INTEGRABLE REPRESENTATIONS OF A KAC-MOODY LIE ALGEBRA.

IF G IS A LIE GROUP, $\mathfrak{g}$ ITS LIE ALGEBRA, ANY REP'N OF $\pi : G \to GL(V)$ REP'N OF $\mathfrak{g}$ AND V IS A $\mathfrak{g}$-MODULE.

CONVERSE IS NOT TRUE.

FOR EXAMPLE IF $V = M(\lambda)$ $\quad \lambda \in \mathfrak{h}^*$

THIS IS AN INFINITE DIMENSIONAL REP'N THAT CANNOT BE LIFTED TO A REP'N OF G.

EVIDENCE: IF V IS A REP'N OF G

$\mathfrak{h} = \mathrm{LIE}(T)$ $\quad N(T)/T = W$ ACTS ON $\mathfrak{h}^*$

AND THE WEIGHT MULTIPLICITIES MUST BE W-INVARIANT.

WEIGHT MULTIPLICITIES FOR $M(\lambda)$

$$\mathrm{ch}\, V = \sum_{\mu \in \mathfrak{h}^*} \dim(V_\mu)\, e^\mu$$

CHARACTER.

1 DIM'L

$\downarrow$

$$M(\lambda) = U(\mathfrak{g}) \otimes_{\mathfrak{b}} \mathbb{C}_\lambda \qquad \mathfrak{b} = \mathfrak{h} \oplus \mathfrak{n}^+.$$

$\mathfrak{n}_+$ ANNIHILATES $\mathbb{C}_\lambda$ WHILE $\mathfrak{h}$ ACTS BY λ.

$$= U(\mathfrak{n}_-) \otimes \left(U(\mathfrak{b}) \otimes U(\mathfrak{b}) \otimes \mathbb{C}_\lambda \right)$$

$$= U(\mathfrak{n}_-) \otimes \mathbb{C}_\lambda \ .$$

(VECTOR SPACE ISO.) AS AN $\mathfrak{g}$-MODULE

$$M(\lambda) \cong U(\mathfrak{n}_-) \otimes e^\lambda$$

$$\mathrm{CH}\ M(\lambda) = e^\lambda \underbrace{\mathrm{CH}\ U(\mathfrak{n}_-)}_{\text{PBW}}$$

$$U(\mathfrak{n}_-) = \bigoplus \prod_{i=1}^{r} f_i^{k_i} = \prod_{\substack{r\text{-TUPLES} \\ (k_1, \dots, k_r) \in \mathbb{N}^r}} f_i^{k}$$

$$\mathrm{CH}\ U(\mathfrak{n}_-) =$$

$$= \prod_{i=1}^{r} \sum_{k_i=0}^{\infty} e^{-k_i \alpha_i} = \prod (1 - e^{-\alpha_i})^{-1}$$

$$\mathrm{CH}\ M(\lambda) = e^\lambda \prod (1 - e^{-\alpha_i})^{-1}.$$

EXAMPLE: $\mathfrak{sl}_2$

$$\operatorname{ch} M(\lambda) = e^{\lambda}\left(1 + e^{-\alpha_1} + e^{-2\alpha_1} + \dots\right)$$

$s_1 : \alpha_1 \mapsto -\alpha_1$

$$\lambda \to \lambda - \langle \alpha_1^\vee, \lambda \rangle \alpha_1$$

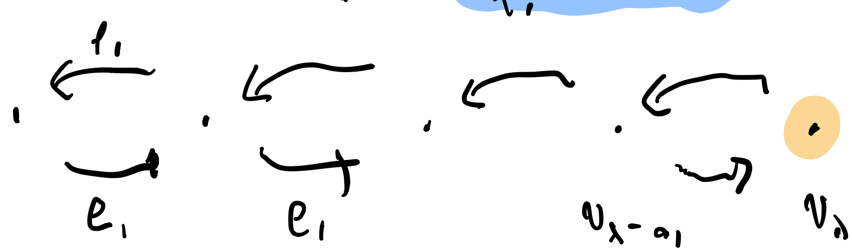
NOT W-INVARIANT.

BUT SUPPOSE $\langle \alpha_1^\vee, \lambda \rangle = n \in \mathbb{N} = \{0, 1, 2, \dots\}$.

THEN I CLAIM THE MAXIMAL IRR QUOTIENT $L(\lambda)$ DOES HAVE W-INVARIANT CHARACTER

v_λ = HIGHEST WEIGHT VECTOR IN $M(\lambda)$ ✓

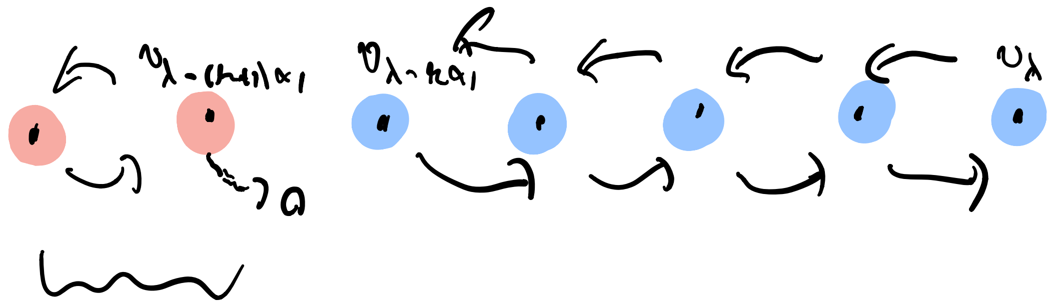
DEFINE $v_{\lambda - n\alpha_1} = \frac{1}{n!} f_1^n v_\lambda$



$$f_1(v_{\lambda-j\alpha_1}) = (j+1)\,v_{\lambda-(j+1)\alpha_1} \quad ✓$$

$$e_1(v_{\lambda-j\alpha_1}) = (k-j+1)\,v_{\lambda-(j-1)\alpha_1} \quad ✓$$

(WRITING k INSTEAD OF $\lambda(\alpha_1^\vee)$.)



MAXIMAL PROPER SUBMODULE.

= IRREDUCIBLE QUOTIENT $L(\lambda)$

THE WEIGHTS OF $L(\lambda)$

$$\lambda - k\alpha_1, \quad \ldots, \quad \lambda-\alpha_1, \quad \lambda$$

$$s_1(\lambda) = \lambda - \langle\alpha_1^\vee, \lambda\rangle\alpha_1 = \lambda - k\alpha_1.$$

THEOREM: IF $\mathfrak{g}$ IS A KAC-MOODY LIE ALGEBRA.

~~FINITE DIM'L SIMPLE LIE ALG.~~

THE WEIGHTS OF $L(\lambda)$ ARE W-INVARIANT IF $\langle\alpha_i^\vee, \lambda\rangle$ IS A NONNEGATIVE INTEGER.

TO PROVE

$$e_1(v_{\lambda - j\alpha_1}) = (n-j+1)\, v_{\lambda-(j-1)\alpha_1}$$

INTRODUCE

$$\Delta = H^2 + 2e_1 f_1 + 2f_1 e_1 \qquad H = \alpha_1^\vee$$

ARGUE THAT IT COMMUTES WITH e_1, f_1 SINCE IT IS IN $Z(U(\mathfrak{g}))$.

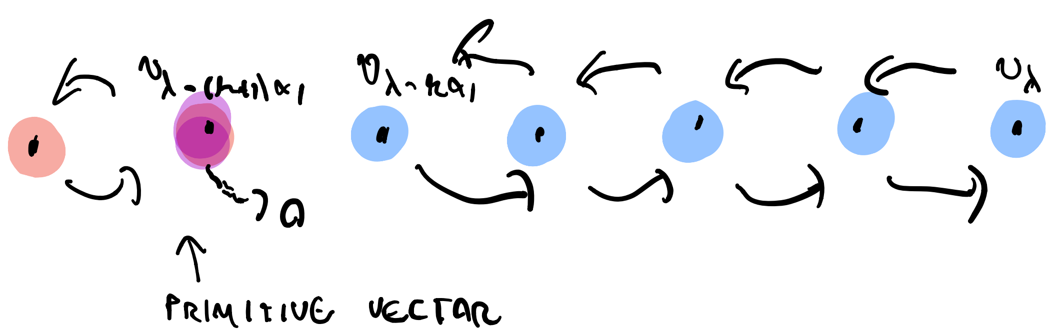
$$\Delta = H^2 + 2H + 4f_1 e_1$$

APPLY TO v_λ USING $Hv_\lambda = n v_\lambda$ AND $e_1 v_\lambda = 0$

$$\Delta v_\lambda = (n^2 + 2n)\, v_\lambda \quad \text{AND}$$

$$\Delta v = (n^2 + 2n)\, v \quad \text{FOR ALL } v.$$

THIS ALLOWS US ENOUGH INFORMATION TO CALCULATE $e_1 v_{\lambda - j\alpha_1} = (n-j+1)\, v_{\lambda-(j-1)\alpha_1}$



DEF: $v \in V_\mu$ IS A PRIMITIVE VECTOR AND μ IS A PRIMITIVE WEIGHT IF THERE EXISTS A SUBMODULE U OF V SUCH THAT $v \notin U$ BUT $e_i v \in U$ FOR ALL e_i.

FOR EXAMPLE SUPPOSE $U = 0$ (IMPORTANT CASE) THEN THE DEFINITION MEANS

$$v \neq 0 \quad \text{BUT} \quad e_i v = 0 \quad \text{ALL } i$$

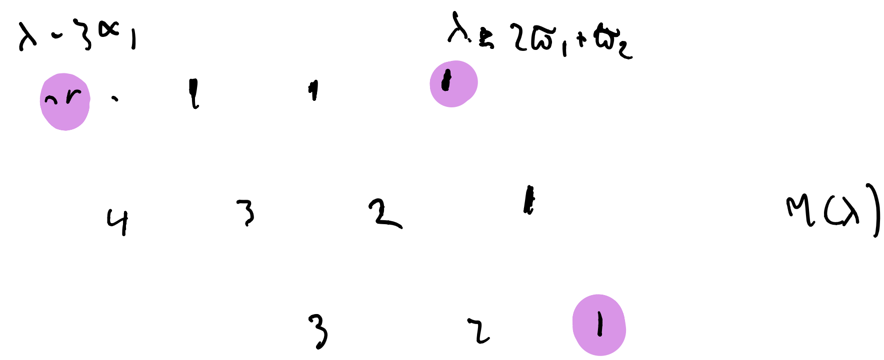
THE REASON FOR MAKING THE DEFINITION THIS WAY (WITH U) IS WE WANT PRIMITIVE VECTORS TO HAVE SOME FUNCTORIAL PROPERTIES

$$0 \to V' \to V \xrightarrow{\varphi} V'' \to 0 .$$

ROUGHLY PRIMITIVE VECTORS OF V SHOULD CORR. TO P.V. IN V', V''.

IF v'' IS A P.V. FOR V'' WANT $\varphi^{-1}(v'')$ TO BE A P.V. FOR V''.

EXAMPLE: $\mathfrak{g} = \mathfrak{sl}_3$ $\quad \langle \alpha_j^\vee, \omega_i \rangle = \delta_{ij}$



CLAIM $\lambda - 3\alpha_1$ IS A P.V.

$\langle \alpha_1^\vee, \lambda \rangle = 2$ SO EMPLOYING

$\mathfrak{sl}_2 \to \mathfrak{sl}_3$

$\{e_1, f_1, \alpha_1^\vee\}$ SPAN AN $\mathfrak{sl}_2$.

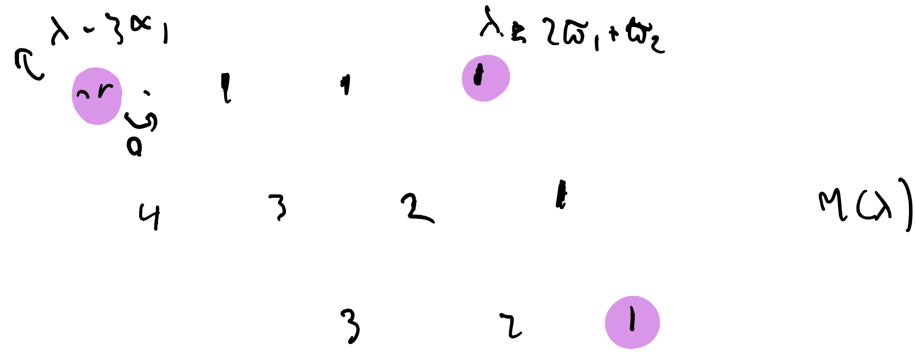
FROM OUR $\mathfrak{sl}_2$ THEORY.

TO PROVE

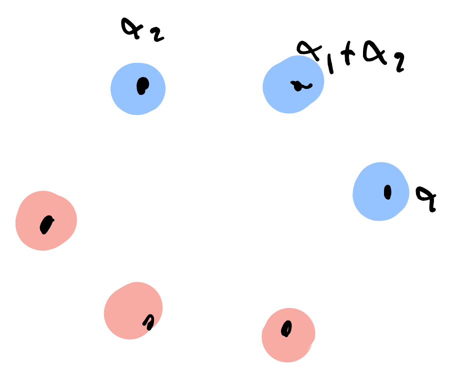
$$e_1(v_{\lambda - j\alpha_1}) = (k - j + 1)\, v_{\lambda - (j-1)\alpha_1}$$

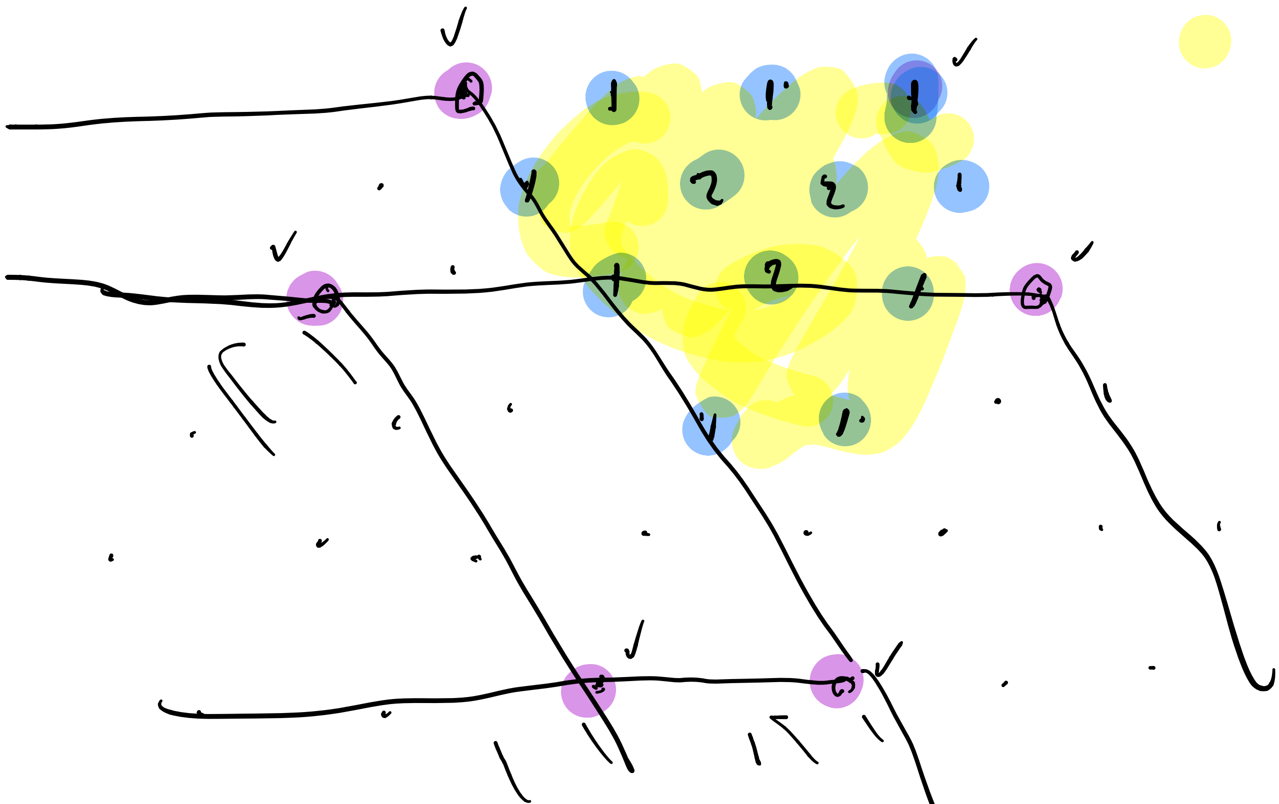
THIS TELLS ME THAT $e_1 v_{\lambda - 3\alpha_1} = 0$

$n = 2, \; j = 3$



$v_{\lambda - 3\alpha_1} = \frac{1}{3!} f_1^2 v_\lambda$





RESOLUTION OF $L(\lambda)$ BY VERMA MODULES

BGG RESOLUTION.

$$0 \to M(\cdot\cdot) \to \begin{matrix} M(\cdots) \\ \oplus M(\cdots \end{matrix} \to M(\lambda - 3\alpha_1) \oplus M(\lambda - 2\alpha_2) \to M(\lambda) \to L(\lambda) \to 0$$

BGG RESOLUTE

$$\mathrm{ch}\, L(\lambda) = \mathrm{ch}\, M(\lambda) - \mathrm{ch}\, M(\lambda - (\langle \alpha_1^\vee, \lambda \rangle + 1)\alpha_1) - \mathrm{ch}\, M(\lambda - (\langle \alpha_2^\vee, \lambda \rangle + 1)\alpha_2) + \cdots -$$

THE LOCATION OF THE PRIMITIVE VECTORS CONTROLS THE VERMA MODULE (BGG) RESOLUTION.

$$\mathrm{ch}\, L(\lambda) = e^{\lambda} \prod_{\alpha \in \Delta^+} (1 - e^{-\alpha})$$

$$- e^{\lambda - (\langle \alpha_1^\vee, \lambda \rangle + 1)\alpha} \prod (\cdots) - \cdots$$

$$= \prod (1 - e^{-\alpha})^{-1} \sum (-1)^{\ell(w)} e^{w(\lambda+\rho)-\rho}.$$

COMPARING THE CHARACTERS IN BGG RESOLUTION GIVES THE WEYL CHARACTER FORMULA. IT IS NOT REALLY NECESSARY TO USE THE BGG RESOLUTION. THIS IDEA (DUE TO BGG) IS THE ESSENCE OF HOW KAC PROVED THE KAC-WEYL C.F.

$$s_1(\lambda + \alpha) - \rho = \lambda - (\langle \alpha_1^\vee, \lambda \rangle + 1)\alpha_1$$

ETC.

$$w(\lambda + \rho) - \rho = w(\lambda) - \overbrace{\sum_{\substack{\alpha \in \Delta^+ \\ w\alpha \in \Delta^-}} \alpha_i}^{w(\rho) - \rho} +$$