Modular Forms and Affine Lie Algebras

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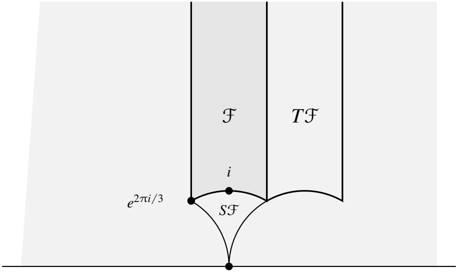


The plan of this course

This course will cover the representation theory of a class of Lie algebras called affine Lie algebras. But they are a special case of a more general class of infinite-dimensional Lie algebras called Kac-Moody Lie algebras. Both classes were discovered in the 1970's, independently by Victor Kac and Robert Moody. Kac at least was motivated by mathematical physics. Most of the material we will cover is in Kac' book Infinite-dimensional Lie algebras which you should be able to access on-line through the Stanford libraries.

In this class we will develop general Kac-Moody theory before specializing to the affine case. Our goal in this first part will be Kac' generalization of the Weyl character formula to certain infinite-dimensional representations of infinite-dimensional Lie algebras.

Affine Lie algebras and modular forms

The Kac-Moody theory includes finite-dimensional simple Lie algebras, and many infinite-dimensional classes. The best understood Kac-Moody Lie algebras are the affine Lie algebras and after we have developed the Kac-Moody theory in general we will specialize to the affine case.

We will see that the characters of affine Lie algebras are modular forms. We will not reach this topic until later in the course so in today's introductory lecture we will talk a little about modular forms, without giving complete proofs, to show where we are headed.

The upper half plane

The simplest modular forms are modular forms for $\mathrm{SL}(2,\mathbb{Z})$, and we start with those. The group $\mathrm{SL}(2,\mathbb{R})$ acts on the Poincaré upper half plane

$$\mathcal{H} = \{\tau \in \mathbb{C} | \operatorname{im}(\tau) > 0\}$$

by linear fractional transformations:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} : \tau \longmapsto \frac{a\tau + b}{c\tau + d}.$$

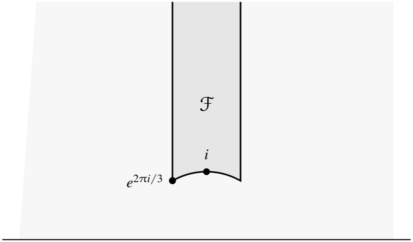
The discrete subgroup $\mathrm{SL}(2,\mathbb{Z})$ then acts discontinuously.

The fundamental domain

A fundamental domain consists of

$$\mathcal{F} = \{\tau = x + iy \mid -\frac{1}{2} \leqslant x \leqslant \frac{1}{2}, |\tau| = \sqrt{x^2 + y^2} \geqslant 1\},$$

The subgroup $\{-I\}$ acts trivially. The $\mathrm{SL}(2, \mathbb{Z})$ translates of this domain tile $\mathcal{H}$.



S and T

Two useful generators of $\mathrm{SL}(2,\mathbb{Z})$ are

$$T = \begin{pmatrix} 1 & 1 \\ & 1 \end{pmatrix}, \qquad S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

They satisfy the relations

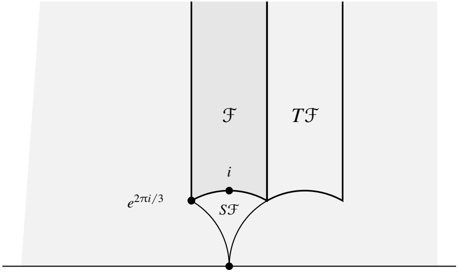
$$S^2 = -I, \qquad (ST)^3 = I.$$

They have the effect:

$$T : \tau \longmapsto \tau + 1, \qquad S : \tau \longmapsto -\frac{1}{\tau}.$$

The fundamental domain

Here is the fundamental domain $\mathcal{F}$ with the two translates $T\mathcal{F}$ and $S\mathcal{F}$.



$$T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

$$S = \begin{pmatrix} & -1 \\ 1 & \end{pmatrix}$$

- $\mathrm{SL}(2, \mathbb{Z})$, notes by Keith Conrad

Cusps

We may extend the action of $\mathrm{SL}(2,\mathbb{Z})$ to $\mathcal{H}\cup\mathbb{R}\cup\{\infty\}$, where the action is by linear fractional transformations, and we define $\frac{a\tau+b}{c\tau+d}=\infty$ if $c\tau+d=0$, and $\frac{a\tau+b}{c\tau+d}=\frac{a}{c}$ if $\tau=\infty$. We think of $\mathbb{R}\cup\{\infty\}$ as the projective line $\mathbb{P}^1(\mathbb{R})$.

Alternatively (and better) we can just add $\mathbb{P}^1(\mathbb{Q})=\mathbb{Q}\cup\{\infty\}$ and consider $\mathrm{SL}(2,\mathbb{Z})\backslash(\mathcal{H}\cup\mathbb{P}^1(\mathbb{Q}))$. There is just one orbit of $\mathrm{SL}(2,\mathbb{Z})$ on $\mathbb{P}^1(\mathbb{Q})$ but for subgroups such as

$$\Gamma_0(N)=\left\{\begin{pmatrix} a & b \\ c & d\end{pmatrix}\in\mathrm{SL}(2,\mathbb{Z})|c\equiv 0 \text{ mod } N\right\}$$

there may be several. These orbits are called cusps.

Modular curves

Shimura showed that adjoining the cusps to the quotient $\Gamma_0(N)\backslash\mathcal{H}$ produces an algebraic curve that can naturally be defined over $\mathbb{Q}$. This is the modular curve $X_0(N)$.

One may also consider quotients

$$X(N) = \Gamma(N)\backslash\mathcal{H}\cup\{\text{cusps}\}, \qquad \Gamma(N) = \{\gamma \in \mathrm{SL}(2,\mathbb{Z}) | \gamma \equiv I \bmod N\}.$$

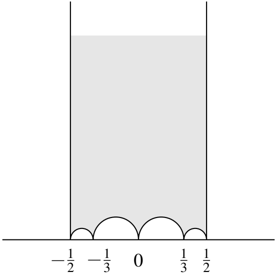
For these the field of definition is $\mathbb{Q}(e^{2\pi i/N})$ in Shimura's theory of canonical models.

The following web link discusses the field of definition of $X_0(N)$:

- Modular Functions and Modular Forms by James Milne

Example: $X_0(11)$

Here is a fundamental domain for $\Gamma_0(11)$:



The cusps $\pm\frac{1}{2}, \pm\frac{1}{3}, 0$ are actually $\Gamma_0(11)$ equivalent, so this group has only two cusps, $\{0, \infty\}$. As an algebraic curve, $X_0(11)$ has the equation $y^2 + y = x^3 - x^2 - 10x - 20$.

- Notes on $X_0(11)$ by Tom Weston

Elliptic modular forms

Now we may introduce modular forms. They may be thought of as sections of line bundles over the modular curve.

The simplest type of modular forms we will call elliptic modular forms of level 1 and weight k. These are holomorphic functions of $\tau \in \mathcal{H}$ that satisfy

$$f\left(\frac{a\tau+b}{c\tau+d}\right) = (c\tau+d)^k f(\tau), \qquad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2,\mathbb{Z}).$$

Here k is an even positive integer called the weight.

There is another condition that we want to impose...

Holomorphy at the cusp

Since T is the transformation $\tau \mapsto \tau + 1$ any modular form satisfies $f(\tau + 1) = f(\tau)$. Hence it is well defined as a function of $q = e^{2\pi i\tau}$ and has a Laurent expansion

$$f(\tau) = \sum_{n\in\mathbb{Z}} a_n q^n.$$

We require $a_n = 0$ if $n < 0$. This expression is called the q-expansion (or Fourier expansion at ∞).

We express the requirement that $a_n < 0$ for $n < 0$ by saying that f is holomorphic at the cusp at ∞. We could relax the condition by only requiring $a_n = 0$ for n sufficiently negative, and then we would say that f is meromorphic at ∞.

Modular forms and cusp forms

If $a_0 = 0$ we say that f vanishes at the cusp at ∞ or that f is a cusp form.

The spaces of modular forms of weight k and cusp forms are respectively denoted

$$M_k(\mathrm{SL}(2,\mathbb{Z})), \qquad S_k(\mathrm{SL}(2,\mathbb{Z})).$$

These definitions can be extended to $\Gamma_0(N)$ noting:

- One must formulate the notions of holomorphy and vanishing at cusps $\neq \infty$;
- For $\Gamma_0(N)$ it is possible to include a Dirichlet character χ mod N (“Nebentypus”) in the definition.

$$f\left(\frac{a\tau+b}{c\tau+d}\right) = \chi(N)(c\tau+d)^k f(\tau), \qquad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N).$$

Forms with odd weight

Modular forms (resp. cusp forms) of level N and weight k with Nebentypus character χ for $\Gamma_0(N)$ are denoted:

$$M_k(\Gamma_0(N),\chi), \qquad S_k(\Gamma_0(N),\chi).$$

From the definition:

$$f\left(\frac{a\tau+b}{c\tau+d}\right)=\chi(N)(c\tau+d)^k f(\tau), \qquad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N)$$

If $\chi(-1)=1$ then k must be even, but if $\chi(-1)=-1$ then k must be odd for consistency.

Eisenstein series and Ramanujan's Δ

The simplest elliptic modular forms of level 1 are Eisenstein series:

$$G_k(\tau) = \frac{1}{2} \sum_{\substack{(c,d)\in\mathbb{Z}^2 \\ \gcd(c,d)=1}} (c\tau + d)^{-k}.$$

Elliptic modular forms of level 1 form a ring, and G_4, G_6 are generators. A cusp form of weight 12:

$$\Delta(\tau) = \frac{1}{1728}(G_4^3 - G_6^2).$$

This is Ramanujan's discriminant function, a very famous and important modular form. It has a remarkable product expansion.

$$\Delta(\tau) = q \prod_{n=1}^{\infty} (1 - q^n)^{24} = \sum_{n=1}^{\infty} \tau(n) q^n, \qquad q = e^{2\pi i \tau}$$

The L-function of Δ

The infinite product expansion shows that Δ is nonvanishing on $\mathcal{H}$ but has a zero of order 1 at the cusp. Ramanujan made two famous conjectures (1916). The first conjecture was the Euler product

$$L(s,\Delta) := \sum_{n=1}^{\infty} \tau(n) n^{-s} = \prod_{p} (1 - \tau(p) p^{-s} + p^{11-2s})^{-1}.$$

This may be complemented by the functional equation

$$\Lambda(s,\Delta) := (2\pi)^{-s} \Gamma(s) L(s,\Delta) = \Lambda(12-s,\Delta).$$

Euler products with functional equations

Compare the Euler product and functional equation for $L(s, \Delta)$ to the Riemann zeta function, which also has an Euler product and a functional equation:

$$\zeta(s) := \sum_{n=1}^{\infty} n^{-s} = \prod_{p} (1 - p^{-s})^{-1},$$

$$\zeta^*(s) := \pi^{-s/2} \Gamma\left(\frac{s}{2}\right) \zeta(s) = \zeta^*(1-s)$$

we see that $\zeta(s)$ and $L(s, \Delta)$ are both objects of the same general type, and from today's perspective, they are automorphic forms on $\mathrm{GL}(n)$ for $n = 1, 2$ respectively.

Glimpses of the bigger picture

Ramanujan's conjectured Euler product was proved by Mordell (1917). A second conjecture, that $\tau(p) \leqslant 2p^{11/2}$ for p prime, was not proved until 1970 by Deligne, and is a manifestation of deep connections between the theory of modular forms and algebraic geometry.

The Ramanujan-Mordell Euler product was generalized by Hecke (1937) to other modular forms. This Hecke theory extends to automorphic forms on $\mathrm{GL}(n)$, and is an important connection between modular forms and representation theory. It shows that one may associate with any automorphic form on $\mathrm{GL}(n)$ an L-series that has an Euler product and functional equation (and, we might add, an unproven Riemann hypothesis).

Hecke theory and Atkin-Lehner theory

Hecke (with a later important complement by Atkin and Lehner) generalized Ramanujan's Euler products and showed that $M_k(\mathrm{SL}(2,\mathbb{Z}))$ or $M_k(\Gamma_0(N),\chi)$ has a basis of modular forms whose L-functions have Euler products. Thus a Hecke eigenform

$$f(\tau) = \sum_{n=0}^{\infty} a_n e^{2\pi i n\tau}$$

satisfies

$$L(s,f) := \sum_{n=1}^{\infty} a_n n^{-s} = \prod_p (1 - a(p)p^{-s} + \chi(p)p^{k-1-2s})^{-1}.$$

There are some subtleties concerning $p|N$ and this is what was clarified by Atkin and Lehner.

Modular forms of weight 2

If $ad - bc = 1$ then

$$\frac{d}{d\tau}\frac{a\tau + b}{c\tau + d} = (c\tau + d)^{-2}.$$

This implies that the holomorphic differential form

$$(c\tau + d)^{-2}\, d\tau$$

is invariant under $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$.

Thus if f is a modular form of weight 2 for $\Gamma_0(N)$ (Nebentypus $\chi = 1$) then $\omega = f(\tau)\, d\tau$ is a holomorphic differential form on $\Gamma_0(N)\backslash H$. The condition for ω to be holomorphic for the cusps is that f is a cusp form. So modular forms of weight 2 are precisely holomorphic differentials on $X_0(N)$.

Cusp forms of weight 2

Since the modular curve $X_0(N)$ is a smooth complex curve it has a genus g, and the Riemann-Roch theorem implies that g is the dimension of the space of holomorphic differentials.

Therefore

$$g = \dim\ S_2(\Gamma_0(N)).$$

For example if $N = 11$ the genus $g = 1$ and it may be checked that the unique cusp form of weight 2 is

$$q \prod_{n=1}^{\infty} (1 - q^n)^2 (1 - q^{11n})^2.$$

The arithmetic theory (Eichler, Shimura) is simplest for cusp forms of weight 2.

The cusp form of weight 2 for $\Gamma_0(11)$

Now consider the unique cusp for of weight 2 for $\Gamma_0(11)$:

$$q \prod_{n=1}^{\infty} (1 - q^n)^2 (1 - q^{11n})^2 = \sum_{n=1}^{\infty} a(n)$$

$$= q - 2q^2 - q^3 + 2q^4 + q^5 + 2q^6 - 2q^2 - \ldots$$

This, like Ramanujan's modular form Δ of weight 12 is a "Hecke eigenform" which implies that the L-series made with the same coefficients has an Euler product:

$$\sum_{n=1}^{\infty} a(n) = \prod_{p} (1 - a(p)p^{-s} + p^{1-2s})^{-1}.$$

We will relate the coefficients $a(p)$ to a diophantine problem.

Eichler-Shimura Theory

We saw that an equation for $X_0(11)$ is

$$y^2 + y = x^3 - x^2 - 10x - 20.$$

Let $|E(\mathbb{F}_p)|$ be the number of solutions to this equation over the finite field $\mathbb{F}_p$, plus one for the point at infinity on the curve.

Remarkably, Eichler-Shimura theory shows

$$\#E(\mathbb{F}_p) = 1 - a(p) + p.$$

Here is some data:

p	2	3	5	7	13	17	19	23	29
$\#E(\mathbb{F}_p)$	5	5	5	10	10	20	20	25	30
$a(p)$	-2	-1	1	-2	4	-2	0	-1	0

The modularity theorem

The last slide depended on the fact that the elliptic curve

$$y^2 + y = x^3 - x^2 - 10x - 20$$

is the modular curve $X_0(11)$. It is too much to expect that every elliptic curve E defined over $\mathbb{Q}$ can be realized thus as a modular curve $X_0(N)$. But remarkably, every elliptic curve $E/\mathbb{Q}$ admits a morphism $X_0(N) \to E$. The genus of $X_0(N)$ might be > 1. This was roughly conjectured by Taniyama and Shimura (1956). The conjecture was popularized when it was restated by Weil (1967). In 1986 Ribet showed that the Taniyama-Shimura conjecture implies Fermat's Last Theorem. The Taniyama-Shimura modularity conjecture was proved by Wiles and Taylor, and Breuil, Conrad, Diamond and Taylor.

Modular forms of half-integral weight

Modular forms of level $k = 1, 2, 3, \cdots$ are automorphic forms on $\mathrm{GL}(2)$ and fit into the Langlands program.

However we will often be concerned with modular forms that are not automorphic forms on $\mathrm{GL}(n)$ in this sense. We may consider automorphic forms of half-integral weight. For these the Hecke theory is more subtle and was not understood until Shimura (1973) and Waldspurger (1980).

In the adele language, modular forms of integral weight may be associated with functions on $\mathrm{GL}(2, \mathbb{A})$ where $\mathbb{A}$ is the adele ring of $\mathbb{Q}$. Modular forms of half integral weight live not on $\mathrm{GL}(2, \mathbb{A})$ but on a double cover (central extension) $\widetilde{\mathrm{GL}}(2, \mathbb{A})$ called the metaplectic group.

The Dedekind eta function

An example of a modular form of half-integral weight may be obtained by taking Δ and raising it to the $1/24$ power. In view of the formula

$$\Delta(\tau) = q \prod_{n=1}^{\infty} (1 - q^n)^{24}$$

the function Δ never vanishes on $\mathcal{H}$ and we may consider

$$\eta(\tau) = q^{1/24} \prod_{n=1}^{\infty} (1 - q^n)$$

where $q^{1/24} = e^{2\pi i \tau/24}$.

The Dedekind eta function as a modular form

Note that η is a modular form of weight $1/2$ for $\mathrm{SL}(2,\mathbb{Z})$ because

$$\eta\left(\frac{a\tau+b}{c\tau+d}\right)=(*)(c\tau+d)^{1/2}\eta(\tau),\qquad \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

where $(*)$ is a 24-th root of unity. This is the eta function of Dedekind (1877). Without the $q^{1/24}$, this goes back to Euler and the theory of partitions.

The Jacobi-Riemann theta function

For another automorphic form of half integral weight consider

$$\theta(\tau) = \sum_{-\infty}^{\infty} q^{n^2} = 1 + 2q + 2q^4 + 2q^9 + \cdots$$

introduced by Jacobi. This plays a role in Riemann's second proof of the functional equation of $\zeta(s)$.

This is a modular form of half-integral weight for $\Gamma_0(4)$, with a slightly complicated multiplier system.

$$\theta\left(\frac{a\tau + b}{c\tau + d}\right) = (c\tau + d)^{1/2} \gamma_d^{-1} \left(\frac{c}{d}\right) \theta(\tau), \qquad \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_0(4),$$

where $\gamma(d) = 1$ or i according as $d \equiv 1, 3$ mod 4 for an appropriate branch of the square root, and $\left(\frac{c}{d}\right)$ is Shimura's version of the quadratic residue symbol.

Poisson summation

To prove an automorphicity for θ one may use the Poisson summation formula. A function f on $\mathbb{R}$ is Schwartz class if it is smooth and f and all of its derivatives are of facter than polynomial decay. We may define the Fourier transform

$$\hat{f}(x) = \int_{-\infty}^{\infty} f(y) e^{2\pi i x y}\, dy$$

which is also Schwartz.

Proposition (Poisson summation)

Let f be a Schwartz function on $\mathbb{R}$. Then

$$\sum_{n=-\infty}^{\infty} f(n) = \sum_{n=-\infty}^{\infty} \hat{f}(n).$$

Proof of Poisson summation

To prove the Poisson summation formula define

$$F(x) = \sum_{n=-\infty}^{\infty} f(x+n).$$

This is smooth and periodic hence has a Fourier expansion

$$F(x) = \sum_{n=-\infty}^{\infty} a_n e^{2\pi inx}$$

where

$$a_n = \int_0^1 F(x) e^{-2\pi inx}\, dx.$$

Proof of Poisson summation (continued)

Substituting the definition $F(x) = \sum_{n=-\infty}^{\infty} f(x+n)$ and telescoping the sum

$$a_n = \int_{-\infty}^{\infty} f(x) e^{-2\pi i n x} dx = \hat{f}(-n).$$

Now substituting $x = 0$ in

$$\sum_{n=-\infty}^{\infty} a_n e^{2\pi i n x} = F(x) = \sum_{n=-\infty}^{\infty} \hat{f}(-n) e^{2\pi i n x}$$

gives the Poisson summation formula.

Applying the Poisson summation formula to $f(x) = e^{2\pi i\tau x^2}$,

$$\hat{f}(x) = \frac{1}{\sqrt{-2i\tau}} e^{-2\pi i x^2/(4\tau)}$$

we obtain

$$\theta(\tau) = \frac{1}{\sqrt{-2i\tau}} \theta\left(-\frac{1}{4\tau}\right).$$

This gives a transformation property with respect to

$$S' = \begin{pmatrix} & -1 \\ 4 & \end{pmatrix} : \tau \mapsto -\frac{1}{4\tau}.$$

Note that S' is not in $\Gamma_0(4)$ but it normalizes it. It may be seen that $\Gamma_0(4)$ is generated by $T = \left(\begin{smallmatrix} 1 & 1 \\ & 1 \end{smallmatrix}\right)$, $-I$ and $S'T(S')^{-1}$ so the modularity of θ may be deduced.

Elliptic functions

The early history of modular forms was intimately tied up with the theory of elliptic functions. If $\tau \in \mathcal{H}$ we may consider meromorphic functions of $z \in \mathbb{C}$ that are doubly periodic:

$$f(z + c\tau + d) = f(z).$$

These are elliptic functions. The quotient $\mathbb{C}/\Lambda$ where Λ is the lattice generated by $1, \tau$ can be identified with an elliptic curve E over the complex numbers, and elliptic functions are just meromorphic functions on E.

Theta functions

We cannot require elliptic functions to be holomorphic: they always have poles unless they are constant. But it is possible to replace the transformation invariance by something more general and consider holomorphic functions θ that satisfy a quasiperiodicity such as

$$\theta(z+u) = \theta(z)e^{2\pi i(L(z,u)+J(u))}, \qquad u \in \mathbb{Z} \oplus \mathbb{Z}\tau,$$

where L is linear in z. By taking the ratio of two theta functions with the same theta multiplier $e^{2\pi i(L(z,u)+J(u))}$ we obtain an elliptic function.

Jacobi's theta functions

For example, consider the function

$$\theta(z,\tau) = \sum_{n=-\infty}^{\infty} q^{n^2} e^{2\pi i n z}, \qquad q = e^{2\pi i \tau}.$$

Thus this theta function (from Jacobi's 1829 treatis Fundamenta Nova Theoria Functionum Ellipticarum) has periodicity properties for both 1 and τ and is a theta function.

This is a theta function whose periodicity properties with respect to the periods 1 and τ are $\theta(z+1) = \theta(z)$ and

$$\theta(z+\tau,\tau) = \sum_{n=-\infty}^{\infty} q^{n^2} q^n e^{2\pi i n z} = q^{-1/4} e^{-2\pi i z} \theta(z,\tau),$$

which is easily proved by reindexing the summation and completing the square.

Double quasi-periodicity of θ

From these two periodicities we see that

$$\theta(z + n + m\tau, \tau) = (*)\theta(z, \tau)$$

where the factor $(*)$ is nonzero (hence of the form $e^{2\pi i M(z, n+m\tau)}$) and with $u = n + m\tau$ fixed, $M(z, u)$ is linear in z. Thus

$$\theta(z + u) = e^{2\pi i (L(z,\tau) + J(u))}, \qquad u \in \mathbb{Z} + \mathbb{Z}\tau$$

satisfies the definition of a theta function.

Modular transformation of θ

A deeper transformation property

$$\theta(z,\tau) = (-2i\tau)^{-1/2} e^{-i\pi z^2/2\tau} \theta\left(-\frac{z}{2\tau}, -\frac{1}{4\tau}\right)$$

may be proved by the Poisson summation formula. Note that $\theta(0,\tau)$ is the function we called θ earlier. From our previous discussion, this transformation property implies that $\theta(\tau) = \theta(0,\tau)$ is a modular form. Many other identities may be deduced from this that have applications in the theory of partitions going back to Euler.

The Jacobi triple product identity

The Jacobi triple product identity is a famous infinite product for $\theta(z,\tau)$. Let us write $e^{2\pi iz} = -w$ for a parameter w. Then

$$\theta(z,\tau) = \sum_{-\infty}^{\infty} (-w)^n q^{n^2}.$$

Theorem (Jacobi triple product identity)

We have

$$\sum_{-\infty}^{\infty} (-w)^n q^{n^2} = \prod_{n=1}^{\infty} (1 - wq^{2n-1})(1 - q^{2n})(1 - w^{-1}q^{2n-1}).$$

Partial proof

Many proofs of this may be given. It is easy to show that

$$\sum_{-\infty}^{\infty}(-w)^n q^{n^2} = (*)\prod_{n=1}^{\infty}(1-wq^{2n-1})(1-w^{-1}q^{2n-1})$$

where the factor $(*)$ is independent of w. Indeed, both sides satisfy the same recursion

$$F(w,q) = -wqF(q^2w,q)$$

from which it may be deduced that the quotient is independent of w. But evaluating the ratio

$$\frac{\sum(-w)^n q^{n^2}}{\prod(1-wq^{2n-1})(1-w^{-1}q^{2n-1})}$$

as $\prod(1-q^{2n})$ is harder.

η is a theta function

The Dedekind η function is also a theta function. Let us relate it to $\theta(z, \tau)$. We consider

$$\theta\left(z, \frac{3\tau}{2}\right) = \prod_{n=1}^{\infty}(1 - wq^{3n-3/2})(1 - q^{3n})(1 - w^{-1}q^{3n-3/2}).$$

Choose z so that $w = q^{1/2}$:

$$\theta\left(\frac{\tau}{2}, \frac{3\tau}{2}\right) = \prod_{n=1}^{\infty}(1 - q^{3n-1})(1 - q^{3n})(1 - w^{-1}q^{3n-2}) = \prod_{n=1}^{\infty}(1 - q^{n})$$

η as a theta function (condinued)

Now remembering the definition of θ:

$$\theta\left(\frac{\tau}{2},\frac{3\tau}{2}\right)=\sum_{-\infty}^{\infty}(-q)^{n/2}q^{3n^2/2}=\sum_{n=-\infty}^{\infty}(-1)^n q^{(3n^2+n)/2}.$$

Multiply this by $q^{1/24}$ to complete the square. Since $3n^2+n+\frac{1}{12}=3\left(n+\frac{1}{6}\right)^2$ we get

$$\eta(\tau)=q^{1/24}\prod_{n=1}^{\infty}(1-q^n)=q^{1/24}\theta\left(\frac{\tau}{2},\frac{3\tau}{2}\right)=\sum_{n=-\infty}^{\infty}(-1)^n q^{(6n+1)^2/24}$$

Review of Lie theory

Let $\mathfrak{g}$ be a finite-dimensional simple Lie algebra and $\mathfrak{h}$ a Cartan subalgebra. Then we may write

$$\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha \in \Phi} \mathfrak{X}_\alpha$$

where for $\alpha \in \mathfrak{h}^*$

$$\mathfrak{X}_\alpha = \{X \in \mathfrak{g} | [H, X] = \alpha(H)X \text{ for } H \in \mathfrak{h}\}$$

and the root system Φ is the set of nonzero $\alpha \in \mathfrak{h}^*$ such that $\mathfrak{X}_\alpha$ is nonzero. We may find a nondegenerate symmetric bilinear form $(\ |\ )$ on $\mathfrak{h}$ that is invariant meaning that

$$([x, y]|z) = -(y|[x, z]).$$

Now we may define an isomorphism $\nu : \mathfrak{h} \longrightarrow \mathfrak{h}^*$ by $\langle \nu(H), H' \rangle = (H|H')$. By means of this isomorphism, we may transfer $(\ |\ )$ to an inner product on $\mathfrak{h}^*$.

The Weyl group

We may find a vector $\rho \in \mathfrak{h}^*$ called the Weyl vector such that if

$$\Phi^+ = \{\alpha \in \Phi | (\rho|\alpha) > 0\}$$

then

$$\rho = \frac{1}{2} \sum_{\alpha \in \Phi^+} \alpha.$$

The set Φ^+ is called the set of positive roots.

If $\alpha \in \Phi$ let $r_\alpha : \mathfrak{h}^* \longrightarrow \mathfrak{h}^*$ be the reflection in the hyperplane orthogonal to α. The group W generated by the r_α is called the Weyl group. It has a character $\mathrm{sgn} : W \longrightarrow \{\pm 1\}$ such that $\mathrm{sgn}(r_\alpha) = -1$.

Comparison

If $\mu \in \mathfrak{h}^*$ let e^μ be a formal symbol such that $e^{\mu+\lambda} = e^\mu e^\lambda$. The Weyl denominator formula is the identity

$$\prod_{\alpha\in\Phi^+}(1-e^{-\alpha}) = \sum_{w\in W} \mathrm{sgn}(w) e^{w(\rho)-\rho}.$$

This is an adjunct to the Weyl character formula.

Now compare this to the Jacobi triple product identity:

$$\prod_{n=1}^{\infty}(1-q^n)(1-zq^n)(1-z^{-1}q^n) = \sum_{n=-\infty}^{\infty} (-1)^n z^n q^{n^2}.$$

Both identities equate a product with an alternating sum. In both cases the sum is quite sparse, showing that the product has many cancellations.

Affine Lie algebras and modular forms

It turns out that the Weyl theory, including the Weyl character formula and the Weyl denominator formula, generalize to the class of Kac-Moody Lie algebras, which includes infinite-dimensional Lie algebras with infinite root systems and infinite Weyl groups. The simplest infinte-dimensional Kac-Moody Lie algebra is the affine Lie algebra $\widehat{\mathfrak{sl}}(2)$. And the Weyl denominator formula for $\widehat{\mathfrak{sl}}(2)$ is the Jacobi triple product identity.

This is just the beginning. Motivated by examples in mathematical physics, an extensive theory was developed by Kac and Peterson (1984) showing how characters and "string functions" of representations of affine Lie algebras are modular forms.