AFFINE LIE ALGEBRAS:

CHAPTERS 6, 7, 12, 13 IN KAC' BOOK

PROOF OF KAC-WEYL CHARACTER FORMULA, FINISHING THURSDAY'S LECTURE.

LET $\lambda \in P^+$ (DOMINANT WEIGHTS) FOR $\mathfrak{g}$ MOMENTARILY A FD SIMPLE LIE ALG. BUT WE'LL CONSIDER KAC-MOODY CASE ALSO

SINCE $$\mathrm{ch}\, M(\lambda) = \sum_{\substack{|\mu+\rho|^2 = |\lambda+\rho|^2 \\ \mu \leq \lambda}} \underline{d(\lambda,\mu)}\, \mathrm{ch}\, L(\mu)$$

$d(\lambda, \mu)$ = # OF TIMES $L(\mu)$ APPEARS IN COMPOSITION SERIES

$$\mathrm{ch}\, L(\lambda) = \sum_{\substack{|\mu+\rho|^2 = |\lambda+\rho|^2 \\ \mu \leq \lambda}} c(\lambda,\mu)\, \mathrm{ch}\, M(\mu)$$

$$= \sum_{\cdots} c(\lambda,\mu)\, e^{\mu} \underbrace{\prod_{\alpha\in\Delta^+} (1-e^{-\alpha})^{-1}}_{\text{CH OF } M(\mu) \cong e^{\mu}\, \text{ch}(U(\mathfrak{n}_-))}$$

WE CAN CHECK $\Delta = e^{\rho}\prod(1-e^{-\alpha})$

$$w(\Delta) = (-1)^{\ell(w)}\Delta$$

$$= \sum_{\cdots} c(\lambda,\mu)\, e^{\mu+\rho}\, \Delta^{-1}$$

WE LEARN $c(\lambda,\mu) = (-1)^{\ell(w)} c(\lambda, w\cdot\mu)$

$$w\cdot\mu = w(\mu+\rho)-\rho.$$

USING $L(\lambda)$ IS W-INVARIANT, Δ^{-1} ANTI-SYMMETRIC SO

$$\sum c(\lambda,\mu)\, e^{\mu+\rho}$$

IS ANTI SYMMETRIC.

$$\sum C(\lambda, w\cdot\mu)\, e^{\mu+\rho}$$

$$= (-1)^{\ell(w)} \sum C(\lambda, w\cdot\mu)\, e^{w(\mu+\rho)}$$

$$= (-1)^{\ell(w)}\, C(\lambda, w\cdot\mu)\, e^{w\cdot\mu+\rho}$$

COMPARING COEFFS.

$$C(\lambda, w\cdot\mu) = (-1)^{w}\, C(\lambda,\mu).$$

WE ALSO KNOW $C(\lambda,\lambda) = 1$.

WE THUS MAY ISOLATE THE TERMS WE WANT;

$$A^{-1} \sum_{w\in W} C(\lambda, w\cdot\lambda)\, e^{w\cdot\lambda+\rho} + \text{~~POSSIBLE OTHER TERMS~~}$$

SOME OF THE μ WITH $|\mu+\rho|^2 = |\lambda+\rho|^2$ AND $\mu \leq \lambda$.

$$= \Delta^{+} \sum (-1)^{\ell(w)} e^{w(\lambda+\rho)}$$ + ~~OTHER TERMS.~~

WE HAVE TO SHOW THAT IF

$$c(\lambda,\mu) \neq 0 \quad \text{THEN} \quad \mu = \omega\cdot\lambda$$

FOR SOME $\omega \in W$. FIND $w \in W$ SUCH THAT $\omega(\mu+\rho)$ IS DOMINANT.

$$c(\lambda,\mu) = (-1)^{\ell(\omega)}\, c(\lambda, \omega\cdot\rho)$$

$$\omega\cdot\mu = \underbrace{\omega(\mu+\rho)}_{\text{DOMINANT}} - \rho.$$

FROM SUPPORT CONSIDERATIONS $c(\lambda, w\cdot\mu) = 0$ UNLESS $w\cdot\mu < \lambda$.

WE HAVE TO HAVE

$$|\omega\cdot\mu+\rho|^2 = |\mu+\rho|^2 = |\lambda+\rho|^2.$$

WE MAY REPLACE μ BY $\omega \cdot \mu$ FOR SAKE OF SIMPLICITY. μ MAY NOT BE DOMINANT BUT $\mu + \rho$ IS DOMINANT.

$$|\lambda + \rho|^2 = |\mu + \rho|^2$$

$$|a|^2 - |b|^2 = (a+b \mid a-b).$$

$$0 = (\lambda + \mu + 2\rho \mid \lambda - \mu)$$

$\lambda - \mu = \sum_i k_i \alpha_i$ $\quad k_i \geq 0$ SINCE $\lambda \geq \mu$

$\lambda + \mu + 2\rho$ IS STRONGLY DOMINANT.

$(\lambda + \mu + 2\rho \mid \alpha_i) > 0$ FOR ALL i.

SO $\sum k_i \times$ POSITIVE $= 0 \Rightarrow$ ALL $k_i = 0 \Rightarrow \lambda = \mu$.

IN KM CASE REAL ISSUE IS THAT COMPOSITION SERIES MAY NOT HAVE FINITE LENGTH.

UNTWISTED AFFINE LIE ALGEBRAS.

$$0 \to \mathbb{C} \hookrightarrow \hat{\mathfrak{g}}' \to \underbrace{\mathbb{C}[t,t^{-1}] \otimes \mathfrak{g}}_{\mathfrak{g}_t} \to 0$$

$\mathfrak{g}$: F.D. SIMPLE L.A.

EXPLICIT COCYCLE.

$$X_n = t^n \otimes X \qquad X \in \mathfrak{g},\ n \in \mathbb{Z}$$

$$\phi : \mathfrak{g}_t \times \mathfrak{g}_t \to \mathbb{C}$$

$$\phi(X_n, Y_m) = \delta_{n,-m} \cdot \underbrace{n(X|Y)}$$

INVARIANT B.F. ON $\mathfrak{g}$.

IN LECTURE 3 CHECKED $\phi \in Z^2(\mathfrak{g}_t, \mathbb{C})$.

HENCE

$\hat{\mathfrak{g}}'$ = SPAN OF X_n, K

$[K, \text{ANY}] = 0$

$[X_n, Y_m] = \phi(X_n, Y_m) \cdot K$.

LET $\hat{\mathfrak{g}} = \hat{\mathfrak{g}}' \oplus \mathbb{C} \cdot d$

$d : \hat{\mathfrak{g}}' \to \hat{\mathfrak{g}}'$ IS THE DERIVATION

$d(X_n) = nX_n$ $\qquad$ $d = t\frac{d}{dt}$.

$[d, X_n] = nX_n$ IN $\hat{\mathfrak{g}}$.

THE CARTAN SUBALGEBRA

$$\hat{\mathfrak{h}} = \mathfrak{h} \oplus \mathbb{C} \cdot K \oplus \mathbb{C} \cdot d$$

IS AN ABELIAN SUBALGEBRA.

$$\hat{n}_+ = \underbrace{n_+}_{\text{POS PART OF } \mathfrak{g}} \oplus \text{ SPAN OF } X_n,\ n > 0$$

$$n_- = n_- \oplus \text{ SPAN OF } X_n,\ n < 0.$$

$$\hat{\mathfrak{g}} = \mathfrak{h} \oplus n_+ \oplus n_- .$$

WE ARGUED IN LECTURE 3 THAT

$\hat{\mathfrak{g}} = \underbrace{\mathfrak{g}(\hat{A})}$ $\quad \hat{A}$ IS THE

KAC MOODY LIE ALGEBRA.

EXTENDED CARTAN MATRIX OF $\mathfrak{g}$

(THE MAIN THING TO CHECK IS GENERATORS SATISFY RIGHT RELATIONS AND THERE ARE NO NONZERO IDEALS THAT DO NOT MEET $\mathfrak{h}$.)

EMBED $\mathfrak{h}^* \to \hat{\mathfrak{h}}^*$: IF WE HAVE A LINEAR FUNCTIONAL ON $\mathfrak{h}$ EXTEND BY ZERO ON K AND d. TO L.F. ON $\hat{\mathfrak{h}}$.

$\alpha_1, \dots, \alpha_r$ ARE SIMPLE ROOTS OF $\mathfrak{g}$, $\hat{\alpha}_1, \dots, \hat{\alpha}_r$ ARE SIMPLE COROOTS THESE ARE IN $\hat{\mathfrak{g}}$ ALSO AND ARE SIMPLE ROOTS AND COROOTS. BUT IS ONE MORE SIMPLE ROOT α_0 (AND COROOT $\alpha_0^\vee$).

Let δ be the "null root":

$$\delta : \hat{\mathfrak{h}} \to \mathbb{C}.$$

δ is zero on $\mathfrak{h}$, $\delta(k) = 0$ but $\delta(d) = 1$.

This is a root since if $H \in \mathfrak{h}$.

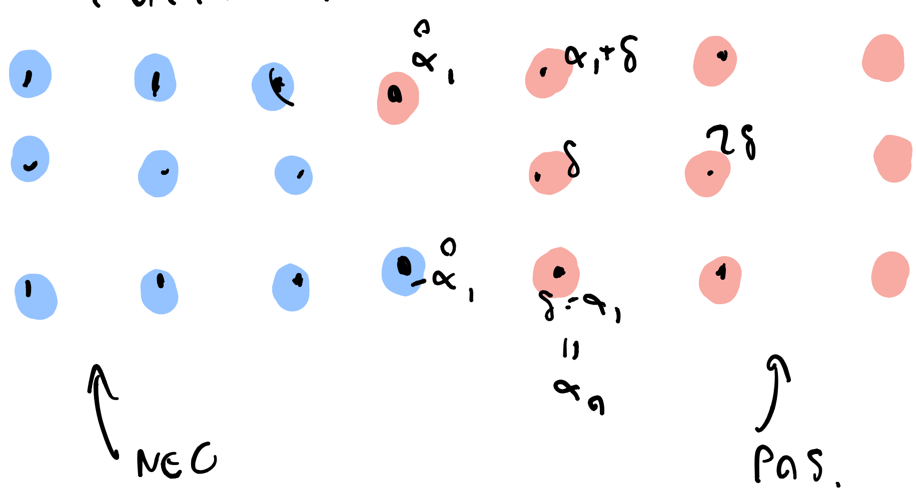
$$\approx \quad H_1 \in \mathfrak{X}_\delta = \{ X \mid [H, X] = \delta(H) X \text{ for } H \in \mathfrak{h} \}.$$

The root system $\hat{\Delta}$ consists of

$\{ \mathring{\alpha} + n\delta \mid \mathring{\alpha} \in \Delta,\ n \in \mathbb{Z} \}$ (real roots)

$\{ n\delta \mid n \in \mathbb{Z},\ n \neq 0 \}$.

Roots of $\hat{\mathfrak{sl}}_2$:



THERE IS ONE SIMPLE POSITIVE ROOT NOT IN $\{\alpha_1, \alpha_2, \dots, \alpha_r\}$.

(A ROOT IS SIMPLE IF IT IS NOT THE SUM OF OTHER POSITIVE ROOTS.

$$\alpha_0 = \delta - \theta \qquad \theta \in \Delta \text{ FINITE R.S. OF } \mathfrak{g}$$

θ IS THE LONGEST ROOT.

IF $\delta + \gamma$ IS POS. ROOT AND $\gamma \neq -\theta$ YOU CAN WRITE THIS $(\delta + \gamma_1) + \gamma_2$, γ_2 A POSITIVE ROOT OF $\mathfrak{g}$.

$$\theta = \sum_{i=1}^{r} a_i \alpha_i$$

a_i "MARKS" OR "LABELS"

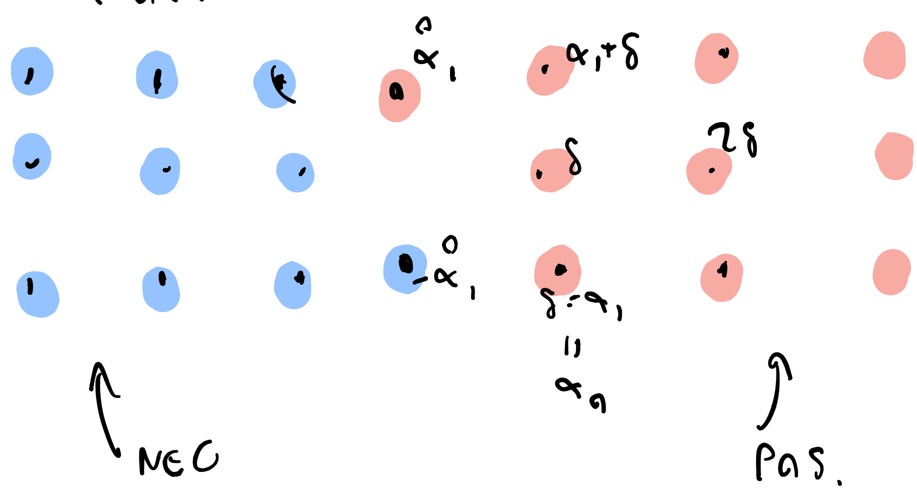
(IN LEC. 3 SHOWED SAGE CODE TO COMPUTE a_i) IN TABLES IN CH. 4 OF KAC.

DEFINE $a_0 = 1$.

THEN $\displaystyle\sum_{i=0}^{r} a_i \alpha_i = (\delta - \theta) + \theta = \delta.$

$$\boxed{\sum a_i \alpha_i = \delta.}$$

ROOTS OF $\hat{sl}_2$:



$$\delta = a_0 \alpha_0 + a_1 \alpha_1 = \alpha_0 + \alpha_1$$

IN TYPE A_1 ALL $a_i = 1$.

ALSO LET $\theta^\vee$ = COROOT CORR TO θ

$\hat{\mathfrak{g}}^\sigma$

$$\theta^\vee = \sum_{i=1}^{r} a_i^\vee \alpha_i^\vee$$

AND ALSO $a_0^\vee = 1$, $\alpha_0^\vee$ THE MISSING SIMPLE COROOT.

TO BE PROVED

$$\boxed{K = \sum_{i=0}^{r} a_i^\vee \alpha_i^\vee} = \theta^\vee + \alpha_0^\vee.$$

$\alpha_0^\vee$ IS SUPPLIED TO US BY THE REALIZATION OF THE CARTAN MATRIX AND OUR IDENTIFICATION OF $\hat{\mathfrak{g}} = \mathfrak{g}(\hat{A})$

I WILL ASSUME $\mathfrak{g}$ IS SIMPLY LACED SO CARTAN MATRIX IS SYMMETRIC:

$$a_{ij} = \langle \alpha_i^\vee, \alpha_j \rangle = a_{ji}.$$

MORE GENERALLY $A = (a_{ij}) = D \cdot B$

WHERE B IS SYMMETRIC AND

$$D = \mathrm{DIAG}\left(a_i a_i^{\vee\,-1}\right).$$

I WILL ARGUE THAT

LEMMA: $\sum a_i^\vee a_i$ IS CENTRAL.

$$\alpha_j\left(\sum a_i^\vee \alpha_i^\vee\right) = 0$$

INDEED LHS $= \sum_{i=0}^{r} \langle \alpha_i^\vee, \alpha_j \rangle a_i^\vee = \sum a_{ij} a_i^\vee$

$$= \sum a_{ji} a_i = \sum \langle \alpha_j^\vee, \alpha_i a_i \rangle = \langle \alpha_j^\vee, \delta \rangle = 0$$

(IN CASE $a_{ij} = a_{ji}$, $\Delta \cong \Delta^\vee$ SO
$a_i = a_i^\vee$)

δ KILLS $\mathfrak{g}$. LEMMA PROVED.

LET $K' = \sum_{i=0}^{r} a_i^\vee \alpha_i^\vee$

$\alpha_j(K') = 0$ SO

$$[K', e_i] = \alpha_i(K') \cdot e_i = 0$$

SIMILARLY $[K', f_i] = 0$

SO K' COMMUTES WITH GENERATORS

$$K' \in Z(\hat{\mathfrak{g}}') = \mathbb{C} \cdot K.$$

SO BY ADJUSTING A CONSTANT WMA $K' = K$.

TO REITERATE!

$$\sum_{i=0}^{r} a_i \alpha_i = \delta \qquad \sum_{i=0}^{r} a_i^\vee \alpha_i^\vee = K.$$

$$\sum_{i=1}^{r} a_i \alpha_i = \theta, \quad \alpha_0 = \delta - \theta.$$

DEFINE FUNDAMENTAL WEIGHTS

$\Lambda_i \quad (i = 0, \cdots, r) \qquad \Lambda_i \in \hat{h}^+$

$\langle \alpha_j^\vee, \Lambda_i \rangle = \delta_{ij}$

$\langle d, \Lambda_i \rangle = 0.$

$P = \{ \lambda \in \hat{h}^+ \mid \langle \alpha_i^\vee, \lambda \rangle \in \mathbb{Z} \} = \bigoplus \mathbb{Z} \Lambda_i \oplus \mathbb{C} \cdot \delta.$

$P^+ = \{ \qquad \in \mathbb{N} \} = \sum \mathbb{N} \cdot \Lambda_i \oplus \mathbb{C}\delta.$

IF $\lambda - \lambda' \in \mathbb{C}\delta$ THEN

$L(\lambda), L(\lambda')$ DIFFER BY

TENSORING WITH A ONE-DIMENSIONAL MODULE.

$\lambda' = \lambda + a \cdot \delta$

ABELIAN SUBALG

$\varepsilon_a : g' \to 0 \qquad \varepsilon_a : g \to \mathbb{C}$

$d \to a$

$L(\lambda') = \mathbb{C}_{\varepsilon_a} \otimes L(\lambda).$

SO THERE IS NO LOSS OF GENERALITY IN JUST STUDYING $L(\lambda)$.

Ch, $L(\lambda)$ IS DESCRIBED BY STRING FUNCTIONS, MODULAR FORMS.