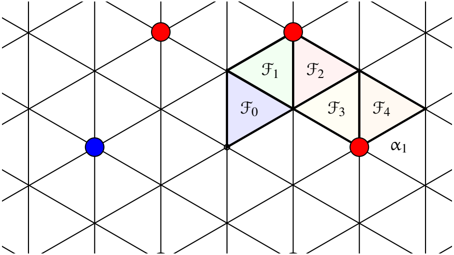
Lecture 3: affine Lie algebras

Daniel Bump



Our story begins

Let $\mathfrak{g}$ be a finite-dimensional simple Lie algebra. Today we will construct the associated untwisted affine Lie algebra $\hat{\mathfrak{g}}$ in two steps: first we will make a central extension of $\mathbb{C}[t, t^{-1}] \otimes \mathfrak{g}$, then adjoin a derivation.

This construction appeared in the current algebra introduced by Gell-Mann in quantum field theory in the 1960's. Affine Lie algebras occur in many areas of physics and mathematics, such as conformal field theory.

The Killing form

We will need is a symmetric bilinear form $(\ |\ )$ on $\mathfrak{g}$ that is ad-invariant. This means that

$$(\mathrm{ad}(x)y|z) = -(y|\,\mathrm{ad}(x)z)$$

for $x, y, z \in \mathfrak{g}$. Since $\mathrm{ad}(x)y = [x, y]$ is skew-symmetric and $(|)$ is symmetric, this is equivalent to

$$([x, y]|z) = (z|[x, y]) = ([z, x]|y).$$

If $\mathfrak{g}$ is finite-dimensional, the Killing form is an ad-invariant symmetric bilinear form. This is

$$(x|y) = \mathrm{tr}(\mathrm{ad}(x)\,\mathrm{ad}(y)).$$

Proposition

The Killing form is invariant.

Proof

To check invariance, remember the property of the trace $\mathrm{tr}(AB) = \mathrm{tr}(BA)$ if A, B are endomorphisms of some vector space. Using this and the fact that ad is a representation

$$([x, y]|z) = \mathrm{tr}(\mathrm{ad}[x, y]\,\mathrm{ad}(z)) =$$

$$\mathrm{tr}(\mathrm{ad}(x)\,\mathrm{ad}(y)\,\mathrm{ad}(z)) - \mathrm{tr}(\mathrm{ad}(y)\,\mathrm{ad}(x)\,\mathrm{ad}(z))$$

$$= \mathrm{tr}(\mathrm{ad}(y)\,\mathrm{ad}(z)\,\mathrm{ad}(x)) - \mathrm{tr}(\mathrm{ad}(y)\,\mathrm{ad}(x)\,\mathrm{ad}(z)) =$$

$$\mathrm{tr}(\mathrm{ad}(y)\,\mathrm{ad}([z, x])) = (y|[z, x]).$$

The cocycle

The tensor product of a commutative (and associative) algebra A with a Lie algebra $\mathfrak{g}$ is a Lie algebra. To make $A \otimes \mathfrak{g}$ into a Lie algebra, define

$$[a \otimes X, b \otimes Y] = ab \otimes [X, Y].$$

The skew-symmetry and Jacobi identities are easy to check.

Thus we may consider the Lie algebra $\mathfrak{g}_t = \mathbb{C}[t, t^{-1}] \otimes \mathfrak{g}$ with spanned by elements of the form $X_n = t^n \otimes X$ $(n \in \mathbb{Z})$ subject to the bracket

$$[t^n \otimes X, t^m \otimes Y] = t^{n+m} \otimes [X, Y].$$

Now we can define a 2-cocycle $\phi : \mathfrak{g}_t \otimes \mathfrak{g}_t \longrightarrow \mathbb{C}$ by

$$\boxed{\phi(X_n, Y_m) = n\delta_{n,-m}(X|Y)}.$$

The cocycle relation

Let us check that this is a cocycle. We compute

$$\phi([X_n, Y_m], Z_p) = \delta_{n+m+p,0}(n+m)([X, Y], Z).$$

Now $\phi([X_n, Y_m], Z_p)$ vanishes unless $n+m+p=0$. Assuming this,

$$\phi([X_n, Y_m], Z_p) = (n+m)([X, Y]|Z).$$

Now

$$\phi([X_n, Y_m], Z_p) + \phi([Y_m, Z_p], X_n) + \phi([Z_p, X_n], Y_m)$$

$$= (n+m)([X, Y]|Z) + (m+p)([Y, Z]|X) + (p+n)([Z, X]|Y).$$

This vanishes because $([X, Y]|Z)$ is invariant under cyclic permutations of the indices, and $n+m+p=0$. Also ϕ is skew-symmetric, so it is a 2-cocycle.

The affine Lie algebra

Now let $\hat{\mathfrak{g}}'$ be the Lie algebra obtained as a central extension using this cocycle:

$$0 \longrightarrow \mathbb{C} \longrightarrow \hat{\mathfrak{g}}' \longrightarrow \mathfrak{g}_t \longrightarrow 0.$$

We will denote the image of $1 \in \mathbb{C}$ as K.

Finally, there is a derivation $d : \mathfrak{g}_t \longrightarrow \mathfrak{g}_t$ defined by $d = t\frac{d}{dt}$, so $d(X_n) = nX_n$. This may be considered a derivation of $\hat{\mathfrak{g}}'$. Adjoining this gives the affine Lie algebra $\hat{\mathfrak{g}}$. Thus

$$[X_n + \lambda K + \mu d, Y_m + \rho K + \nu d]$$

$$= [X, Y]_{n+m} + \delta_{n,-m} n(X|Y)K + \mu m Y_m - \nu n X_n \ .$$

$\hat{\mathfrak{g}}'$ is the derived Lie algebra $[\hat{\mathfrak{g}}, \hat{\mathfrak{g}}]$. It has all the interesting representations of $\hat{\mathfrak{g}}$ but $\hat{\mathfrak{g}}$ is better to work with.

Orthogonality of root spaces

Lemma

Suppose $\mathfrak{g}$ is a Lie algebra with a weight space decomposition

$$\mathfrak{g} = \bigoplus_{\alpha} \mathfrak{g}_\alpha$$

with respect to the abelian subalgebra $\mathfrak{h}$. Let $(\ |\ )$ be an ad-invariant bilinear form on $\mathfrak{g}$. Then $\mathfrak{g}_\alpha$ and $\mathfrak{g}_\beta$ are orthogonal unless $\alpha = -\beta$.

Proof. If $a \neq -\beta$ find $H \in \mathfrak{h}$ such that $\alpha(H) \neq -\beta(H)$. If $X \in \mathfrak{g}_\alpha$ and $y \in \mathfrak{g}_\beta$ then

$$\alpha(H)(X|Y) = ([H,X]|Y) = -(X|[H,Y]) = -\beta(H)(X|Y).$$

Therefore $(X|Y) = 0$.

Embedding $\mathfrak{g}$ in $\hat{\mathfrak{g}}$

The map $X \longmapsto X_0$ is a Lie algebra homomorphism $\mathfrak{g} \longrightarrow \hat{\mathfrak{g}}$ so the affine Lie algebra $\hat{\mathfrak{g}}$ contains a copy of $\mathfrak{g}$.

Now suppose that the finite-dimensional Lie algebra $\mathfrak{g}$ is a simple Lie algebra with root system Φ with respect to a Cartan subalgebra $\mathfrak{h}$. Thus $\mathfrak{h}$ is a maximal abelian subalgebra $\mathfrak{h}$ and $\mathfrak{g}$ has a weight space decomposition

$$\mathfrak{g} = \bigoplus_{\alpha} \mathfrak{g}_\alpha, \qquad \mathfrak{g}_0 = \mathfrak{h}.$$

The root spaces $\mathfrak{g}_\alpha$ with $\alpha \in \Phi$ are one-dimensional. We will generally denote them $\mathfrak{X}_\alpha$ instead of $\mathfrak{g}_\alpha$. We have a triangular decomposition

$$\mathfrak{g} = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}_+, \qquad \mathfrak{n}_\pm = \bigoplus_{\alpha \in \Phi_\pm} \mathfrak{X}_\alpha.$$

Embedding $\mathfrak{h}$ in $\hat{\mathfrak{h}}$

We identify $\mathfrak{g}$ and in particular $\mathfrak{h}$ with their images under the map $X \mapsto X_0$, which is a Lie algebra homomorphism $\mathfrak{g} \longrightarrow \hat{\mathfrak{g}}$. Let $\hat{\mathfrak{h}} = \mathfrak{h} \oplus \mathbb{C}K \oplus \mathbb{C}d$, which is a maximal abelian subalgebra of $\hat{\mathfrak{g}}$. Its dimension is $\dim(\mathfrak{h}) + 2$.

The Cartan subalgebra $\mathfrak{h}$ of the finite dimensional Lie algebra $\mathfrak{g}$ is a subalgebra of the Cartan subalgebra $\hat{\mathfrak{h}} = \mathfrak{h} \oplus \mathbb{C}K \oplus \mathbb{C}d$. It will be convenient to consider $\mathfrak{h}^*$ to be a subspace of $\hat{\mathfrak{h}}^*$. This means we want a canonical extension of any linear functional λ on $\mathfrak{h}$ to $\hat{\mathfrak{h}}$. We define the extension to be zero on $\mathbb{C}K \oplus \mathbb{C}d$.

The null root δ

Note that if α is a root of $\mathfrak{g}$, this extension makes it a root of $\hat{\mathfrak{g}}$. Let us consider why this is true. If $X_\alpha \in \mathfrak{g}$ spans the root space for α, so $[H, X_\alpha] = \alpha(H)X_\alpha$ for $H \in \mathfrak{h}$, then we are identifying X_α with $(X_\alpha)_0 \in \mathfrak{g}$. The identity $[H, (X_\alpha)_0] = \alpha(H)(X_\alpha)_0$ defines an extension to $\mathfrak{h}$ and from the formula

$$[X_n + \lambda K + \mu d, Y_m + \rho K + \nu d]$$

$$= [X, Y]_{n+m} + \delta_{n,-m} n(X|Y)K + \mu m Y_m - \nu n X_n \ .$$

we see that $\alpha(K) = \alpha(d) = 0$.

We define $\delta \in \hat{\mathfrak{h}}^*$ to be the linear functional that is zero on $\hat{\mathfrak{h}}' = \mathfrak{h} \oplus \mathbb{C}K$ but $\delta(d) = 1$. This too is a root, with multiplicity $r = \dim(\mathfrak{h})$. Indeed, $\mathfrak{h}_1$ is the root eigenspace.

The root space decomposition of $\widehat{\mathfrak{g}}$

Proposition

The space $\hat{\mathfrak{h}}$ *is a maximal abelian subalgebra of* $\hat{\mathfrak{g}}$*. We have a weight space decomposition*

$$\hat{\mathfrak{g}} = \hat{\mathfrak{h}} \oplus \bigoplus_{\substack{n\in\mathbb{Z}\\ n\neq 0}} \hat{\mathfrak{X}}_{n\delta} \oplus \bigoplus_{\substack{n\in\mathbb{Z}\\ \alpha\in\Phi}} \hat{\mathfrak{X}}_{\alpha+n\delta}.$$

Here $\hat{\mathfrak{X}}_{n\delta}$ $(n \neq 0)$ *equals* $\mathfrak{h}_n$ *and has dimension* $\dim(\mathfrak{h})$*. The space* $\hat{\mathfrak{X}}_{\alpha+n\delta}$ *is one-dimensional and is spanned by* $(X_\alpha)_n$*.*

Proof

This is easy to prove from the formula

$$[X_n + \lambda K + \mu d, Y_m + \mu K + \nu d]$$

$$= [X, Y]_{n+m} + \delta_{n,-m} n(X|Y)K + \mu m Y_m - \nu n X_n \ .$$

From this $H \in \mathfrak{h}$ we have (identifying $H = H_0$). So if $X \in \mathfrak{X}_\alpha$ we have

$$[H, X_m] = \alpha(H) X_m, \qquad [K, X_m] = 0, \qquad [d, X_m] = m X_m.$$

Thus for $\hat{H} \in \hat{\mathfrak{h}}$ we have

$$[\hat{H}, X_m] = (\alpha + m\delta)(\hat{H}) X_m.$$

For the spaces $\hat{\mathfrak{X}}_{n\delta} = \mathfrak{h}_n$ we also check easily that

$$[\hat{H}, (H')_n] = n\delta(\hat{H})(H')_n.$$

The root space decomposition

Thus $\hat{\mathfrak{g}}$ has the following roots with respect to the Cartan subalgebra $\hat{\mathfrak{h}}$:

- The real roots $\alpha + n\delta$ with $\alpha \in \Phi$ and $n \in \mathbb{Z}$; these have multiplicity 1;
- The imaginary roots $n\delta$ with $0 \neq n \in \mathbb{Z}$; these have multiplicity $\dim(\mathfrak{h})$.

So let $\hat{\Phi}$ be the set of roots of $\hat{\mathfrak{g}}$ as enumerated above. We wish to partition these into positive and negative roots. A real root $\alpha + n\delta$ is defined to be positive if either $n > 0$ or $n = 0$ and $\alpha \in \Phi^+$. An imaginary root $n\delta$ ($n \neq 0$) is positive if $n > 0$. Let $\widehat{\Phi}^+$ and $\hat{\Phi}^-$ be the positive and negative roots.

The simple roots

A positive root α is simple if it is real, and if there is no decomposition of α as a sum of a sum of other positive roots. Let $\alpha_1, \cdots, \alpha_r$ be the simple roots of $\mathfrak{g}$. It is easy to see that these are simple roots of $\hat{\mathfrak{g}}$. However there is one more simple root that we will now define. Let $\theta \in \Phi$ be the highest root of $\mathfrak{g}$. We will call the positive root $\delta - \theta$ the affine root.

Proposition

The root $\alpha_0 = \delta - \theta$ is a simple root. The simple roots are

$$\{\alpha_0, \alpha_1, \cdots, \alpha_r\}.$$

Proof

We are slightly modifying the definition of α_0 that we used in discussing finite-dimensional simple Lie algebras, where we defined $\alpha_0 = -\theta$. Shifting by δ has important advantages, the first one being that the α_i are now linearly independent. Recall that

$$\theta = \sum_{i=1}^{r} a_i \alpha_i$$

in terms of the marks (or labels) discussed in Lecture 3. Thus (with $a_0 = 1$)

$$\delta = \sum_{i=0}^{r} a_i \alpha_i$$

It is easy to see that there is no way to write $\delta - \theta$ as a sum of positive roots, but any positive root that is not in the above list can be decomposed as a sum of other positive roots.

The coroots

We also have coroots $\alpha_i^\vee \in \mathfrak{h}$ defined for $i = 1, \cdots, r$. We define $\alpha_0^\vee$ by requiring

$$K = \sum_{i=0}^{r} a_i^\vee \alpha_i^\vee .$$

With $a_0^\vee = 1$, this means

$$\alpha_0^\vee = K - \sum_{i=1}^{r} a_i^\vee \alpha_i^\vee = K - \theta^\vee$$

where $\theta^\vee$ is the coroot of $\mathfrak{g}$ associated to the highest root θ. (Note: $\theta^\vee$ is not the highest root of $\Phi^\vee$ if Φ is not simply laced. Indeed, $\theta^\vee$ is a short root, and the highest root is a long root.

The Cartan matrix

Lemma

Let $(a_{ij})_{i,j=0}^{n}$ be the extended Cartan matrix of the finite-dimensional Lie algebra $\mathfrak{g}$. Then

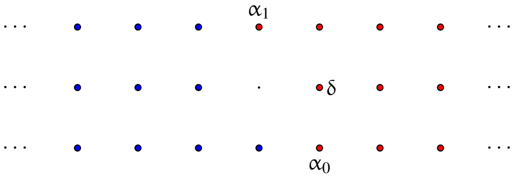
$$a_{ij} = \langle \alpha_i^{\vee}, \alpha_j \rangle.$$

We gave definitions of α_0 and $\alpha_0^{\vee}$ as elements of $\mathfrak{h}^*$ and $\mathfrak{h}$ in Lecture 3, but now we have changed these definitions. Let $\mathring{\alpha}_0$ and $\mathring{\alpha}_0^{\vee}$ be the Lecture 3 affine root and coroot, so now $\alpha_0 = \mathring{\alpha}_0 + \delta$ and $\alpha_0^{\vee} = \mathring{\alpha}_0^{\vee} + K$.

By definition $a_{ij} = \langle \mathring{\alpha}_i^{\vee}, \mathring{\alpha}_j \rangle$. Now $\langle K, \lambda \rangle = 0$ if $\lambda \in \mathfrak{h}^*$ since we defined the extension of $\lambda \in \mathfrak{h}^*$ to a functional on $\hat{\mathfrak{h}}$ by making it zero on K and d. Also $\langle H, \delta \rangle = 0$ if $H \in \mathfrak{h}' = \mathfrak{h} \oplus \mathbb{C}K$. The Lemma follows.

Example

The simplest affine Lie algebra is $\hat{\mathfrak{sl}}(2)$. Here is the root system.



The red nodes are positive roots, the blue nodes are negative.

The Cartan matrix is $\begin{pmatrix} 2 & -2 \\ -2 & 2 \end{pmatrix}$.

The Weyl group

Our goal is to define the affine Weyl group, with actions on $\hat{\mathfrak{h}}$ and $\hat{\mathfrak{h}}^*$. These are to be infinite Coxeter groups generated by simple reflections s_i $(i = 0, 1, \cdots, r)$ to be given by the formulas

$$s_i(x) = x - \langle \alpha_i^\vee, x \rangle \alpha_i, \qquad x \in \hat{\mathfrak{h}}^*,$$

$$s_i(x) = x - \langle x, \alpha_i \rangle \alpha_i^\vee, \qquad x \in \hat{\mathfrak{h}}.$$

The group these generate is isomorphic to the affine Weyl group defined in Chapter 3, though now they are acting on a larger vector space $\hat{\mathfrak{h}}^*$.

Review: finite-dimensional semisimple Lie algebras

Recall that if $\mathfrak{g}$ is a finite-dimensional semisimple Lie algebra then the integral weights are the elements $\lambda \in \mathfrak{h}^*$ such that $\langle \alpha_i^\vee, \lambda \rangle \in \mathbb{Z}$. They form a lattice, $P \subseteq \mathfrak{h}^*$. The dominant weights λ are the elements of P such that $\langle \alpha_i^\vee, \lambda \rangle$ is a nonnegative integer number for $i = 1, \cdots, r$. Let P^+ be the cone of dominant weights.

Integral weights and dominant weights

We adapt this definition to the affine Lie algebra $\hat{\mathfrak{g}}$. We call $\lambda \in \hat{\mathfrak{h}}^*$ an integral weight if $\lambda(\alpha_i^\vee) \in \mathbb{Z}$. Let P be the set of integral weights. This is not quite a lattice since $\mathbb{C}\delta \subseteq P$. However $P/\mathbb{C}\delta$ is a lattice an by abuse of language we will call P the weight lattice. An integral weight λ is called dominant if $\lambda(\alpha_i^\vee)$ are nonnegative integers. Let P^+ be the set of dominant weights.

The dominant weights are special because the irreducible highest weight representation $L(\lambda)$ is integrable in that is can be lifted to the simply-connected Lie group G such that $\mathrm{Lie}(G) = \mathfrak{g}$. This is a more basic property than the fact that $L(\lambda)$ is finite-dimensional.

Integrability and the Weyl group

Integrability implies that $V = L(\lambda)$ has a weight space decomposition

$$V = \bigoplus_{\lambda \in P} V_\lambda$$

where the weight multiplicities $\dim(V_\lambda)$ are W-invariant.

Affine Lie algebras do have group analogs, the loop groups. However there is no need to construct these in order to generalize the notion of integrability. For a general Kac-Moody Lie algebra $\mathfrak{g}$ there is a Weyl group, and a notion of integrability that implies that the weight multiplicities are W-invariant.

Integrability will be discussed in a later lecture. Today we will focus on the Weyl group and its action on the weights.

The weight lattice

Returning to the affine algebra $\hat{\mathfrak{g}}$ we defined the weight lattice $\hat{P}$ to be the set of $\lambda \in \hat{\mathfrak{h}}^*$ such that $\lambda(\alpha_i^\vee) \in \mathbb{Z}$.
We define elements $\Lambda_0, \cdots, \Lambda_r$ of $\hat{P}$ by

$$\langle \alpha_i^\vee, \Lambda_j \rangle = \delta_{i,j}, \qquad \langle d, \Lambda_j \rangle = 0.$$

P contains $\mathbb{C}\delta$ and the Λ_i are a basis of the free abelian group $P/\mathbb{C}\delta$. Remembering that

$$K = \sum_{i=0}^{r} a_i^\vee \alpha_i^\vee,$$

we have

$$\langle K, \Lambda_j \rangle = a_j^\vee.$$

The dominant weights have the form $\sum n_i \Lambda_i + c\delta$ with $n_i \in \mathbb{N}$ and $c \in \mathbb{C}$.

Action of W on roots

For now let us investigate the effect of the Weyl group on roots. We begin by classifying the roots as positive and negative. First the real roots $\alpha + n\delta$ ($\alpha \in \Phi$) are positive if either $n > 0$ or $n = 0$ and $\alpha \in \Phi^+$. An imaginary root $n\delta$ ($n \neq 0$) is positive if $n > 0$. Any root that is not positive is negative. Let $\hat{\Phi}^+$ be the positive roots, and $\hat{\Phi}^-$ be the set of negative roots.

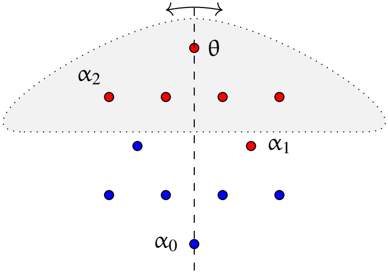
Recall that the imaginary root δ is orthogonal to $\hat{\mathfrak{h}}' = \hat{\mathfrak{h}} \oplus \mathbb{C}K$ while $\delta(d) = 1$. Since the $\alpha_i^\vee \in \hat{\mathfrak{h}} \oplus \mathbb{C}K$, we see that

$$s_i(\delta) = \delta - \langle \alpha_i^\vee, \delta \rangle \alpha_i = \delta.$$

Thus the Weyl group does not move the imaginary roots!

A property of root systems

Let us consider a root system Φ with a finite Weyl group W. It is a well-known, and very important, property of Φ that if s_i is the simple reflection corresponding to the simple root α_i, then α_i is the unique positive root α such that $s_i(\alpha) \in \Phi^-$. Thus s_i permutes $\Phi^+ - \{\alpha_i\}$.



Showing that s_1 permutes $\Phi - \{\alpha_1\}$ for G_2

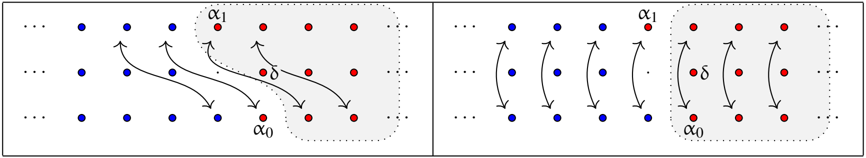
The case of the affine Weyl group

The same is true of the affine Weyl group. Of course $s_i(\alpha_i) = -\alpha_i$ (since $\langle \alpha_i^\vee, \alpha_i \rangle = 2$). But this is the only positive root that is mapped to a negative root by s_i.

Proposition

If $\alpha \in \hat{\Phi}^+$ is a positive root distinct from α_i then $s_i(\alpha)$ is a positive root.

Below: the simple reflections s_0, s_1 for $\widehat{\mathfrak{sl}}(2)$.



Proof: the case $1 \leqslant i \leqslant r$

Write $\alpha = \mathring{\alpha} + n\delta$ where $\mathring{\alpha} \in \Phi$. For $i \geqslant 1$ we have

$$s_i(\alpha) = s_i(\mathring{\alpha} + n\delta) = s_i(\mathring{\alpha}) + n\delta.$$

Since $\mathring{\alpha} \in \Phi$ and $s_i \in W$ (the finite Weyl group) $s_i(\mathring{\alpha}) + n\delta \in \Phi + n\delta$. If $n > 0$ this is a positive root by definition since $\Phi + n\delta \subseteq \hat{\Phi}^+$. On the other hand if $n = 0$ we remember that $\mathring{\alpha}$ is a positive root of Φ that is distinct from Φ_i, and so $s_i(\alpha)$ is also a positive root. The case $i > 0$ is proved.

The case $i = 0$

Now consider the case where $i = 0$. As before

$$s_0(\alpha) = s_0(\mathring{\alpha}) + n\delta = \mathring{\alpha} + n\delta - \langle \alpha_0^\vee, \mathring{\alpha} + n\delta \rangle \alpha_0.$$

We recall that $\alpha_0^\vee = K - \theta^\vee$ where θ is the highest root, and $\alpha_0 = \delta - \theta$. Since $\langle K, \mathring{\alpha} + n\delta \rangle = 0$ and $\langle \alpha_0^\vee, \delta \rangle = 0$

$$s_0(\alpha) = \mathring{\alpha} + n\delta + \langle \theta^\vee, \mathring{\alpha} \rangle \alpha_0 = \mathring{\alpha} + n\delta - \langle \theta^\vee, \mathring{\alpha} \rangle \theta + \langle \theta^\vee, \mathring{\alpha} \rangle \delta$$

$$= r_\theta(\mathring{\alpha}) + (n + \langle \theta^\vee, \mathring{\alpha} \rangle)\delta.$$

We will argue that either $n + \langle \theta^\vee, \mathring{\alpha} \rangle \geqslant 0$ and $r_\theta(\mathring{\alpha})$ is a positive root, or $n + \langle \theta^\vee, \mathring{\alpha} \rangle \geqslant 1$. In either case, this is enough to show that $s_0(\alpha) \in \hat{\Phi}^+$.

Proof (continued)

Lemma

The inner product $\langle\theta^\vee, \mathring{\alpha}\rangle = -2$ *if* $\mathring{\alpha} = -\theta$*. Otherwise* $\langle\theta^\vee, \mathring{\alpha}\rangle \geqslant -1$*. If* $\langle\theta^\vee, \mathring{\alpha}\rangle = -1$*, then* $r_\theta(\mathring{\alpha})$ *is a positive root. If* $\mathring{\alpha}$ *is a positive root, then* $\langle\theta^\vee, \mathring{\alpha}\rangle \geqslant 0$*.*

Of course $\langle\theta^\vee, -\theta\rangle = -2$. To show that $\langle\theta^\vee, \mathring{\alpha}\rangle \geqslant -1$ if $\mathring{\alpha} \neq -\theta$, we note that θ is a long root. Thus it is enough to show that if $\theta \in \Phi$ is a long root, and $\mathring{\alpha} \neq -\theta$, then $\langle\theta^\vee, \mathring{\alpha}\rangle \geqslant -1$. To prove this, note that θ and $\mathring{\alpha}$ can be embedded in a root system of rank 1 or 2, so it is sufficient to check this for the Cartan types A_2, B_2, G_2, and this is easy to do.

Proof of the Lemma (concluded)

Suppose that $\langle \theta^\vee, \mathring{\alpha} \rangle = -1$. Then $r_\theta(\mathring{\alpha}) = \mathring{\alpha} - \langle \theta^\vee, \mathring{\alpha} \rangle \theta = \mathring{\alpha} + \theta$. This is a root since $r_\theta(\Phi) = \Phi$, and it cannot be a negative root since if $\mathring{\alpha} + \theta = -\gamma$ with $\gamma \in \Phi^+$ then $\theta + \gamma = -\mathring{\alpha} \in \Phi$, which is a contradiction beause $\theta + \gamma$ cannot be a root because θ is the highest root. So $r_\theta(\mathring{\alpha})$ is a positive root.

It remains to show that if $\mathring{\alpha}$ is a positive root then $\langle \theta^\vee, \mathring{\alpha} \rangle \geqslant 0$. Clearly we may assume that $\mathring{\alpha} = \alpha_i$ is a simple reflection. Consider $s_{\alpha_i}(\theta) = \theta - \langle \alpha_i^\vee, \theta \rangle \alpha_i$. Since θ is the highest root, we must have $\langle \alpha_i^\vee, \theta \rangle \geqslant 0$. (Otherwise $s_{\alpha_i}(\theta)$ would be higher.)

The end of the proof

Returning to the identity

$$s_0(\alpha) = r_\theta(\mathring{\alpha}) + (n + \langle \theta^\vee, \mathring{\alpha} \rangle)\delta,$$

the Lemma contains all the information we need to check that this is in $\hat{\Phi}^+$. Indeed, since $n > 0$, the only way $n + \langle \theta^\vee, \mathring{\alpha} \rangle$ can be negative is if $n = 1$ and $\mathring{\alpha} = -\theta$, but in that case $\alpha = \delta - \theta = \alpha_0$ and this case is excluded by the hypothesis of the Proposition. Therefore $n + \langle \theta^\vee, \mathring{\alpha} \rangle \geqslant 0$. Now $n + \langle \theta^\vee, \mathring{\alpha} \rangle > 0$ unless $n = 1$ and $\langle \theta^\vee, \mathring{\alpha} \rangle = -1$. In this case $r_\theta(\mathring{\alpha})$ is a positive root and in every case we are done.

Formulaire

Here are the pairings of particular elements of $\hat{\mathfrak{h}}^*$ with $\hat{\mathfrak{h}}$.

	α_0	$\alpha_j (j > 0)$	δ	Λ_0	$\Lambda_j (j > 0)$
$\alpha_0^\vee$	1	a_{0j}	0	1	0
$\alpha_i^\vee (i > 0)$	a_{i0}	a_{ij}	0	0	δ_{ij}
K	0	0	0	1	$a_j^\vee$
d	1	0	1	0	0

$$\alpha_0 = \delta - \theta, \qquad \alpha^\vee = K - \theta^\vee.$$

$$\sum_{i=0}^{r} a_i \alpha_i = \delta, \qquad \sum_{i=0}^{r} a_i^\vee \alpha_i^\vee = K,$$

$$s_i(x) = x - \langle \alpha_i^\vee, x \rangle \alpha_i, \qquad x \in \hat{\mathfrak{h}}^*,$$

$$s_i(x) = x - \langle x, \alpha_i \rangle \alpha_i^\vee, \qquad x \in \hat{\mathfrak{h}}.$$

Sage

```
CT = CartanType("B3~")
RS = RootSystem(CT)
WL = RS.weight_lattice(extended=True)
alpha = WL.alpha()
alphacheck = WL.alphacheck()
delta = WL.null_root()
Lambda = WL.fundamental_weights()
K = WL.null_coroot()
a = CT.col_annihilator()
ac = CT.row_annihilator()
```

Here is Sage code to create most of the elements of $\hat{\mathfrak{h}}$ and $\hat{\mathfrak{h}}^*$ on the previous page. Be sure and use `extended=True` in creating $\hat{\mathfrak{h}}^*$. Unfortunately Sage does not create $\hat{\mathfrak{h}}$ but rather $\hat{\mathfrak{h}}'$ which omits d.

Sage

Then the inner products can be checked for Type B3:

```
sage: [Lambda[i].scalar(K) for i in [0..3]]
[1, 1, 2, 1]
sage: [ac[i] for i in [0..3]]
[1, 1, 2, 1]
sage: m = Matrix([[alpha[j].scalar(alphacheck[i])
    for j in [0..3]] for i in [0..3]]); m
[ 2  0 -1  0]
[ 0  2 -1  0]
[-1 -1  2 -1]
[ 0  0 -2  2]
sage: m == CT.cartan_matrix()
True
```

Level

If $\lambda \in \mathfrak{h}^*$, the value $\langle \lambda, K \rangle$ is called the level of λ. Thus every root has level 0, including the null root δ. But the fundamental weight Λ_j has value $a_j^\vee$. In particular Λ_0 has level 1, and in Type A all Λ_i have level 1.

Note that $\langle K, \alpha_i \rangle = 0$. From this and the formula

$$s_i(x) = x - \langle \alpha_i^\vee, x \rangle \alpha_i, \qquad x \in \hat{\mathfrak{h}}^*,$$

it follows that λ and $w(\lambda)$ are of the same level for $w \in W$.

Affine Weyl groups

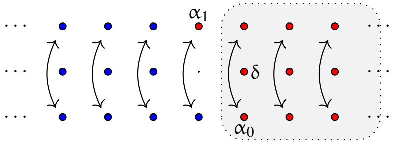
We have introduced two notions of the affine Weyl group. In Lecture 3 we defined the affine Weyl group as a group of reflections in $\mathfrak{h}^*$, where $\mathfrak{h}$ is the Cartan subalgebra of the finite-dimensional simple Lie algebra $\mathfrak{g}$. This affine Weyl group contains translations by roots as a normal subgroup and is actually the semidirect product of the finite Weyl group by the translations.

In this Lecture we have another group that we are also calling the affine Weyl group. This W_{aff} acts linearly on $\hat{\mathfrak{h}}^*$.

Why are the two affine Weyl groups the same?

Action on Level 0

Since the level k elements of $\hat{\mathfrak{h}}^*$ are preserved by W_{aff}, we may study the action of W_{aff} on weights of fixed level. The weights of level 0 are spanned by the roots, and W_{aff} acts by permuting them (but without moving the imaginary roots).



Action on level 1

Let us do an experiment, and compute part of the Weyl orbit of Λ_0 for the $\hat{\mathfrak{sl}}(2)$. We could do this by hand using information in the formulaire, or we can use Sage.

```
RS = RootSystem("A1~")
WL = RS.weight_lattice(extended=True)
Lambda = WL.fundamental_weights()
alpha = WL.alpha()
delta = WL.null_root()
W = WL.weyl_group(prefix="s")
(s0,s1) = W.simple_reflections()
ws = [s1*s0*s1*s0*s1,s0*s1*s0*s1,s1*s0*s1,s0*s1,s1,
W.one(),s0,s1*s0,s0*s1*s0,s1*s0*s1*s0,s0*s1*s0*s1*s0]
```

The orbit of Λ_0

```
sage: for w in ws:
....:     print ("%s : %s"%(w,Lambda[0].weyl_action(w)))

s1*s0*s1*s0*s1 : 5*Lambda[0] - 4*Lambda[1] - 4*delta
s0*s1*s0*s1 : -3*Lambda[0] + 4*Lambda[1] - 4*delta
s1*s0*s1 : 3*Lambda[0] - 2*Lambda[1] - delta
s0*s1 : -Lambda[0] + 2*Lambda[1] - delta
s1 : Lambda[0]
1 : Lambda[0]
s0 : -Lambda[0] + 2*Lambda[1] - delta
s1*s0 : 3*Lambda[0] - 2*Lambda[1] - delta
s0*s1*s0 : -3*Lambda[0] + 4*Lambda[1] - 4*delta
s1*s0*s1*s0 : 5*Lambda[0] - 4*Lambda[1] - 4*delta
s0*s1*s0*s1*s0 : -5*Lambda[0] + 6*Lambda[1] - 9*delta
```

The stabilizer

In the previous slide we considered the dominant weight Λ_0 of level 1 and calculated its Weyl group orbit. We note that every value appears twice in that table. This is because Λ_0 has a nontrivial stabilizer, which happens to be the subgroup $W = \langle s_1 \rangle$.

The affine Weyl group is the semidirect product of W and the infinite cyclic group $\langle s_1 s_0 \rangle$. So let us repeat the calculation with this in mind.

Calculating the orbit

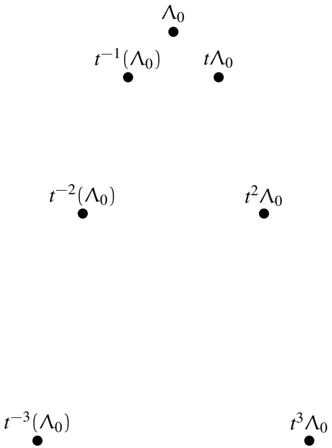
```
sage: t = s1*s0
sage: for k in [-4..4]:
....:     print ("t^%s : %s"%(k,Lambda[0].weyl_action(t^k)))
t^-4 : -7*Lambda[0] + 8*Lambda[1] - 16*delta
t^-3 : -5*Lambda[0] + 6*Lambda[1] - 9*delta
t^-2 : -3*Lambda[0] + 4*Lambda[1] - 4*delta
t^-1 : -Lambda[0] + 2*Lambda[1] - delta
t^0 : Lambda[0]
t^1 : 3*Lambda[0] - 2*Lambda[1] - delta
t^2 : 5*Lambda[0] - 4*Lambda[1] - 4*delta
t^3 : 7*Lambda[0] - 6*Lambda[1] - 9*delta
t^4 : 9*Lambda[0] - 8*Lambda[1] - 16*delta
sage: alpha[1]
-2*Lambda[0] + 2*Lambda[1]
```

So with $t = s_1 s_0$,

$$t^k(\Lambda_0) = \Lambda_0 - k\alpha_1 - k^2\delta.$$

The orbit of Λ_0 in $\hat{\mathfrak{h}}^*$

$$t^k(\Lambda_0) = \Lambda_0 + k\alpha_1 - k^2\delta$$
$$t = s_1 s_0$$



The level k action

We defined an affine Weyl group in Lecture 3 acting on $\mathfrak{h}^*$. This group is generated by $s_0, s_1, \cdots, s_r$ where $s_1, \cdots, s_r$ are the simple reflections in the finite Weyl group (acting on $\mathfrak{h}^*$) and s_0 is the reflection in the hyperplane

$$\{x \mid \langle \theta^\vee, x \rangle = 1\}.$$

We will call this the classical level 1 affine Weyl group.

We could equally well consider the group generated by $s_1, \cdots, s_r$ and the reflection in the hyperplane

$$\{x \mid \langle \theta^\vee, x \rangle = k\}.$$

The action would be similar but the fundamental alcove would be larger, and contain more roots.

The relationship between the two affine Weyl groups

The affine Weyl group acts on $\hat{\mathfrak{h}}^*$. This vector space is two dimensions bigger than $\mathfrak{h}^*$, but we will cut it down in two ways.

Since the affine Weyl group fixes δ, there is an induced action on $\hat{\mathfrak{h}}^*/\mathbb{C}\delta$.

Moreover we may fix the level k and consider the action on the level k (affine) subspace of $\hat{\mathfrak{h}}^*/\mathbb{C}\delta$.

Theorem

The action of W_{aff} on the level k subspace of $\hat{\mathfrak{h}}^*/\mathbb{C}\delta$ is equivalent to the classical level k action. The equivalence is the map

$$\lambda \mapsto \lambda + k\Lambda_0 \quad \textit{mod}\ \mathbb{C}\delta\ .$$

Proof

If $1 \leqslant i \leqslant r$, then

$$s_i(\Lambda_0) = \Lambda_0 - \langle \alpha_i^\vee, \Lambda_0 \rangle \alpha_i = \Lambda_0,$$

because $\langle \alpha_i^\vee, \Lambda_0 \rangle = 0$. So this map is equivariant for s_i if $i \neq 0$.

We must check equivariance for s_0. Indeed

$$s_0(\lambda + k\Lambda_0) = \lambda + k\Lambda_0 - \langle \alpha_0^\vee, \lambda + k\Lambda_0 \rangle \alpha_0 =$$

$$\lambda + k\Lambda_0 - \big(\langle \theta^\vee, \lambda \rangle + k\big)(\delta - \theta).$$

We may discard the δ since we are quotienting by $\mathbb{C}\delta$. Thus

$$s_0(\lambda + k\Lambda_0) = r_\theta(\lambda) + k\theta + k\Lambda_0.$$