Lecture 1: Lie algebras with triangular decomposition

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The triangular decomposition

Let us consider a semisimple complex Lie algebra $\mathfrak{g}$, for example $\mathfrak{sl}_n(\mathbb{C})$. This has a triangular decomposition

$$\mathfrak{g} = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}_+,$$

where $\mathfrak{h}$ is a Cartan subalgebra, and $\mathfrak{n}_+$, $\mathfrak{n}_-$ are the spans of the positive and negative root spaces in $\mathfrak{g}$. If $\mathfrak{g} = \mathfrak{sl}_n$ then $\mathfrak{h}$ is the diagonal subalgebra and $\mathfrak{n}_+$, $\mathfrak{n}_-$ are the upper and lower triangular nilpotent subalgebras. We have two Lie subalgebras

$$\mathfrak{b}^+ = \mathfrak{h} \oplus \mathfrak{n}_+, \qquad \mathfrak{b}^- = \mathfrak{h} \oplus \mathfrak{n}_-.$$

There are similar triangular decompositions in other Lie algebras, particularly the Kac-Moody Lie algebras, so today we will look at the triangular decomposition more generally.

The meaning of the triangular decomposition

Still in the case of a semisimple Lie algebra, assume that $\mathfrak{g}$ is the Lie algebra of the complex analytic Lie group G. Then $\mathfrak{n}_\pm$, $\mathfrak{h}$, $\mathfrak{b}_\pm$ are Lie algebras of Lie subgroups. For example if $G = \mathrm{SL}(2,\mathbb{C})$

$$N_+ = \left\{ \begin{pmatrix} 1 & x \\ & 1 \end{pmatrix} \right\}, \qquad N_- = \left\{ \begin{pmatrix} 1 & \\ x & 1 \end{pmatrix} \right\}, \qquad T = \left\{ \begin{pmatrix} t & \\ & t^{-1} \end{pmatrix} \right\},$$

$$B_+ = \left\{ \begin{pmatrix} t & x \\ & t^{-1} \end{pmatrix} \right\}, \qquad B_- = \left\{ \begin{pmatrix} t & x \\ & t^{-1} \end{pmatrix} \right\}.$$

The Bruhat decomposition asserts $G = \bigcup B_- w B_+$ where w runs over the Weyl group. The big cell $B_- B = N_- T N_+$ is open and $\mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}_+$ is its tangent space, so this must be all of $\mathfrak{g}$.

General triangular decompositions

We see that the triangular decomposition is a local substitute for the Bruhat decomposition.

In Kac-Moody theory, it is possible to work with a group, but it is also feasible and often preferrable to work directly with the Lie algebra.

The triangular decomposition is central in Lie theory. It divides the roots into positive and negative roots, thereby imposing a length function on the Weyl group.

Triangular decompositions occur for many other Lie algebras such as the Heisenberg Lie algebra that plays an important role in the theory. Today we will proceed axiomatically.

Representations of Lie algebras

If $\mathfrak{g}$ is a complex Lie algebra, a representation on a (possibly infinite-dimensional) vector space V is a linear map $\pi : \mathfrak{g} \to \mathrm{End}(V)$ such that

$$\pi([x,y]) = \pi(x)\pi(y) - \pi(y)\pi(x).$$

We also write $x \cdot v = \pi(x)\,v$. We refer to V as a $\mathfrak{g}$-module.

A particular example is the adjoint representation. For this, $V = \mathfrak{g}$ and define

$$\mathrm{ad}(x)y = [x,y].$$

Then $\mathrm{ad} : \mathfrak{g} \to \mathrm{End}(\mathfrak{g})$ is a representation, as follows from the Jacobi identity.

The Cartan subalgebra

Therefore let $\mathfrak{g}$ be a complex Lie algebra, possibly infinite-dimensional. Let $\mathfrak{h}$ be an abelian subalgebra of $\mathfrak{g}$. This means that $[x, y] = 0$ for $x, y \in \mathfrak{h}$.

- We assume that $\mathfrak{h}$ is finite-dimensional.
- We assume that $\mathfrak{h}$ is maximal abelian.

This means that:

- If $X \in \mathfrak{g}$ such that $[H, X] = 0$ for all $H \in \mathfrak{h}$ then $X \in \mathfrak{h}$.

Otherwise $\mathfrak{h} \oplus \mathbb{C}H$ would be a strictly larger abelian subalgebra, contradicting maximality of $\mathfrak{h}$. Also assume

- $\mathrm{ad}(\mathfrak{h})$ acts diagonally on $\mathfrak{g}$.

Without this assumption, $\mathbb{C} \cdot \left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right)$ would be a maximal abelian subalgebra of $\mathfrak{sl}(2, \mathbb{C})$. We'll elaborate later.

Weight space decomposition

Let $\pi : \mathfrak{g} \to \mathrm{End}(V)$ be a representation. We will often write $X \cdot v$ instead of $\pi(X)v$ and refer to V as a $\mathfrak{g}$-module. Let λ be an element of the dual space $\mathfrak{h}^*$. Let

$$V_\lambda = \{v \in V | \pi(H)v = \lambda(H)v \text{ for all } H \in \mathfrak{h}\}.$$

We call V_λ the weight space of λ.

We are only interested in representations of $\mathfrak{g}$ for which the abelian Lie algebra $\mathfrak{h}$ can be diagonalized. Thus assume that

$$V = \bigoplus_{\lambda \in \mathfrak{h}^*} V_\lambda.$$

If this is true, we say that V has a weight space decomposition with respect to the abelian subalgebra $\mathfrak{h}$.

Kac's Lemma on weight space decompositions

Lemma (Kac)

Suppose that V is a $\mathfrak{g}$-module that has a weight space decomposition with respect to $\mathfrak{h}$. Then so does any submodule of V.

Proof. Suppose that U is a submodule. We show

$$U = \bigoplus_{\lambda \in \mathfrak{h}^*} U_\lambda,$$

where $U_\lambda = U \cap V_\lambda$. Let $u \in U$. Write $u = \sum v_{\lambda_j}$ as a finite sum of elements v_{λ_i} of weight spaces $V_{\lambda_1}, \cdots, V_{\lambda_m}$. Since the linear functionals $\lambda_j \in \mathfrak{h}^*$ are distinct, find $H \in \mathfrak{h}$ such that the values $\lambda_j(H)$ are all distinct.

Proof, continued

Then note that

$$\sum_{j=1}^{m} \lambda_j(H)^i v_j = H^i u \in U$$

for all i, in particular $i = 0, \cdots, m-1$. The matrix $(\lambda_j(H)^i)$ with $1 \leqslant j \leqslant m$ and $0 \leqslant i \leqslant m-1$ is invertible since its determinant is a Vandermonde determinant and the $\lambda_j(H)$ are distinct. So $v_j \in U$ proving $u \in \bigoplus U_\lambda$.

Roots

Note that $\mathfrak{g}$ is itself a $\mathfrak{g}$-module under the adjoint representation.

- We will assume that $\mathfrak{g}$ itself has a weight space decomposition.

Thus

$$\mathfrak{g} = \bigoplus_{\alpha \in \mathfrak{h}^*} \mathfrak{g}_\alpha, \qquad \mathfrak{g}_\alpha = \{X \in \mathfrak{g} | [H, X] = \alpha(H) \text{ for } H \in \mathfrak{h}\}.$$

Because $\mathfrak{h}$ is maximal abelian, $\mathfrak{g}_0 = \mathfrak{h}$. Nonzero elements $\alpha \in \mathfrak{h}^*$ such that $\mathfrak{g}_\alpha \neq 0$ are called roots.

- We will assume that the set Φ of roots is a discrete subset of $\mathfrak{h}^*$ that spans a lattice Q, called the root lattice.

Root operators shift root spaces

Lemma

Suppose that (π, V) *is a representation of* V *and that* $X \in \mathfrak{g}_\alpha$*. Then* $X \cdot V_\mu \subseteq V_{\alpha+\mu}$*.*

Indeed, let $v \in V_\mu$ and $H \in \mathfrak{h}$. We need to show that $Xv \in V_{\alpha+\mu}$, that is,

$$H(Xv) = (\alpha + \mu)(H)\, Xv.$$

We have

$$HXv - XHv = [H, X]v.$$

Now $XHv = X\,\mu(H)v = \mu(H)Xv$ since $v \in V_\mu$. Also

$$[H, X] = \mathrm{ad}(H)X = \alpha(H)X$$

since $X \in \mathfrak{g}_\alpha$. Thus

$$HXv = XHv + [H, X]v = (\alpha + \mu)(H)Xv.$$

Closed sets of roots and Lie subalgebras

As a special case

$$[\mathfrak{g}_\alpha, \mathfrak{g}_\mu] \subseteq \mathfrak{g}_{\alpha+\mu}.$$

In particular if $\alpha + \mu \notin \Phi \cup \{0\}$ then $[\mathfrak{g}_\alpha, \mathfrak{g}_\mu] = 0$.
Let S be a subset of $\Phi \cup \{0\}$ such that if $\alpha, \beta \in S$ and $\alpha + \beta \in \Phi \cup \{0\}$ then $\alpha + \beta \in S$. We will express this assumption by saying that S is closed (or convex).

Lemma

If S is a closed subset of $\Phi \cup \{0\}$ then

$$\bigoplus_{\alpha \in S} \mathfrak{g}_\alpha$$

is a Lie subalgebra of $\mathfrak{g}$.

This follows from $[\mathfrak{g}_\alpha, \mathfrak{g}_\mu] \subseteq \mathfrak{g}_{\alpha+\mu}$.

The triangular decomposition

Let us find a hyperplane $H \subseteq \mathfrak{h}^*$ that does not intersect the root lattice Q except at the origin. The roots on one side of H will be designated as positive, the roots on the other side will be designated as negative. Let Φ^+ and Φ^- be the sets of positive and negative roots. These are obviously closed subsets of Φ so

$$\mathfrak{n}_+ = \bigoplus_{\alpha \in \Phi^+} \mathfrak{g}_\alpha, \qquad \mathfrak{n}_- = \bigoplus_{\alpha \in \Phi^-} \mathfrak{g}_\alpha$$

are Lie subalgebras of $\mathfrak{g}$. They are normalized by $\mathfrak{h}$ and so

$$\mathfrak{b}^+ = \mathfrak{h} \oplus \mathfrak{n}_+, \qquad \mathfrak{b}^- = \mathfrak{h} \oplus \mathfrak{n}_-$$

are also subalgebras. By the weight space decomposition of $\mathfrak{g}$

$$\boxed{\mathfrak{g} = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}_+ \ .}$$

The universal enveloping algebra

If A is an associative algebra, then $\mathrm{Lie}(A)$ is the Lie algebra that is equal to A as a set with bracket $[x, y] = xy - yx$.

If $\mathfrak{g}$ is a Lie algebra, the universal enveloping algebra $U(\mathfrak{g})$ is the quotient of the tensor algebra $\mathfrak{g}$ by the two-sided ideal generated by elements of the form $x \otimes y - y \otimes x - [x, y]$. It is an associative algebra with a Lie algebra homomorphism $j : \mathfrak{g} \to \mathrm{Lie}(U(\mathfrak{g}))$. This means $j(x)j(y) - j(y)j(x) = j([x, y])$.

Proposition (Universal property of $U(\mathfrak{g})$)

If A is an associative algebra and $f : \mathfrak{g} \longrightarrow \mathrm{Lie}(A)$ is a Lie algebra homomorphism, then there is a unique algebra homomorphism $F : U(\mathfrak{g}) \longrightarrow A$ such that $f = F \circ j$.

Poincaré-Birkhoff Witt (I)

If V is a $\mathfrak{g}$-module then the universal property applied with $A = \mathrm{End}(V)$ gives an algebra homomorphism $U(\mathfrak{g}) \longrightarrow \mathrm{End}(V)$, so V becomes a $U(\mathfrak{g})$-module homomorphism.

The Poincaré-Birkhoff-Witt theorem (PBW) is a fundamental fact about Lie algebras, describing a basis of the universal enveloping algebra. Let X_i ($i \in I$) be an ordered basis of a Lie algebra $\mathfrak{g}$. We call an element of the universal enveloping algebra $U(\mathfrak{g})$ a standard monomial if it is of the form

$$X_{i_1} \cdots X_{i_k}, \qquad i_1 \leqslant i_2 \leqslant \cdots \leqslant i_k.$$

We did not assume that $\mathfrak{g}$ is finite-dimensional. If it is we may identify $I = \{1, \cdots, n\}$, in which case we could equivalently say that a standard monomial is an element of the form

$$X_1^{m_1} \cdots X_n^{m_n}, \qquad m_i \geqslant 0.$$

PBW (continued)

Theorem (PBW)

The standard monomials are a basis of $U(\mathfrak{g})$.

A proof may be found at Paul Garrett's web page (web link):

http://www-users.math.umn.edu/~garrett/m/algebra/pbw.pdf

Or see Humphrey's book Introduction to Lie Algebras and Representation Theory, available on-line through the Stanford Libraries for proof and discussion of PBW.

It is quite easy to prove that the standard monomials span $U(\mathfrak{g})$, but nontrivial to show that they are linearly independent. This implies that the map $j : \mathfrak{g} \longrightarrow U(\mathfrak{g})$ is injective.

Triangular decomposition of $U(\mathfrak{g})$

Returning to the case where $\mathfrak{g}$ has a triangular decomposition:

$$\mathfrak{g} = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}_+$$

Choose the basis in the PBW theorem so that $I = I_- \cup I_0 \cup I_+$ where X_i with $i \in I_-, I_0, I_+$ respectively are bases of $\mathfrak{n}_-$, $\mathfrak{h}$ and $\mathfrak{n}_+$ respectively. Order I so $I_- < I_0 < I_+$.

Then PBW implies that

$$U(\mathfrak{g}) \cong U(\mathfrak{n}_-) \otimes U(\mathfrak{h}) \otimes U(\mathfrak{n}_+).$$

Indeed with these preparations every standard monomial in $U(\mathfrak{g})$ is a tensor product of standard monomials in $U(\mathfrak{n}_-)$, $U(\mathfrak{h})$ and $U(\mathfrak{n}_+)$ showing that the multiplication map $U(\mathfrak{n}_-) \otimes U(\mathfrak{h}) \otimes U(\mathfrak{n}_+) \longrightarrow U(\mathfrak{g})$ is a vector space isomorphism.

Highest weight modules

Let $\lambda \in \mathfrak{h}^*$. Regard this as a character of the abelian Lie algebra $\mathfrak{h}$. Extend it to a character of $\mathfrak{b} = \mathfrak{b}^+ = \mathfrak{h} \oplus \mathfrak{n}_+$ by letting $\mathfrak{n}_+$ act by zero.

Thus we have a homomorphism $\phi_\lambda : \mathfrak{b} \longrightarrow \mathbb{C}$ such that $\phi_\lambda(H) = \lambda(H)$ for $H \in \mathfrak{h}$ and $\phi_\lambda(X) = 0$ for $X \in \mathfrak{n}_+$. This is a homomorphism (where $\mathbb{C}$ is an abelian Lie algebra) since $\phi_\lambda([x, y]) = 0$ for all $x, y \in \mathfrak{b}$. This is because $[x, y] \in \mathfrak{n}_+$.

Definition

A module V with a weight space decomposition is called a highest weight module for λ if $V_\lambda = \mathbb{C}v_\lambda$ is one-dimensional spanned by a vector v_λ such that $H \cdot v_\lambda = \lambda(h)v_\lambda$ for $H \in \mathfrak{h}$, $X \cdot v_\lambda = 0$ for $X \in \mathfrak{n}_+$, and $V = \mathfrak{g} \cdot v_\lambda$.

The finite-dimensional case

For example, if $\mathfrak{g}$ is a finite-dimensional semisimple Lie algebra, then any irreducible module is a highest-weight module for a unique highest weight $\lambda \in \mathfrak{h}^*$. The λ that occur as highest weights are precisely the dominant weights.

On the other hand, we will see soon that there are other highest weight modules form $\mathfrak{g}$ that are infinite-dimensional. The finite-dimensional modules are special since they are integrable meaning (roughly) that they lift to representations of the Lie group G. We say “roughly” since integrability can be defined without introducing the Lie group.

Universal highest weight modules

We return to the general case.

Proposition

Let V and U be highest weight modules for the same highest weight λ. Then $\mathrm{Hom}_{\mathfrak{g}}(V, U)$ is either zero or one-dimensional.

Proof. Since $\mathfrak{g} \cdot v_\lambda = V$, if $T \in \mathrm{Hom}_{\mathfrak{g}}(V, U)$ annihilates v_λ it is zero. But $\phi(v_\lambda) \in U_\lambda$ which is one-dimensional, so $T(v_\lambda)$ must be a constant multiple of u_λ, and after adjusting by a constant we may assume $T(v_\lambda) = u_\lambda$. But now ϕ is determined since v_λ generates V.

We will say that a highest weight module universal if $\mathrm{Hom}_{\mathfrak{g}}(V, U)$ is one-dimensional for all V.

Universal highest weight modules (continued)

Theorem

Let $\mathfrak{g}$ be a Lie algebra with a triangular decomposition, and let $\lambda \in \mathfrak{h}^*$. There is a universal highest weight module $M(\lambda)$ for λ. It is unique up to isomorphism. The map $\xi \longmapsto \xi \cdot v_\lambda$ (where v_λ is the highest weight vector) is a vector space isomorphism $U(\mathfrak{n}_-) \longrightarrow M(\lambda)$.

The universal highest weight module $M(\lambda)$ is called the Verma module for λ.

Proof. Uniqueness follows from the fact that the definition of a universal highest weight module amounts to a universal property. The universal highest weight module is an initial object in the category of highest weight modules for λ.

Proof (continued): Constructing the Verma module

To prove existence, we give a construction, called the Verma module construction. Denote by $\mathbb{C}_\lambda$ the $\mathfrak{b}$-module $\mathbb{C}$ with the corresponding $\mathfrak{b}$-module structure; that is, $x \cdot a = \phi_\lambda(x)a$ for $x \in \mathfrak{b}$ and $a \in \mathbb{C}$. This becomes a module for $U(\mathfrak{b})$.

Now $M(\lambda)$ is the $\mathfrak{g}$-module induced from the $\mathfrak{b}$-module $\mathbb{C}$. This can be defined as $U(\mathfrak{g})/J_\lambda$ where J_λ is the left ideal generated by elements of the form $b - \phi_\lambda(b)$ with $b \in U(\mathfrak{b})$. Let v_λ be the coset $1 + J_\lambda$ which is obviously a highest weight vector for λ.

Proof (continued): the universal property

We show that $M(\lambda)$ is a universal highest weight module. Let U be a highest weight module with highest weight vector u_λ. Consider the map $U(\mathfrak{g}) \longrightarrow U$ that sends $\xi \in U(\mathfrak{g})$ to $\xi \cdot u_\lambda$. By construction $J_\lambda \cdot u_\lambda = 0$ so this map factors through $M(\lambda) = U(\mathfrak{g})/J_\lambda$. Hence $M(\lambda)$ is a universal highest weight module.

Finally note that

$$M(\lambda) = U(\mathfrak{g})v_\lambda = U(\mathfrak{n}_-)U(\mathfrak{b})v_\lambda = U(\mathfrak{n}_-)v_\lambda$$

so the map $\xi \longmapsto \xi \cdot v_\lambda$ is surjective $U(\mathfrak{n}_-) \longrightarrow M(\lambda)$. Moreover it is it is not hard to deduce from the PBW theorem that $U(\mathfrak{n}_-) \cap J_\lambda = 0$ so this map is also injective.

Maximal submodules

Lemma

Let V be a highest weight module. Then V has a unique maximal proper submodule.

Proof. Since the highest weight vector v_λ generates V, and $V_\lambda = \mathbb{C}v_\lambda$ is one dimensional, it is clear that a submodule U is proper if and only if $U_\lambda = 0$. Now let K be the sum of all proper submodules. Then $K_\lambda = \sum U_\lambda = 0$ and so K is proper. Obviously it is the unique maximal proper submodule.

Irreducibles

Proposition

Let $\mathfrak{g}$ be a Lie algebra with a triangular decomposition, and let $\lambda \in \mathfrak{h}^*$. Then there is a unique highest weight module $L(\lambda)$ for λ that is irreducible. If V is a highest weight module for λ then $L(\lambda)$ is a quotient of V.

Proof. By the Lemma, $M(\lambda)$ has a maximal proper submodule K and since $v_\lambda \notin K$, the quotient $L(\lambda) = M(\lambda)/K$ is a highest weight module. It is irreducible by the maximality of K.

If V is another highest weight module for λ, it is a quotient of $M(\lambda)$. Writing $V \cong M(\lambda)/U$ for some proper submodule U, $K \supseteq U$ so $L(\lambda)$ is also a quotient of V. Since $L(\lambda)$ is a terminal object in the category of highest weight modules for λ, it is unique up to isomorphism.

The finite-dimensional semisimple case

We specialize now to the case where $\mathfrak{g}$ is a semisimple Lie algebra. If $\lambda \in \mathfrak{h}^*$ is a dominant weight, then we know from the Weyl theory that the irreducible highest weight module $V = L(\lambda)$ with highest weight λ is finite-dimensional. Its character

$$\chi_\lambda = \sum_\mu \dim(V_\mu) e^\mu$$

is given by the Weyl character formula:

$$\chi_\lambda = \left[\prod_{\alpha\in\Phi^+} (1 - e^{-\alpha})^{-1} \right] \sum_{w\in W} (-1)^{\ell(w)} e^{w(\lambda+\rho)-\rho}.$$

Here W is the Weyl group, Φ is the root system and Φ^+ is the set of positive roots and ρ is the Weyl vector, half the sum of the positive roots.

The character of the Verma module

On the other hand, let us consider the character of $M(\lambda)$. We noted that the map $\xi \longmapsto \xi \cdot v_\lambda$ is a vector space isomorphism $U(\mathfrak{n}_-)$ to $M(\lambda)$.

Proposition

The character of $M(\lambda)$ is

$$e^\lambda \prod_{\alpha \in \Phi^+} (1 - e^{-\alpha})^{-1}.$$

Proof. We start with the fact that $\xi \mapsto \xi \cdot v_\lambda$ is a vector space isomorphism $U(\mathfrak{n}_-) \to M(\lambda)$. Taking into account the fact that the weight of v_λ is λ,

$$M(\lambda) \cong \mathbb{C}_\lambda \otimes U(\mathfrak{n}_-)$$

as $\mathfrak{h}$-modules.

Proof (continued)

The character of $U(\mathfrak{n}_-)$ may be computed using the PBW theorem. The weights in $\mathfrak{n}_-$ are $-\alpha$ where $\alpha \in \Phi^+$. Let $X_{-\alpha} \in \mathfrak{n}_-$ be the corresponding generators, a basis of $\mathfrak{n}_-$. By PBW a basis of $\mathfrak{n}_-$ consists of $\prod_{\alpha\in\Phi^+} X_{-\alpha}^{k_\alpha}$ with $k_\alpha \in \mathbb{N}$. This vector has weight $-\sum k_\alpha \alpha$, so the character of $U(\mathfrak{n}_-)$ is

$$\prod_{\alpha\in\Phi^+} \sum_{k_\alpha=0}^{\infty} e^{-k_\alpha \alpha} = \prod_{\alpha\in\Phi^+} (1-e^{-\alpha})^{-1}.$$

Thus the character of $M(\lambda)$ is

$$e^{\lambda} \prod_{\alpha\in\Phi^+} (1-e^{-\alpha})^{-1}.$$

Comparison

We may now identify every term in the Weyl character formula:

$$\chi_\lambda = \left[\prod_{\alpha\in\Phi^+}(1-e^{-\alpha})^{-1}\right]\sum_{w\in W}(-1)^{\ell(w)}e^{w(\lambda+\rho)-\rho}.$$

We see that

$$\chi_{L(\lambda)} = \sum_{w\in W}(-1)^{\ell(w)}\chi_{M(w(\lambda+\rho)-\rho)}.$$

This identity is a reflection of an algebraic fact, the BGG resolution of $L(\lambda)$. We will not fully explain this now, but we will look at this a little more closely in a special case.

The case of $\mathfrak{sl}(2,\mathbb{C})$

Let us specialize to the case of $\mathfrak{sl}(2,\mathbb{C})$. There is a unique positive root α_1 and if s_1 is the corresponding simple reflection,

$$w(\lambda+\rho)-\rho = \begin{cases} \lambda & \text{if } w = 1_W, \\ s_1(\lambda)-\alpha_1 & \text{if } w = s_1. \end{cases}$$

Thus

$$\chi_{L(\lambda)} = \chi_{M(\lambda)} - \chi_{M(s_1\lambda-\alpha_1)}.$$

We know from our previous results that there is a surjective homomorphism $M(\lambda) \longrightarrow L(\lambda)$. This shows that the character of the kernel is $M(s_1\lambda-\alpha_1)$, and indeed, there is a short exact sequence

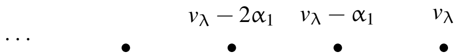
$$0 \longrightarrow M(s_1\lambda-\alpha_1) \longrightarrow M(\lambda) \longrightarrow L(\lambda) \longrightarrow 0.$$

$\mathfrak{sl}(2,\mathbb{C})$, continued

Let us visualize the character of the Verma module $M(\lambda)$ as follows. Note that $\lambda \in \mathfrak{h}^*$ can be arbitrary. The weights are

$$\lambda, \lambda - \alpha_1, \lambda - 2\alpha_1, \cdots .$$

If v_μ is the basis vector spanning $M(\lambda)_\mu$ visualize:



The Verma module $M(\lambda)$ and its maximal irreducible quotient $L(\lambda)$ are both defined for all $\lambda \in \mathfrak{h}^*$. Unless λ is a dominant weight, $M(\lambda)$ is irreducible, so for λ in general position $L(\lambda) = M(\lambda)$.

The root operators shift between the root spaces

We may visualize the effects of E and F thus. The weight space $\mathbb{C}v_\mu$ is the same as the H-eigenspace for eigenvalue $\mu(H)$. And

$$F(v_\mu) = (*)v_{\mu-\alpha_1}, \qquad E(v_\mu) = (*)v_{\mu+\alpha_1}$$

where $(*)$ are constants, usually nonzero:

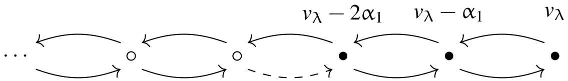
$$\cdots \underset{F}{\overset{E}{\rightleftarrows}} v_{\lambda-2\alpha} \underset{F}{\overset{E}{\rightleftarrows}} v_{\lambda-\alpha} \underset{F}{\overset{E}{\rightleftarrows}} v_\lambda$$

The finite-dimensional quotient

The dominant weights are $\frac{k}{2}\alpha_1$ with $k = 0, 1, 2, \cdots$. If $\lambda = \frac{k}{2}\alpha_1$ is a dominant weight, then $k = \langle \alpha_1^\vee, \lambda \rangle$ is a nonnegative integer. Then we may calculate

$$E_1(v_{\lambda-(k+1)\alpha_1}) = 0$$

and $v_{\lambda-(k+1)\alpha_1}$ generates a proper submodule K and the irreducible quotient $L(\lambda) = M(\lambda)/Q$ is finite ($k+1$-dimensional).



White: the submodule K. Dashed arrow is zero.
Black: the finite dimensional quotient $L(\lambda)$.

BGG resolutions

Now let $\mathfrak{g}$ be a general semisimple complex Lie algebra. Since

$$\chi_{L(\lambda)} = \sum_{w\in W} (-1)^{\ell(w)} \chi_{M(w(\lambda+\rho)-\rho)}$$

we may hope for a resolution of $L(\lambda)$ in terms of Verma modules. Such a resolution is called a BGG resolution. Specifically, Bernstein, Gelfand and Gelfand proved that if N is the number of positive roots of $\mathfrak{g}$, which also equals the length of the long Weyl group element there is an exact sequence

$$0 \longrightarrow C_N \longrightarrow C_{N-1} \longrightarrow \ldots \longrightarrow C_0 = M(\lambda) \longrightarrow L(\lambda) \longrightarrow 0$$

with

$$C_k = \bigoplus_{\substack{w\in W \\ \ell(w)=k}} M(w(\lambda+\rho)-\rho).$$

References for the BGG resolution

Web link to Bernstein, Gelfand and Gelfand, Differential operators on the affine space base space and a study of $\mathfrak{g}$-modules at Joseph Bernstein's web page:

(Web link to BGG paper)

See Theorem 10.1.

A good reference for the BGG resolution (following arguments of Rocha) is Humphreys, Representations of Semisimple Lie algebras in the BGG Category $\mathcal{O}$, Chapter 6.

Weights and dominant weights

The finite-dimensional semisimple Lie algebra $\mathfrak{g}$ is the Lie algebra of a complex analytic Lie group G, which we assume to be simply-connected. If $L(\lambda)$ is finite-dimensional, then it may be "integrated" to obtain a representation of G. Hence we call these definitions integrable though when we discuss the Kac-Moody theory we will want a notion of integrability that does not require us to construct the group G.

Let

$$P = \left\{ x \in \mathfrak{h}^* | \langle \alpha_i^\vee, x \rangle \in \mathbb{Z} \text{ for all } i \right\},$$

$$P^+ = \left\{ x \in P | \langle \alpha_i^\vee, x \rangle \geqslant 0 \text{ for all } i \right\}.$$

Elements of the weight lattice P are called integral weights. Elements of P^+ are called dominant weights. The weight lattice P contains the root lattice Q.

Class functions on G

Let T be the maximal torus of G whose Lie algebra is $\mathfrak{h}$. Assuming that G is simply connected, it P may be identified with the group $X^*(T)$ of rational characters of G (Bump, Lie groups, 2nd ed., Proposition 23.12). So if $\mathbf{z} \in T$ and $\mu \in P$ we will write $\mathbf{z}^\mu$ for the value of μ at $\mathbf{z} \in T$.

Let f be a continuous class function f on G. The conjugates of T are dense in G, so it is enough to describe $f(\mathbf{z})$ for $\mathbf{z} \in T$. We may expand

$$f(\mathbf{z}) = \sum_{\mu \in P} a_\mu \mathbf{z}^\mu$$

Let W be the Weyl group $N(T)/T$. If $\mathbf{z}, \mathbf{z}' \in T$ they are conjugate in G if and only if they are equivalent under W. So a necessary and sufficient condition for this function on T to be a class function is that $a_{w(\mu)} = a_\mu$ for $w \in W$, $\mu \in P$.

If a representation π of $\mathfrak{g}$ can be integrated to a representation of G then the character

$$\chi_\pi = \sum_\mu \dim(V_\mu) e^\mu$$

which is a priori just a formal sum, can be interpreted as the function

$$\chi_\pi(\mathbf{z}) = \sum_\mu \dim(V_\mu) \mathbf{z}^\mu$$

of $\mathbf{z} \in T$. And from the above discussion, the weight multiplicities $\dim(V_\mu)$ must be invariant under W.

Considering the application of this to $L(\lambda)$ and $M(\lambda)$ if $\lambda \in P^+$ then $L(\lambda)$ is finite-dimensional, so naturally the weight multiplicities are W-invariant. But the weight multiplicities of $M(\lambda)$ are never Weyl invariant, so $M(\lambda)$ can never be lifted to G.

The Weyl group action

The Weyl group W can be described without reference to G. Let $\alpha_i \in \mathfrak{h}^*$ be the simple roots, and $\alpha_i^\vee \in \mathfrak{h}$ the corresponding simple coroots. These are vectors such that the generating simple reflections $s_i \in W$ have the defining relation

$$s_i(x) = x - \langle \alpha_i^\vee, x \rangle \alpha_i, \qquad x \in \mathfrak{h}^*.$$

Since we want $s_i(\alpha_i) = -\alpha_i$ this means that $\langle \alpha_i^\vee, \alpha_i \rangle = 2$.

The Cartan matrix

The adjoint action of W on $\mathfrak{h}$ is similarly described:

$$s_i(x) = x - \langle x, \alpha_i \rangle \alpha_i^{\vee}.$$

The matrix $A = (a_{ij})$ where $a_{ij} = \langle \alpha_i^{\vee}, \alpha_j \rangle$ is called the Cartan matrix of $\mathfrak{g}$. As we will see in the next lecture, one many start with the Cartan matrix and reconstruct the Lie algebra. Following Kac, this procedure is very general and produces infinite-dimensional Lie algebras with much of the theory for finite-dimensional Lie algebras going through.