Affine Integrable Representations

Here are the pairings of particular elements of $\hat{\mathfrak{h}}^*$ with $\hat{\mathfrak{h}}$.

	α_0	$\alpha_j (j > 0)$	δ	Λ_0	$\Lambda_j (j > 0)$
$\alpha_0^\vee$	1	a_{0j}	0	1	0
$\alpha_i^\vee (i > 0)$	a_{i0}	a_{ij}	0	0	δ_{ij}
K	0	0	0	1	$a_j^\vee$
d	1	0	1	0	0

$$\alpha_0 = \delta - \theta, \qquad \alpha^\vee = K - \theta^\vee.$$

$$\sum_{i=0}^{r} a_i \alpha_i = \delta, \qquad \sum_{i=0}^{r} a_i^\vee \alpha_i^\vee = K,$$

$$s_i(x) = x - \langle \alpha_i^\vee, x \rangle \alpha_i, \qquad x \in \hat{\mathfrak{h}}^*,$$

$$s_i(x) = x - \langle x, \alpha_i \rangle \alpha_i^\vee, \qquad x \in \hat{\mathfrak{h}}.$$

IF $\mathfrak{g}$ IS A KM LIE ALGEBRA P^+ : DOMINANT WEIGHTS

$\lambda \in P^+$ $\quad \operatorname{CH} L(\lambda) = \Delta^{-1} \sum_{w \in W} (-1)^{\ell(w)} e^{w(\lambda+\rho)-\rho}$.

SUPPOSE $\lambda = 0$. THE TRIVIAL MODULE

$\mathbb{C}$ WITH $X \cdot v = 0$ FOR ALL $X \in \mathfrak{g}$, $v \in \mathbb{C}$

IS A HIGHEST WEIGHT MODULE WITH HW

SO $L(\lambda) = \mathbb{C}$. SO $\operatorname{CH} L(\lambda) = 1$.

$$\operatorname{CH} V = \sum_{\mu \in \mathfrak{h}^*} \operatorname{DIM}(V)_\mu e^\mu.$$

$$\Delta = \sum_{w \in W} (-1)^{\ell(w)} e^{w(\lambda+\rho)+\rho}$$

WEYL DENOMINATOR FORMULA.

JTP IDENTITY:

$$\prod_{n=1}^{\infty} (1-\omega q^{2n-1})(1-q^{2n})(1-\omega^{-1} q^{2n-1})$$

$$= \sum_{-\infty}^{\infty} (-\omega)^n q^{n^2}$$

$$q^2 = e^{-\alpha_1}, \quad \omega = e^{-\delta}$$

LEFT SIDE OF JTP BECOMES THIS EXPRESSION.

AFFINE WEYL GROUP.

$K \in Z(\hat{g})$ $\qquad K = \sum_{i=0}^{r} a_i^\vee \alpha_i^\vee$

WE WILL SAY A WEIGHT λ IS OF LEVEL k IF $\langle K, \lambda \rangle = k$.

PROPOSITION: THE WEIGHTS OF LEVEL k ARE INVARIANT UNDER W.

$W = \langle s_0, \ldots, s_r \rangle$

$s_i(x) = x - \langle \alpha_i^\vee, x \rangle \alpha_i$ $\qquad$ ACTION ON $\mathfrak{g}^*$

WE HAVE $\langle \alpha_i^\vee, K \rangle = 0$ FOR ALL i.

$$0 = [K, e_i] = \alpha_i(K)\, e_i$$

So $\alpha_i(K) = 0$

$$\langle K, s_i(x)\rangle = \langle K, x\rangle - \langle \alpha_i^\vee, x\rangle \underset{\substack{\uparrow \\ \text{ZERO}}}{\overset{\alpha_i(K)}{\langle \alpha_i, K\rangle}} = \langle K, x\rangle.$$

	α_0	$\alpha_j (j>0)$	δ	Λ_0	$\Lambda_j (j>0)$
$\alpha_0^\vee$	1	a_{0j}	0	1	0
$\alpha_i^\vee (i>0)$	a_{i0}	a_{ij}	0	0	δ_{ij}
K	0	0	0	1	$a_j^\vee$
d	1	0	1	0	0

$$\alpha_0 = \delta - \theta, \qquad \alpha^\vee = K - \theta^\vee.$$

$$\sum_{i=0}^{r} a_i \alpha_i = \delta, \qquad \sum_{i=0}^{r} a_i^\vee \alpha_i^\vee = K,$$

$$s_i(x) = x - \langle \alpha_i^\vee, x\rangle \alpha_i, \qquad x \in \hat{\mathfrak{h}}^*$$

LEVEL IS PRESERVED BY ACTION OF W.

WE ALSO HAVE

$$w(\delta) = \delta \qquad (w \in W)$$

$$s_i(\delta) = \delta - \underbrace{\langle \alpha_i^\vee, \delta\rangle}_{\text{ZERO.}} \alpha_i = \delta$$

δ ANNIHILATES $\alpha_i^\vee$ BUT $\delta(d) = 1$.

THEREFORE TO VISUALIZE THE ACTION OF AFFINE WEYL GROUP, WE MAY RESTRICT TO LEVEL k COSET AND IGNORE δ.

$\mathfrak{h}^*$ = SPAN OF $\alpha_1, \dots, \alpha_r$ = DUAL SPACE OF CARTAN SUBALG OF $\mathfrak{g}$.

$$\mathfrak{h}^* \xrightarrow{\phi_k} \left\{ \begin{array}{l} \text{LEVEL} \\ k \text{ SPACE} \\ \text{IN } \mathfrak{h}^* \end{array} \right\}$$

$$\begin{array}{ccccc} \sum c_i \alpha_i & & \sum c_i \alpha_i + k\Lambda_0 & & \\ \lambda & \longrightarrow & \lambda + k\Lambda_0 & \longrightarrow & \mathfrak{h}^* \end{array}$$

Λ_0 IS AFFINE FUNDAMENTAL WEIGHT.

	α_0	$\alpha_j (j>0)$	δ	Λ_0	$\Lambda_j (j>0)$
$\alpha_0^\vee$	1	a_{0j}	0	1	0
$\alpha_i^\vee (i>0)$	a_{i0}	a_{ij}	0	0	δ_{ij}
K	0	0	0	1	$a_j^\vee$
d	1	0	1	0	0

$$\alpha_0 = \delta - \theta, \qquad \alpha^\vee = K - \theta^\vee.$$

$$\sum_{i=0}^{r} a_i \alpha_i = \delta, \qquad \sum_{i=0}^{r} a_i^\vee \alpha_i^\vee = K,$$

$$\langle \alpha_i^\vee, \Lambda_j \rangle = \delta_{ij} \qquad \langle d, \Lambda_i \rangle = 0.$$

$$\mathfrak{h}^* \longrightarrow \text{LEVEL } k \text{ COSET IN } \mathfrak{h}^* \subset \hat{\mathfrak{h}}^* \longrightarrow \hat{\mathfrak{h}}^*/\mathbb{C}\delta$$

$$\lambda \xrightarrow{\phi_k} \lambda + k\Lambda_0 \pmod{\mathbb{C}\delta}$$

ACTION OF W ON LEVEL k COSET IN $\hat{\mathfrak{h}}^*$, WHICH FIXES δ

INDUCES AN ACTION ON $\mathfrak{h}^*$.

LET $\hat{\mathfrak{h}}^*_k$ = AFFINE SPACE OF LEVEL k ELEMENTS $\longrightarrow \underline{\hat{\mathfrak{h}}^*_k}/\underline{\mathbb{C}\delta}$

COSET $k\Lambda_0 + \mathfrak{h}^* + \mathbb{C}\delta$.

$$\mathfrak{h}^* \longrightarrow \hat{\mathfrak{h}}^*_k/\mathbb{C}\delta$$

IS A BIJECTION.

$\langle K, \lambda \rangle = k$

$\dim \mathfrak{h}^* = r \qquad \dim \hat{\mathfrak{h}}^* = r+2, \quad \dim \hat{\mathfrak{h}}^*_k = r+1$

δ HAS LEVEL ZERO

$\hat{\mathfrak{h}}^*_{\mathbb{R}} / \mathbb{C}\delta$ HAS DIM r AGAIN.

$\delta = \alpha_0 + \alpha_1 + \alpha_2$

$\hat{sl}(3)$

$$\mathfrak{h}^* = \langle \alpha_1, \alpha_2 \rangle \longrightarrow \hat{\mathfrak{h}}^* = \langle \alpha_1, \alpha_2, \alpha_0, \Lambda_0 \rangle$$

$$\hat{\mathfrak{h}}^*_{\mathbb{R}} = \underbrace{\mathbb{C}\alpha_0 + \mathbb{C}\alpha_1 + \mathbb{C}\alpha_2}_{\mathbb{C}\alpha_1 \oplus \mathbb{C}\alpha_2 + \mathbb{C}\delta} + \mathbb{R}\Lambda_0$$

FIXED BY W.

THIS IS A BIJECTION.

$$\longrightarrow \mathfrak{h}^*_{\mathbb{R}} / \mathbb{C}\delta$$

$$\mathbb{C}\alpha_1 + \mathbb{C}\alpha_2 \rightsquigarrow (\mathbb{C}\alpha_0 + \mathbb{C}\alpha_1 + \mathbb{C}\alpha_2) + \mathbb{C}\Lambda_0$$

$\mathfrak{h}^*$

$\delta = \alpha_0 + \alpha_1 + \alpha_2$

IT IS δ THAT IS FIXED BY W

$$\hat{\mathfrak{h}}^*_{\mathbb{R}} / \mathbb{C}\delta .$$

CALCULATE ACTION OF s_i

$s_1, \dots, s_r$ $\qquad x \in \hat{\mathfrak{h}}^*_{\mathbb{R}}$

$$s_i(x) = x - \langle \alpha_i^\vee, x \rangle \alpha_i .$$

THIS COMMUTES WITH SHIFT BY $k\Lambda_0$

$$s_i(\Lambda_0) = \Lambda_0 - \langle \alpha_i^\vee, \Lambda_0 \rangle \alpha_i = \Lambda_0$$

$$\langle \alpha_i^\vee, \Lambda_j \rangle = \delta_{ij}$$

$\mathfrak{h}_{\mathbb{R}}$.

$$\underset{\substack{\uparrow \\ \Delta_{i_N} \text{ ACTS}}}{\mathfrak{h}^*} \longrightarrow \hat{\mathfrak{h}}^*_{\mathbb{R}} \longrightarrow \hat{\mathfrak{h}}^* \underset{\substack{\uparrow \\ \Delta_{i_0} \text{ ACTS}}}{/} \mathbb{C}\delta.$$

THESE ACTIONS ARE THE SAME.

$$\Delta_0(\Lambda_0) = \Lambda_0 - \langle \alpha_0^\vee, \Lambda_0 \rangle \alpha_0 = \Lambda_0 - \alpha_0$$

IF θ IS THE LONG ROOT IN THE FINITE ROOT SYSTEM OF $\mathfrak{g}$.

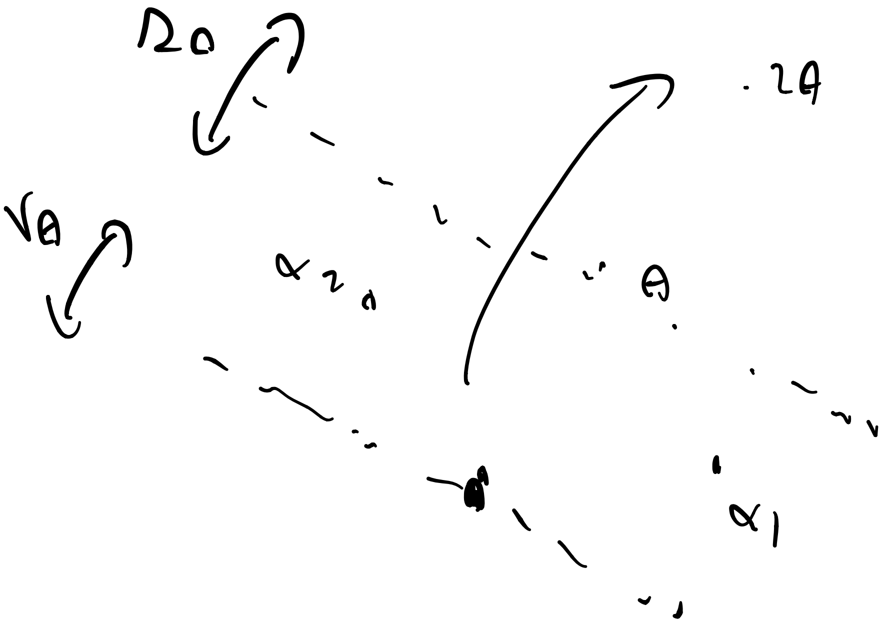
$$\alpha_0 = \delta - \theta$$

$$\Delta_0(\Lambda_0) = \Lambda_0 - \delta + \theta \quad \text{IN } \hat{\mathfrak{h}}^*$$

EFFECT IS $\Delta_0(\Lambda) = \Lambda_0 + \theta$ IN $\hat{\mathfrak{h}}^*_{\mathbb{R}}/\mathbb{C}\delta$.

$s_0(\lambda)$

$$
\begin{tikzcd}
\lambda \in \mathbb{C}\alpha_1 \oplus \cdots \oplus \mathbb{C}\alpha_r \arrow[r, "\phi_k"] \arrow[d, squiggly, "\text{MODIFIED ACTION.}"] & \lambda + k\Lambda_0 \mod \mathbb{C}\delta \arrow[d, "s_0"] \\
r_\theta(\lambda) + k\theta & r_\theta(\lambda) + k\Lambda_0 + k\theta \arrow[l, "\phi_k^{-1}"']
\end{tikzcd}
$$



WEIGHT

k ACTION

$k = 2.$

$k = 2$ $\quad s_0 : \theta \to k\theta.$

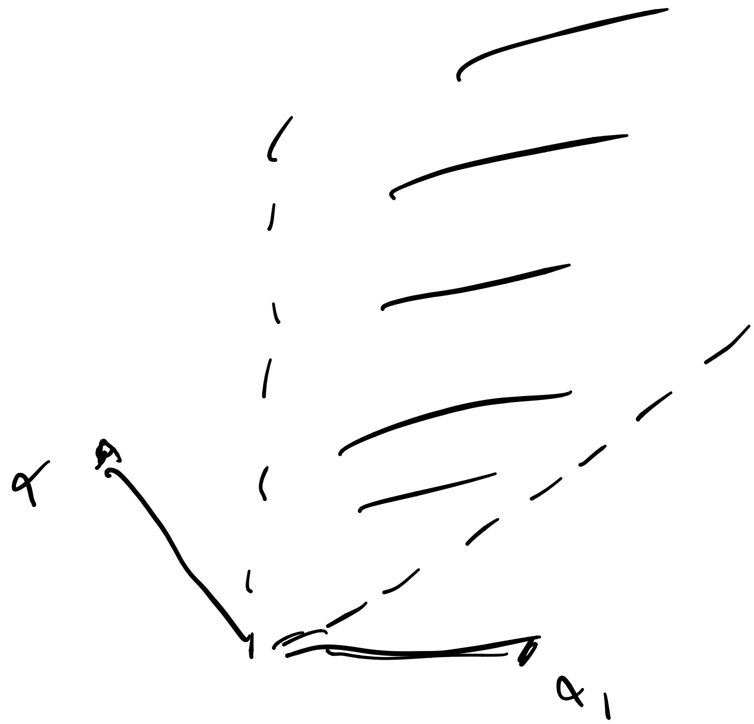
r_θ IS REFLECTION IS HYPERPLANE ORTHOGONAL TO θ.

DOMINANT WEIGHTS

THE WEYL ACTION OF LEVEL k HAS A FUNDAMENTAL DOMAIN, THE LEVEL k FUNDAMENTAL ALCOVE.

$$\langle \alpha_i^\vee, x \rangle \geq 0$$

IF $i = 1, \ldots, r$ THESE ARE AS EXPECTED AND CUT OUT THE POSITIVE WEYL CHAMBER;

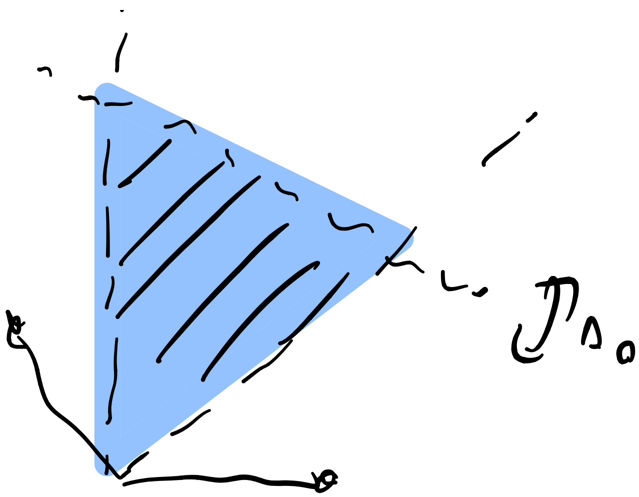


$$\langle \alpha_i^\vee, \lambda \rangle \geq 0 \qquad i = 1, \dots, r.$$

THE REMAINING INEQUALITY IS $\langle \alpha_0^\vee, \lambda + k\Lambda_0 \rangle \geq 0 =$

REMEMBER Φ_L INVOLVES SHIFT

$$\lambda(\theta^\vee) \leq k$$



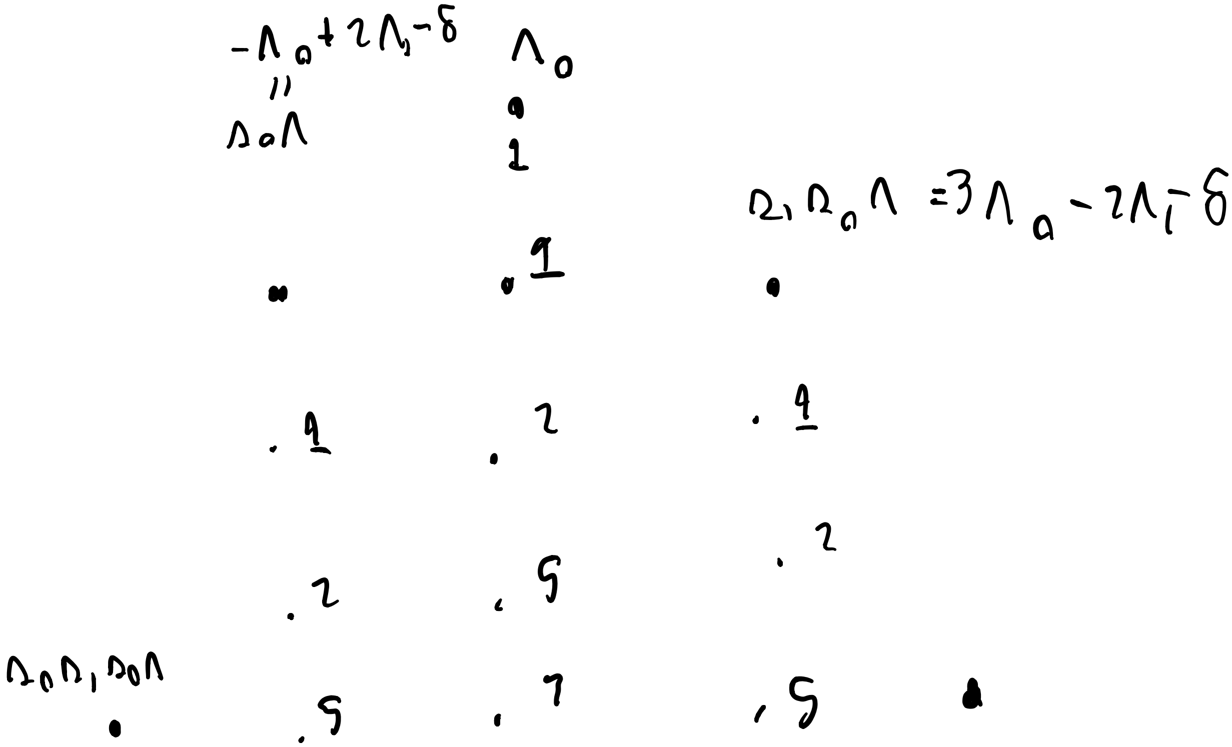
LEVEL k FUNDAMENTAL ALCOVE $\hat{\mathfrak{g}}$.

CONCLUSION: THE DOMINANT WEIGHTS OF LEVEL k CORRESPOND TO WEIGHTS IN THIS ALCOVE.

EXAMPLE: $\lambda = \Lambda_0$

$L(\Lambda_0)$ IS CALLED THE BASIC REPRESENTATION.

$\widehat{\mathfrak{sl}}_2$



STRING FUNCTIONS: THE FUNCTION

$m(t) = \text{MULT } \mu - t\delta$ IS

MONOTONE, ZERO IF $t \ll 0$

SO THERE IS A UNIQUE SMALLEST t SUCH THAT $\mu - t\delta \neq 0$.

THIS $\mu - t\delta$ IS CALLED MINIMAL.

WLOG μ IS ITSELF MINIMAL.

SEQUENCE $\mathrm{MULT}(\mu - t\delta)$ IS CALLED STRING FUNCTION.

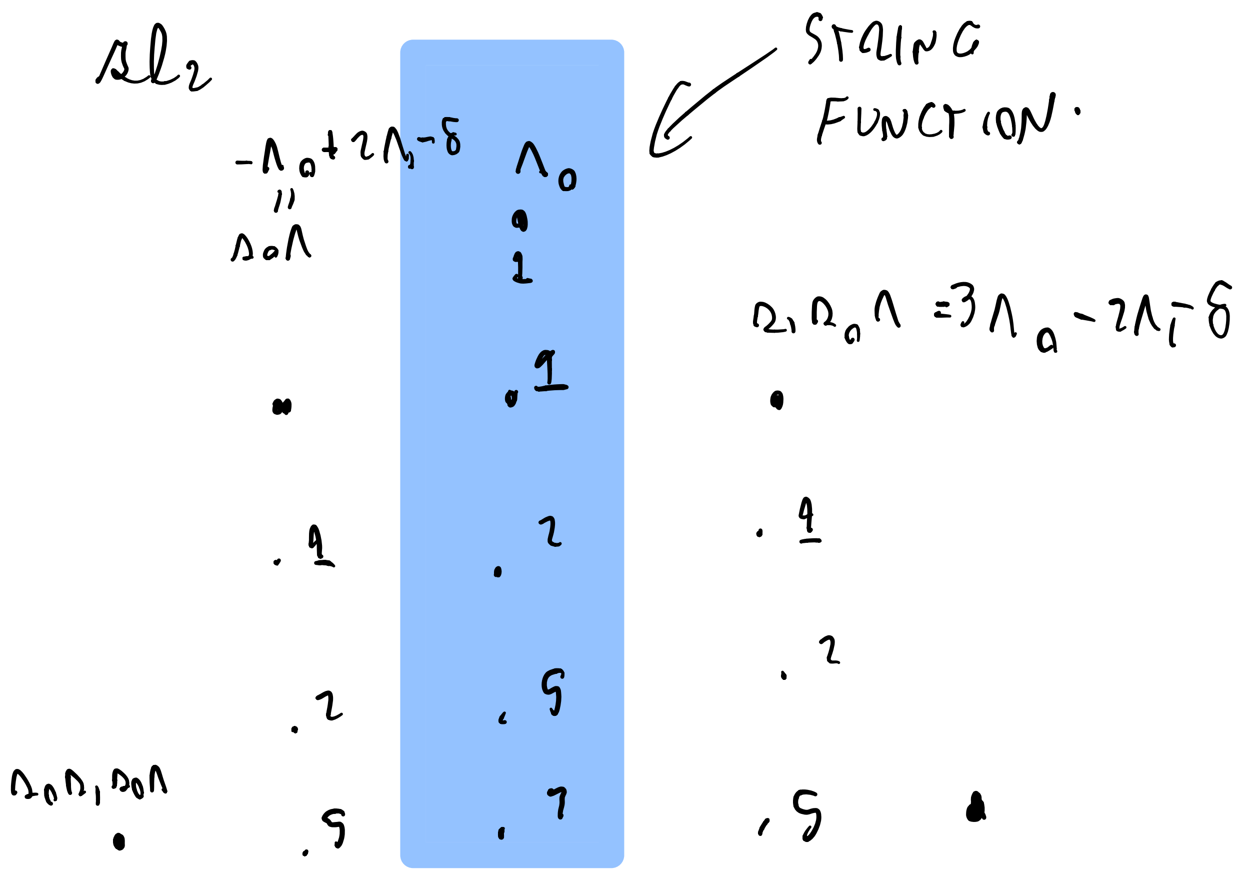
$$a_{\mu} = \sum \mathrm{MULT}(\mu - t\delta)\, q^{t}$$

STRING FUNCTION

FOR THIS REP:

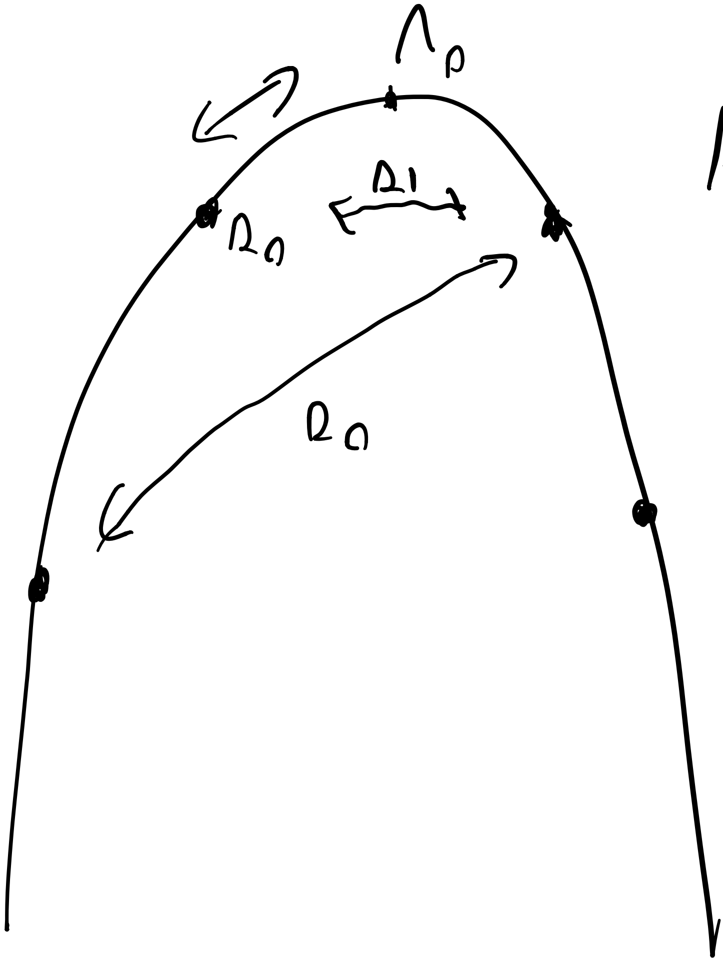
$$\prod_{n=1}^{\infty} (1-q^{n})^{-1} = \sum p(n)\, q^{n}$$

$\mathfrak{sl}_2$



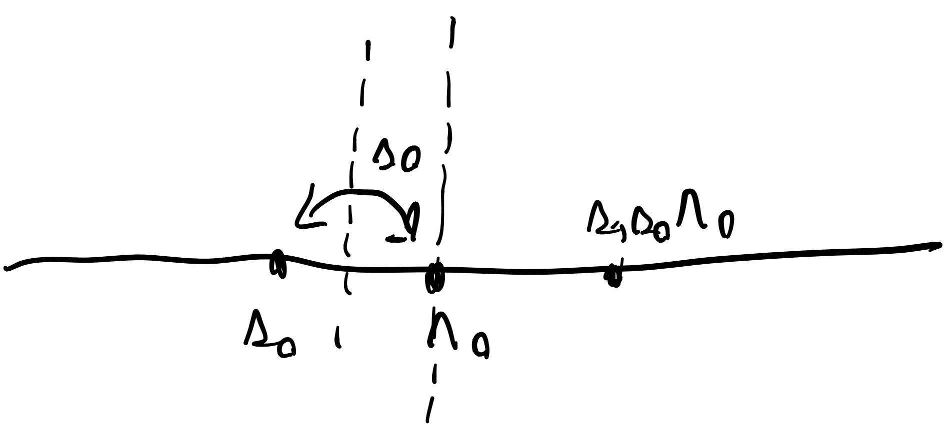
$$w(\delta) = \delta \qquad \delta 0$$

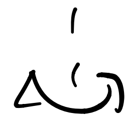
$$B_{w(\mu)} = (B_\mu)$$



$|\mu+\rho|^2 =$
$|\Lambda_0+\rho|^2.$

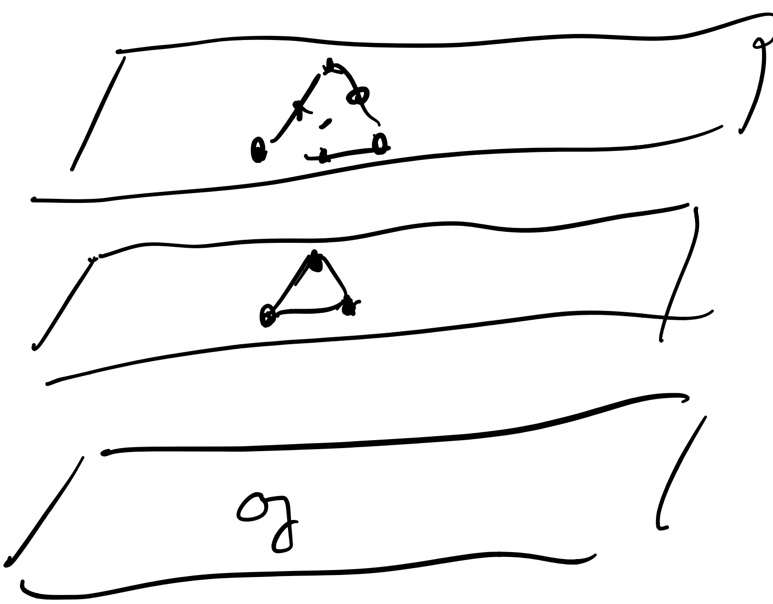
IF WE PROJECT ON $\mathfrak{g}^* \hookrightarrow \mathfrak{g}^{0}_{n}/CS$





For this $\hat{sl}_2$

$$\rho = \Lambda_0 + \Lambda_1 \qquad \left(\rho = \sum \Lambda_i\right)$$



LEVEL k DOMINANT WEIGHTS

$\Updownarrow$

FIELDS IN WZW CFT.

THERE ARE FINITE DIM'L INTEGRABLE REPS OF LEVEL ZERO FOR $\mathfrak{g}'$

THESE ARE KIRILLOV-RESHETIKHIN REPS.

$$\sum q^n \operatorname{mult}(\Lambda_0 - n\delta)$$

$$\parallel$$

$$\prod_{n=1}^{\infty} (1-q^n)^{-1} = \sum p(n) q^n$$

$$= P(q)$$

$$q^{-1/24} P(q) = \frac{1}{\eta(\tau)}$$

$$q = e^{2\pi i \tau} \qquad \eta(\tau) = q^{\frac{1}{24}} \prod (1-q^k)$$

ACTUAL THEOREM

$$\eta^{\dim(\mathfrak{g})} \cdot q^{-m_\Lambda} \, \theta_\Lambda$$

F.D. L.A.

Λ = HIGHEST WEIGHT

THIS IS A MODULAR FORM

SPAN OF THESE FUNCTIONS OVER LEVEL k WEIGHTS IS INVARIANT UNDER

$$\tau \rightarrow -\frac{1}{\tau}.$$

SCATTERING MATRIX = S-MATRIX.

$$C_\lambda = q^{-m_\lambda} \theta_\lambda$$

IS MODULAR BUT ONLY WEAKLY (CAN HAVE POLES E.G. AT CUSPS.)

TO MAKE IT A TRUE MODULAR FORM WE MULTIPLY BY $\eta^{\dim(\mathfrak{g})}$.

KAC-PETERSON (1984)