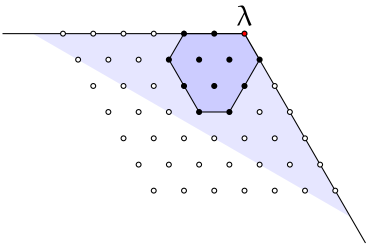
Lecture 2: Finite-dimensional simple Lie algebras

Daniel Bump



Finite-dimensional and affine Lie algebras

Kac-Moody Lie algebras include two special cases that will concern us in this course: the finite-dimensional and affine Lie algebras. There are two reasons for starting with finite-dimensional simple Lie algebras, even though you may already be familiar with their theory.

- They are the first example of Kac-Moody Lie algebras, so they are good for building intuition.
- The second example of affine Lie algebras are canonically constructed from finite-dimensional simple Lie algebras.

There are two kinds of affine Lie algebras: the twisted and untwisted ones. For lack of time we will focus on the untwisted affine Lie algebras.

Review: the root decomposition

Let $\mathfrak{g}$ be a finite-dimensional simple complex Lie algebra. Let $\mathfrak{h}$ be the Cartan subalgebra, a maximal abelian subalgebra such that we have a weight space decomposition of $\mathfrak{g}$ under the adjoint representation:

$$\mathfrak{g} = \bigoplus_{\alpha \in \mathfrak{h}^*} \mathfrak{g}_\alpha,$$

$$\mathfrak{g}_\alpha = \{X \in \mathfrak{g} \mid [H, X] = \alpha(H)X, H \in \mathfrak{h}\}.$$

Thus $\mathfrak{g}_0 = \mathfrak{h}$ and if $\alpha \neq 0$ such that $\mathfrak{g}_\alpha \neq 0$ then α is called a root. We will use the notation $\mathfrak{X}_\alpha$ instead of $\mathfrak{g}_\alpha$. The root spaces $\mathfrak{X}_\alpha$ are one-dimensional.

Review: the triangular decomposition

As usual we decompose the root system Φ into positive and negative roots and write

$$\mathfrak{n}_+ = \bigoplus_{\alpha \in \Phi^+} \mathfrak{X}_\alpha, \qquad \mathfrak{n}_- = \bigoplus_{\alpha \in \Phi^-} \mathfrak{X}_\alpha.$$

Then we have the triangular decomposition

$$\mathfrak{g} = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}_+.$$

This has many consequences that were discussed last time such as (using PBW)

$$U(\mathfrak{g}) = U(\mathfrak{n}_-) \otimes U(\mathfrak{h}) \otimes U(\mathfrak{n}_+),$$

and the theory of highest weight representations.

Review: highest weight representations

Definition

Given any $\lambda \in \mathfrak{h}^*$, a highest weight module V is one generated by a vector v_λ in the λ-weight space V_λ such that

$$\mathfrak{n}_+ v_\lambda = 0, \qquad H v_\lambda = \lambda(H) v_\lambda,\ H \in \mathfrak{h}.$$

The weight space V_λ is one-dimensional. There are two highest weight modules worth mentioning: the universal highest weight module $M(\lambda)$, called a Verma module, and the unique irreducible highest weight module $L(\lambda)$. If V is any highest weight module for λ there are unique (up to scalar) homomorphisms

$$M(\lambda) \to V \to L(\lambda).$$

If λ is in general position $L(\lambda) = M(\lambda)$.

The dual root system

In addition to the root system $\Phi \subset \mathfrak{h}^*$ we have a dual root system $\Phi^\vee \subset \mathfrak{h}$. Indeed there is a bijection $\alpha \mapsto \alpha^\vee$ from $\Phi \subset \mathfrak{h}^*$ to $\Phi^\vee \subset \mathfrak{h}$. If Φ is not a simply-laced root system the map $\alpha \mapsto \alpha^\vee$ is not linear.

The bijection has the property that $\langle \alpha^\vee, \alpha \rangle = 2$. This implies that the bijection takes long roots to short roots, and vice versa.

Elements of $\Phi^\vee$ are called coroots.

The Weyl group action

The coroots encode the Weyl group action in which the simple reflection s_i acts on $\mathfrak{h}^*$ by

$$s_i(\lambda) = \lambda - \langle \alpha_i^\vee, \lambda \rangle \alpha_i$$

and on $\mathfrak{h}$ by the adjoint action

$$s_i(H) = H - \langle H, \alpha_i \rangle \alpha_i^\vee .$$

The Weyl group arises because $\mathfrak{g}$ is the Lie algebra of a Lie group G and $\mathfrak{h}$ is the Lie algebra of its maximal torus T. The group $W = N(T)/T$ then acts on $\mathfrak{h}$ and $\mathfrak{h}^*$. But alternatively we may define W to be the group generated by the s_i determined as above, if the roots and coroots are known.

Reflections

We will always denote a symmetric bilinear form, to be thought of as an inner product, by $(\ |\ )$. We will denote the pairing of a vector space with its dual by $\langle\ |\ \rangle$.

The simple Lie algebra $\mathfrak{g}$ admits an ad-invariant symmetric bilinear form $(\ |\ )$ (the Killing form). The form remains nondegenerate when restricted to $\mathfrak{h}$. We may thus regard $\mathfrak{h}$ as a complexified Euclidean space, and identify it with $\mathfrak{h}^*$. We will not make this identification but it is useful as a psychological device to see that the reflection s_α is a (complexified) Eulidean reflection map.

With the identification of $\mathfrak{h} = \mathfrak{h}^*$

$$\alpha^\vee = \frac{2\alpha}{(\alpha|\alpha)}.$$

Integral and Dominant weights

Let α_i be the simple roots, which are those positive roots that cannot be decomposed into sums of other positive roots. The corresponding coroots $\alpha_i^\vee$ are also simple in this sense.

Definition

(i) $\lambda \in \mathfrak{h}^*$ is an integral weight if $\langle \alpha_i^\vee, \lambda \rangle \in \mathbb{Z}$ for the simple coroots $\alpha_i^\vee \in \mathfrak{H}$.
(ii) if furthermore $\langle \alpha_i^\vee, \lambda \rangle \geqslant 0$ we say that λ is dominant.

The integral weights are the differentials (divided by $2\pi i$) characters of the maximal torus T whose Lie algebra if $\mathfrak{h}$.

In other words, an integral weight is a character of $\mathfrak{h} = \mathrm{Lie}(T)$ that can be integrated to a character of T.

The order relation

There is a partial order on $\mathfrak{h}^*$ in which $\lambda \succcurlyeq \mu$ if

$$\lambda - \mu = \sum_{i=1}^{r} n_i \alpha_i$$

where n_i are nonnegative integers.

Proposition

Let V be a highest weight module for λ. If μ is a weight of V (so $V_\mu \neq 0$) then $\lambda \succcurlyeq \mu$.

Proof. Since V is a quotient of $M(\lambda)$ it is sufficient to prove this for $M(\lambda)$. Last time we computed the character

$$\sum_{\mu} \dim(M(\lambda)_\mu) e^{\mu} = e^{\lambda} \prod_{\alpha \in \Phi^+} (1 - e^{-\alpha})^{-1}.$$

Proof (continued)

Thus

$$\sum_{\mu} \dim(M(\lambda)_{\mu}) e^{\mu} = e^{\lambda} \prod_{\alpha \in \Phi^+} (1 + e^{-\alpha} + e^{-2\alpha} + \cdots)$$

$$= \sum_{\nu} \mathfrak{P}(\nu) e^{\lambda - \nu}$$

where $\mathfrak{P}(\nu)$ is the number of representations of ν in the form

$$\nu = \sum n_{\alpha} \alpha.$$

The function $\mathfrak{P}$ is called the Kostant partition function. If ν is a sum (with integer coefficients) of positive roots, it is also a sum (with integer coefficients) of simple roots, so $\lambda \succcurlyeq \mu$ where $\mu = \lambda - \nu$.

Integrable representations

Suppose $\mathfrak{g} = \mathrm{Lie}(G)$ and that V is a $\mathfrak{g}$-module. We say that V is integrable if it is the differential of a representation of G.

(In the Kac-Moody theory we will modify this definition so that we do not have to construct G, though we will still have the Weyl group.)

Let V be an integrable representation and let

$$V = \bigoplus_{\mu} V_{\mu}$$

be its weight space decomposition. Then the representation of G, restricted to $N(T)$ permutes the weight spaces and so

$$\dim(V_{\mu}) = \dim(V_{w\mu}), \qquad w \in W.$$

The Verma module is not integrable

Let $\mathrm{supp}(V)$ be the set of weights of V:

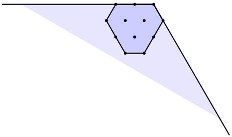
$$\mathrm{supp}(V) = \{\mu \in \mathfrak{h}^* \mid V_\mu \neq 0\}.$$

The support of $M(\lambda)$ is

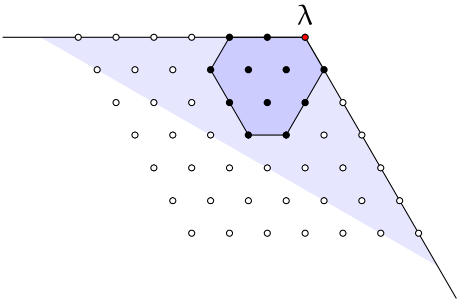
$$\{\mu \in \mathfrak{h}^* \mid \lambda \succcurlyeq \mu\} .$$

This is not stable under W, so $M(\lambda)$ is not integrable.

Visually, here is the convex span of $W\lambda$ inside the cone of μ such that $\lambda \succcurlyeq \mu$ for a dominant weight λ.



Supports of $M(\lambda)$ and $L(\lambda)$



The weights of $L(\lambda)$ with $\lambda = (3,1,0)$ for $\mathfrak{sl}(3)$ are the black dots in the dark shaded hexagon. They are W-invariant. The weights of $M(\lambda)$ are all $\mu \preccurlyeq \lambda$ in the lighter shaded region (white dots). They are not W-invariant so $M(\lambda)$ cannot be integrable.

Finite-dimensional irreducibles

Let $P \subset \mathfrak{h}^*$ be the weight lattice of positive integral weights. Let P^+ be the subset of dominant weights. Assume that the simple Lie algebra $\mathfrak{g}$ is the Lie algebra of a simply connected complex analytic Lie group G.

Theorem (Weyl)

If $\lambda \in P^+$ then there is an irreducible highest weight module for λ. It is a module for G as well as $\mathfrak{g}$.

Since there is a unique irreducible highest weight module for λ the module in Weyl's theorem is the unique irreducible highest weight module $L(\lambda)$. We see that if $\lambda \in P^+$ then $L(\lambda)$ is integrable in the sense that it integrates to a representation of the Lie group G.

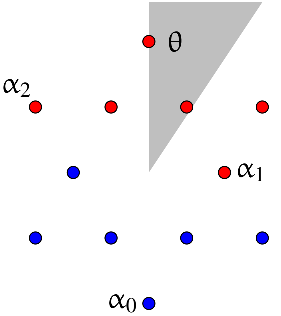
The highest root

By Weyl's theory, every finite-dimensional irreducible has a highest weight. Applying this to the adjoint representation of $\mathfrak{g}$ on itself, there is a highest root θ, and every root satisfies $\lambda \succcurlyeq \alpha$.

Let $\{\alpha_1, \cdots, \alpha_r\}$ be the simple roots. Then $\langle \alpha_i \,|\, \alpha_j \rangle \leqslant 0$ if $i \neq j$, where $(\ |\ )$ is an ad-invariant positive definite inner product on $\mathfrak{g}$, such as the Killing bilinear form.

Let us provisionally define $\alpha_0 = -\theta$. (We will modify this definition later in the context of affine Lie algebras.) It remains true that $(\alpha_i|\alpha_j) \leqslant 0$ when $i \neq j$ on the extended set $\{\alpha_0, \alpha_1, \cdots, \alpha_r\}$.

Example: The G_2 root system



Red: Positive roots Blue: negative roots.

α_1 and α_2 are the simple roots, $\theta = 3\alpha_1 + 2\alpha_2$ is the highest root, and $\alpha_0 = -\theta$ is the affine root.

Cartan matrix and Dynkin diagram

.

The data $\alpha_i \in \mathfrak{h}^\vee$ and $\alpha_i \in \mathfrak{h}$ encode the Weyl group action. There are two useful ways of representing this data. First, the Cartan matrix is the matrix of inner products $a_{ij} = \langle \alpha_i^\vee, \alpha_j \rangle$. In the Kac-Moody theory, the Cartan matrix is the starting point of the construction.

Graphically, the Dynkin diagram is a diagram whose vertices correspond to the α_i, and whose edge structure describes the relationship between two vertices.

There is an extended Cartan matrix and an extended Dynkin diagram in which we add α_0.

Example: the B_4 Cartan matrix

The Lie algebra of $\mathfrak{so}(9)$ is of type B_3 in the Cartan classification. We may identify the root lattice $P \subset \mathfrak{h}^*$ and its dual lattice $P^\vee$ with $\mathbb{Z}^4$. With this identification, the simple roots and coroots are:

$$\alpha_1 = (1,-1,0,0), \quad \alpha_2 = (0,1,-1,0), \quad \alpha_3 = (0,0,1,-3), \quad \alpha_4 = (0,0,0,1),$$

$$\alpha_1^\vee = (1,-1,0,0), \quad \alpha_2^\vee = (0,1,-1,0), \quad \alpha_3^\vee = (0,0,1,-1), \quad \alpha_4^\vee = (0,0,0,2).$$

The highest root

$$\theta = (1,1,0,0) = \alpha_1 + 2\alpha_2 + 2\alpha_3 + 3\alpha_4.$$

The Cartan matrix (a_{ij}) where $a_{ij} = \langle \alpha_i^\vee, \alpha_j \rangle$ is

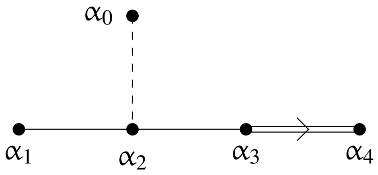
$$\begin{pmatrix} 2 & -1 & & \\ -1 & 2 & -1 & \\ & -1 & 2 & -1 \\ & & -2 & 2 \end{pmatrix}$$

Example: the B_4 extended Cartan matrix

If we add the affine root $\alpha_0 = (-1,-1,0,0)$ to the roots, and $\alpha_0^\vee = (-1,-1,0,0)$ in the dual lattice, we get the extended Cartan matrix

$$\begin{pmatrix} 2 & & -1 & & \\ & 2 & -1 & & \\ -1 & -1 & 2 & -1 & \\ & & -1 & 2 & -1 \\ & & & -2 & 2 \end{pmatrix}.$$

The extended Dynkin diagram (which we will review below) is

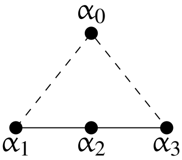


The Dynkin diagram

The Dynkin diagram is a graph whose vertices are the simple roots. Draw an edge connecting α_i to α_j if they are not orthogonal.

For the extended Dynkin diagram, we add a node for α_0.

We use a dashed line for connections of α_0. Here is the extended Dynkin diagram for A_3:



$$\begin{aligned}\alpha_1 &= (1, -1, 0, 0)\\ \alpha_2 &= (0, 1, -1, 0)\\ \alpha_3 &= (0, 0, 1, -1)\\ \alpha_0 &= (-1, 0, 0, 1)\end{aligned}$$

Double and triple bonds

If α_i and α_j have different lengths, we connect them by:

- a double bond if their root lengths are in the ratio $\sqrt{2}$;
- a triple bond if their root lengths are in the ratio $\sqrt{3}$.

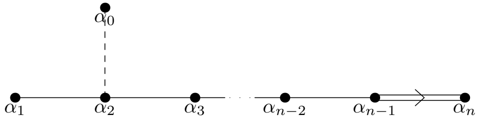
The triple bond only occurs with G_2. Here are the angles of the roots:

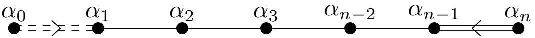
bond	angle	example
no bond	$\frac{\pi}{2}$	$\mathfrak{sl}(2) \times \mathfrak{sl}(2)$
single bond	$\frac{2\pi}{3}$	$\mathfrak{sl}(3)$
double bond	$\frac{3\pi}{4}$	$\mathfrak{sp}(4)$
triple bond	$\frac{5\pi}{6}$	G_2

The direction of the arrow

If the roots are connected by a double or triple bond, they have different lengths. We draw an arrow from the long root to the short root.

Here are the extended Dynkin diagram of type B_n and C_n:

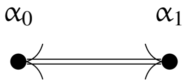




The case of $\mathrm{SL}(2)$

In the case of $\mathrm{SL}(2)$, something happens that does not occur in other cases: the highest root θ happens to coincide with a simple root α_1.

The roots α_0 and α_1 have the same length. We may draw the extended Dynkin diagram with a unique bond $\Longleftrightarrow$:



The nodes α_0 and α_1 are interchangeable by an outer automorphism, so this notation does not distinguish them. This type of bond occurs only in this case.

The marks or labels of the extended diagram

In the table Aff 1 on page 54 of Kac's book *Infinite-dimensional Lie algebras* are the extended Dynkin diagrams of simple Lie algebras with certain coprime positive integers a_i called labels or marks next to each vertex α_i. These satisfy

$$\sum_{i=0}^{r} a_i \alpha_i = 0.$$

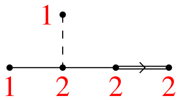
To look at this another way, write

$$\theta = \sum_{i=1}^{r} a_i \alpha_i.$$

We may do this with integers $a_i > 0$ since every positive root is a sum of simple roots. Then we may $a_0 = 1$.

The Coxeter number

Here are the labels a_i for B_4:



the vector $(a_0, \cdots, a_r)$ annihilates the columns of the extended Cartan matrix (since $\sum_{i=0}^{r} a_i \alpha_i = 0$).

The number $h = \sum a_i$ is called the Coxeter number. It has various meanings. For example the product of simple reflections $s_1 \cdots s_r$ (in any order) is called a Coxeter element and h is its order.

The comarks and dual Coxeter number

With θ the longest root, $\theta^\vee$ may be expanded in terms of the simple coroots:

$$\theta^\vee = \sum a_i^\vee \alpha_i^\vee .$$

Then with $a_0^\vee = 1$ we have

$$\sum_{i=0}^{r} a_i^\vee \alpha_i^\vee = 0.$$

Note that if the root system is not simply-laced, $\theta^\vee$ is not the highest element of the dual root system $\Phi^\vee$. Indeed, $\theta^\vee$ is a short root, and the highest root element is a long root.

The vector $(a_0^\vee, \cdots, a_r^\vee)$ annihilates the rows of the extended Cartan matrix. The ubiquitous number $h^\vee = \sum a_i^\vee$ is called the dual Coxeter number.

Informational

In Kac's book the dual Coxeter numbers are tabulated in Chapter 6. Here are Sage incantations to obtain the labels, colabels, Coxeter and dual Coxeter numbers.

```
sage: CartanType("B4~").col_annihilator() #labels
Finite family {0: 1, 1: 1, 2: 2, 3: 2, 4: 2}
sage: CartanType("B4~").row_annihilator() #colabels
Finite family {0: 1, 1: 1, 2: 2, 3: 2, 4: 1}
sage: CartanType("B4").coxeter_number()
8
sage: CartanType("B4").dual_coxeter_number()
7
```

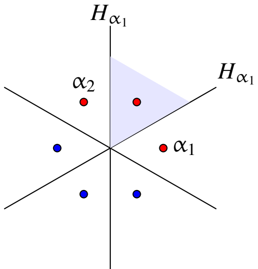
- Coxeter Number (Wikipedia)
- Discussion of dual Coxeter numbers on Math Overflow

Review: the positive Weyl chamber

The Weyl group W of the finite-dimensional semisimple Lie algebra $\mathfrak{g}$ acts on $\mathfrak{h}^*$. If α is a root, let

$$H_\alpha = \{x \in \mathfrak{h}^* \mid \langle \alpha^\vee, x \rangle = 0\}$$

be the hyperplane orthogonal to α. The Weyl group is generated by reflections s_i in the walls of the positive Weyl chamber, which are the hyperplanes $H_i = H_{\alpha_i}$. For $\mathfrak{sl}(3)$ here is the positive Weyl chamber (shaded blue)

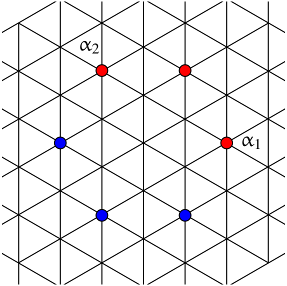


The affine Weyl group

We may expand the configuration of hyperplanes to include

$$H_{\alpha,n} = \{x \in \mathfrak{h}^* \mid \langle \alpha^\vee, x \rangle = n\}, \qquad n \in \mathbb{Z}.$$

The group generated by reflections in these hyperplanes is the affine Weyl group W_{aff}. For $\mathfrak{sl}(3)$:

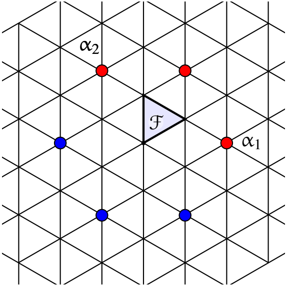


The fundamental alcove

To generate, we only need $s_1, \cdots, s_r$ and one more, s_0 which is the reflection in the hyperplane $H_{\theta,1}$. A fundamental domain $\mathcal{F}$ for W_{aff} is obtained by adjoining the inequality

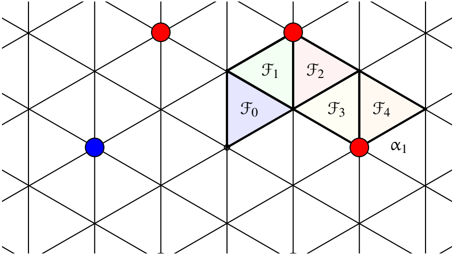
$$\langle \theta^\vee, x \rangle \leqslant 1.$$

For $\mathfrak{sl}(3)$:



Translations in the affine Weyl group

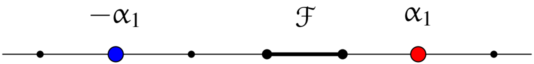
The affine Weyl group is generated by $s_0, \cdots, s_r$, the reflections in the walls of $\mathfrak{H}$. It contains all translations t_β for β in the root lattice Q. Let us illustrate this for t_{α_1}.



The alcoves $\mathcal{F}_0, \cdots, \mathcal{F}_4$ are $\mathcal{F}$, $s_0\mathcal{F}$, $s_0 s_2\mathcal{F}$, $s_0 s_2 s_0\mathcal{F}$ and $s_0 s_2 s_0 s_1\mathcal{F} = t_{\alpha_1}\mathcal{F}$. The operation $s_0 s_2 s_0 s_1$ moves $\mathcal{F}$ to $t_{\alpha_1}\mathcal{F}$ in the correct orientation, so $s_0 s_2 s_0 s_1 = t_{\alpha_1}$.

The case of $\mathrm{SL}(2)$

In the case of $\mathrm{SL}(2)$ the fundamental alcove is a line segment from 0 to $\alpha_1/2$. Let s_0 and s_1 be the simple reflections. Then $s_1 s_0$ is the translation by α_1 and hence has infinite order. There is no other instance in affine Weyl groups where $s_i s_j$ has infinite order.



Affine Lie algebra

After the finite-dimensional semisimple Lie algebras, the best known and arguably simplest Kac-Moody Lie algebras are the affine Lie algebras. These admit a construction independent of the general Kac-Moody construction by affinizing a simple complex Lie algebra.

Although the affine Lie algebras are canonically derived from finite-dimensional simple Lie algebras, their theory is deeper with many new features.

There are two methods of enlarging a Lie algebra. We will make use of both. The first is to make a central extension; the second is to ajoin a derivation.

Central extensions

Let $\mathfrak{g}$ be a Lie algebra and $\mathfrak{a}$ an abelian Lie algebra. By a central extension of $\mathfrak{g}$ by $\mathfrak{a}$ we mean a Lie algebra $\tilde{\mathfrak{g}}$ and a short exact sequence

$$0 \longrightarrow \mathfrak{a} \longrightarrow \tilde{\mathfrak{g}} \longrightarrow \mathfrak{g} \longrightarrow 0$$

such the image of $\mathfrak{a}$ is contained in the center of $\tilde{\mathfrak{g}}$.

Just as group extensions of a group G by an abelian group A are classified by $H^2(G,A)$, extensions of a Lie algebra $\mathfrak{g}$ by an abelian Lie algebra $\mathfrak{a}$ are classified by $H^2(\mathfrak{g},\mathfrak{a})$. We consider only the important special case of central extensions.

The cohomology group H^2

Central extensions of a Lie algebra $\mathfrak{g}$ by an abelian Lie algebra $\mathfrak{a}$ are classified by the Lie algebra cohomology group $H^2(\mathfrak{g}, \mathfrak{a})$ whose definition we review.

A bilinear map $\phi : \mathfrak{g} \times \mathfrak{g} \longrightarrow \mathfrak{a}$ is called a 2-cocycle if $\phi(x, y) = -\phi(y, x)$ and

$$\phi([x, y], z) + \phi([y, z], x) + \phi([z, x], y) = 0.$$

If $f : \mathfrak{g} \longrightarrow \mathfrak{a}$ is a linear map then the map $\phi(x, y) = f([x, y])$ is called a coboundary.

Coboundaries are automatically cocycles. If $Z^2(\mathfrak{g}, \mathfrak{a})$ is the group of cocycles and $B^2(\mathfrak{g}, \mathfrak{a})$ are the coboundaries then $H^2(\mathfrak{g}, \mathfrak{a}) = Z^2(\mathfrak{g}, \mathfrak{a})/B^2(\mathfrak{g}, \mathfrak{a})$.

Central extensions from 2-cocycles

Let $\phi \in Z^2(\mathfrak{g}, \mathfrak{a})$. Define $\tilde{\mathfrak{g}} = \mathfrak{g} \oplus \mathfrak{a}$ with the modified bracket:

$$[(X, A), (Y, B)] = ([X, Y], \phi(X, Y)).$$

This is independent of $A, B \in \mathfrak{a}$ because we want $\mathfrak{a}$ to be central. This bracket defines a Lie algebra. To see that the Jacobi relation is satisfied, note that

$$[[(X, A), (Y, B)](Z, C)] = ([[X, Y], Z], \phi([X, Y], Z)).$$

From this it easy to check that the Jacobi identity for $\tilde{\mathfrak{g}}$ follows from the Jacobi identity for $\mathfrak{g}$ and the cocycle condition.

If we change ϕ by a coboundary then we obtrain an equivalent extension. It may be shown that every extension is of this form, uniquely up to a coboundary, so central extensions are indeed classified by $H^2(\mathfrak{g}, \mathfrak{a})$.

Example: the Virasoro algebra

An example of such a construction is the Virasoro algebra, a Lie algebra that plays a central role in conformal field theory. We start with the "Witt Lie algebra" $\mathfrak{w}$ of vector fields on the circle. This has generators L_n subject to relations

$$[L_n, L_m] = (n-m)L_{n+m}.$$

Define $\phi : \mathfrak{w} \times \mathfrak{w} \longrightarrow \mathbb{C}$ defined by

$$\phi(L_n, L_m) = \frac{1}{12}\delta_{n,-m}(n^3 - n)$$

Proposition

This ϕ is a 2-cocycle.

We are regarding $\mathbb{C}$ as an abelian Lie algebra.

Checking the cocycle identity

From the last slide:

$$\mathfrak{w} = \{L_n (n \in \mathbb{Z}) \mid [L_n, L_m] = (n-m)L_{n+m}\}.$$

$\phi : \mathfrak{w} \times \mathfrak{w} \longrightarrow \mathbb{C}$ is defined by

$$\phi(L_n, L_m) = \frac{1}{12}\delta_{n,-m}(n^3 - n).$$

Clearly $\phi(L_n, L_m) = -\phi(L_m, L_n)$. We must check

$$\phi([L_n, L_m], L_p) + \phi([L_m, L_p], L_n) + \phi([L_p, L_n], L_m) = 0.$$

Both sides vanish unless $n + m + p = 0$, so assume this. A small amount of algebra verifies the cocycle condition.

```
sage: def coc(n,m,p): return (1/12)*(n-m)*(p^3-p)
sage: P.<n,m>=PolynomialRing(QQ)
sage: p = -n-m
sage: coc(n,m,p)+coc(m,p,n)+coc(p,n,m)
0
```

The Virasoro algebra

Moreover, Fuchs and Feigin proved that $H^2(\mathfrak{w}, \mathbb{C})$ is one-dimensional generated by the class of ϕ.

We may define a central extension $\mathcal{V}$ of $\mathfrak{w}$ by $\mathbb{C}$. Denoting the image in $\mathcal{V}$ of $1 \in \mathbb{C}$ by C, $\mathcal{V}$ has a basis L_n ($n \in \mathbb{Z}$) plus one central element C and

$$[L_n, L_m] = (n-m)L_{n+m} + \frac{n^3 - n}{12}\delta_{n,-m}C.$$

The representation theory of $\mathcal{V}$ is much richer than that of $\mathfrak{w}$. If V is a module, then the central element C acts by a scalar c, known as the central charge. Classification of the representations of $\mathcal{V}$ is a key step in the study of conformal field theories.

Adjoining a derivation

In addition to making a central extension, a second construction that we may use is to adjoint a derivation of a Lie algebra. This is a special case of a more general construction, the semidirect product. If $d : \mathfrak{g} \longrightarrow \mathfrak{g}$ is a derivation, so $d[x, y] = [dx, y] + [x, dy]$, then we may put a Lie algebra structure on $\mathfrak{g} \oplus \mathbb{C} \cdot d$ by

$$[x + ad, y + bd] = [x, y] + ad(y) - bd(x).$$

This is a special case of a more general construction, the semidirect product.

The plan (I)

Let $\mathfrak{g}$ be a finite-dimensional simple Lie algebra. In the next lecture we will construct the (untwisted) affine Lie algebra $\widehat{\mathfrak{g}}$. If $\mathfrak{g}$ is a Lie algebra and A is a commutative associative algebra then $A \otimes \mathfrak{g}$ is a Lie algebra with the bracket

$$[(a, X), (b, Y)] = (ab, [X, Y]).$$

Thus t is an indeterminate $\mathbb{C}[t, t^{-1}] \otimes \mathfrak{g}$ is a Lie algebra. The first step is to make a central extension:

$$0 \to \mathbb{C} \to \widehat{\mathfrak{g}}' \to \mathbb{C}[t, t^{-1}] \otimes \mathfrak{g} \to 0.$$

This Lie algebra $\widehat{\mathfrak{g}}$ has a much richer representation theory than $\mathbb{C}[t, t^{-1}] \otimes \mathfrak{g}$.

The plan (II)

We have so far constructed a first version $\widehat{\mathfrak{g}}'$ of the affine Lie algebra by forming a central extension of $\mathbb{C}[t, t^{-1}] \otimes \mathfrak{g}$. Just as with the Virasoro algebra, the central extension has a much richer representation theory than $\mathbb{C}[t, t^{-1}] \otimes \mathfrak{g}$. It has one defect: the weight spaces of its modules are usually infinite-dimensional.

To remedy this defect, we enlarge it one step further by adjoining a derivation d to obtain the untwisted affine Lie algebra $\widehat{\mathfrak{g}}$. The subalgebra $\widehat{\mathfrak{g}}'$ is the derived subalgebra.