direct.

The sum $\tilde{\mathfrak{g}} = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}_+$ is direct

To prove that the sum $\mathfrak{n}_- + \mathfrak{h} + \mathfrak{n}_+$ is direct, let n_-, h, n_+ be elements of $\mathfrak{n}_-$, $\mathfrak{h}$ and $\mathfrak{n}_+$ such that $n_- + h + n_+ = 0$. Apply this identity to the module $1 \in T(V)$. Then $n_+(1) = 0$ while $h(1) = \langle\lambda, h\rangle$. We see that $n_-(1) + \langle\lambda, h\rangle = 0$ for all $\lambda \in \mathfrak{h}^*$. The only way $\langle\lambda, h\rangle$ can be independent of λ is if $h = 0$ and so $n_-(1) = 0$. We have shown that $T(V)$ may be identified with the enveloping algebra of $\mathfrak{n}_-$. With this identification, $n_- \in \mathfrak{n}_-$ maps to the element $n_-(1) \in T(V)$. Since the inclusion of a Lie algebra in its enveloping algebra is injective, $n_-(1) = 0$ implies that $n_- = 0$. We have proved that if $n_- + h + n_+ = 0$ then h and n_- both vanish, and therefore the sum $\tilde{\mathfrak{g}} = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}_+$ is direct.

Weight space decomposition: $\mathfrak{n}_-$

We will argue that $\tilde{\mathfrak{g}}$ has a weight space decomposition. The first step is to show that $\mathfrak{n}_-$ has a weight space decomposition. It is an $\mathfrak{h}$-module via the adjoint representation.

It follows from PBW that $U(\mathfrak{n}_-) \cong T(V)$ has a weight space decomposition with finite multiplicities, since every standard vector $f_{i_1} \cdots f_{i_k}$ with $i_1 \cdots i_k$ is an $\mathfrak{h}$-eigenvector with weight $-\alpha_{i_1} - \ldots - \alpha_{i_k}$, and these span $\mathfrak{n}_-$. Because the α_i are linearly independent, the dimensions of the weight spaces are finite.

Now since $\mathfrak{n}_-$ is an $\mathfrak{h}$-submodule of $T(V)$, $\mathfrak{n}_-$ itself has a weight space decomposition.

The weight space decomposition (continued)

It is enough to see that each of $\mathfrak{n}_-$, $\mathfrak{h}$ and $\mathfrak{n}_+$ separately have weight space decompositions. We have discussed $\mathfrak{n}_-$ and of course $\mathfrak{h}$ itself is a weight space with eigenvalue 0.

This leaves $\mathfrak{n}_+$.

We have a automorphism $\omega : \tilde{\mathfrak{g}} \longrightarrow \tilde{\mathfrak{g}}$ such that $\omega(e_i) = -f_i$, $\omega(f_i) = -e_i$ and $\omega(h) = -h$, which follows from the presentation of $\tilde{\mathfrak{g}}$ by generators and relations. Applying the involution, we may deduce that $\mathfrak{n}_+$ is also a free Lie algebra generated by the e_i.

The ideal $\mathfrak{r}$

The last thing to prove is that $\tilde{\mathfrak{g}}$ has an ideal $\mathfrak{r}$ that is maximal with respect to the condition that $\mathfrak{r} \cap \mathfrak{h} = 0$.

From the weight space decomposition, a necessary and sufficient condtion for $\mathfrak{r} \cap \mathfrak{h} = 0$ is that

$$\mathfrak{r} \subseteq \bigoplus_{\alpha \neq 0} \tilde{\mathfrak{g}}_\alpha.$$

From this characterization it is obvious that the sum of such $\mathfrak{r}$ does not meet $\mathfrak{h}$ and therefore there is an ideal that is maximal for the condition that $\mathfrak{r} \cap \mathfrak{h} = 0$.

The theorem is now proved.

The definition of the Kac-Moody Lie algebra

Now we may define the Kac-Moody Lie algebra associated to a generalized Cartan matrix A as follows. We start with a realization $(\mathfrak{h}, \mathfrak{h}^*)$.

Definition

The Lie algebra $\mathfrak{g}$ is generated by $\mathfrak{h}$ and generators e_i, f_i subject to the relations

$$[e_i, f_j] = \delta_{ij}\alpha_i^\vee \qquad (i, j = 1, \cdots, n)$$

$$[h, h'] = 0 \qquad (h, h' \in \mathfrak{h})$$

$$[h, e_i] = \langle h, \alpha_i \rangle e_i$$

$$[h, f_i] = -\langle h, \alpha_i \rangle f_i \qquad (i = 1, \cdots, n, h \in \mathfrak{h}).$$

Additionally, there is a requirement that $\mathfrak{g}$ has no nonzero ideal $\mathfrak{r}$ such that $\mathfrak{r} \cap \mathfrak{h} = 0$.

Existence and uniqueness of the Kac-Moody Lie algebra

Proposition

There is a unique Lie algebra $\mathfrak{g}$ satisfying this definition.

Clearly we may obtain such a Lie algebra by taking $\tilde{\mathfrak{g}}$ and quotienting by the ideal $\mathfrak{r}$ that is maximal with respect to the condition that $\mathfrak{r} \cap \mathfrak{h} = 0$. Conversely, given a Lie algebra $\mathfrak{g}$ with generators satisfying such relations, there is a homomorphism $\tilde{\mathfrak{g}} \longrightarrow \mathfrak{g}$ mapping the generators in $\tilde{\mathfrak{g}}$ to the generators in $\mathfrak{g}$, and it is easy to see that the kernel of this homomorphism must be the maximal $\mathfrak{r}$ such that $\mathfrak{r} \cap \mathfrak{h} = 0$.

Finite-dimensional simple Lie algebras

Proposition

Let $\mathfrak{g}$ be a simple Lie algebra with Cartan matrix A. Then $\mathfrak{g}$ is the Kac-Moody Lie algebra associated to A.

This is clear since $\mathfrak{g}$ has generators with the given relations and no proper nonzero ideals, a fortiori no nonzero ideals $\mathfrak{r}$ such that $\mathfrak{r} \cap \mathfrak{h} = 0$.

Affine Lie algebras are Kac-Moody Lie algebras

Theorem

Let $\mathfrak{g}$ be a finite-dimensional simple Lie algebra. Let $\hat{\mathfrak{g}}$ be the untwisted affine Lie algebra associated to $\mathfrak{g}$. Then $\hat{\mathfrak{g}}$ is the Kac-Moody Lie algebra associated to the extended Cartan matrix $\hat{A}$ of $\mathfrak{g}$.

We must show that any nonzero ideal $\mathfrak{r}$ intersects $\hat{\mathfrak{h}}$. We know that $\mathfrak{r}$ has a weight space decomposition, so it contains an element X_n of some $\mathfrak{X}_\alpha$ where $\alpha = \mathring{\alpha} + n\delta$ is a root. Here $X \in \mathfrak{g}_{\mathring{\alpha}}$. We may find $Y \in \mathfrak{g}_{-\mathring{\alpha}}$ such that $(X|Y) \neq 0$. We have

$$[X_n, Y_{-n}] = [X, Y]_0 + n(X|Y)K.$$

This is in $\widehat{\mathfrak{h}}$ so it must vanish. Therefore $n = 0$ and so $\mathring{\alpha} \neq 0$. This implies that $[X, Y] \neq 0$, but $[X, Y] \in \mathfrak{h}$, contradiction.