BGG GAVE A PROOF OF WCF FOR SEMISIMPLE $\mathfrak{g}$ BASED ON CATEGORY $\mathcal{O}$ IDEAS. KAC GENERALIZED TO KM LIE ALGEBRAS.

LET $\mathfrak{g}$ BE A SYMMETRIZABLE KM LIE ALG.

$\mathfrak{h}^* \supset P$ (WEIGHT LATTICE)

$P = \{\lambda \in \mathfrak{h}^* \mid \langle \alpha_i^\vee, \lambda \rangle \in \mathbb{Z}\} \supset P^+$

$P^+ = \{\lambda \in \mathfrak{h}^* \mid \langle \alpha_i^\vee, \lambda \rangle \in \mathbb{N}\}$

$\mathbb{N} = \{0, 1, 2, \dots\}$

PROVED IN LECTURE 5 IF $\lambda \in P^+$ THEN $L(\lambda)$ IS INTEGRABLE. SO

- $e_i, f_i$ ACT LOCALLY NILPOTENTLY
- $L(\lambda) = \bigoplus V_\mu$    $\dim(V_\mu)$ IS $W$-INVARIANT.

TODAY WE'LL PROVE

$$\operatorname{ch} L(\lambda) = \Delta^{-1} \sum_{w \in W} (-1)^{\ell(w)} e^{w(\lambda+\rho)}$$

$w \cdot \lambda = w(\lambda+\rho) - \rho$

$$\operatorname{ch} V = \sum_{\mu \in \mathfrak{h}^*} \dim(V_\mu) e^\mu$$

$$\Delta = e^{\rho} \prod_{\alpha \in \Delta^+} (1 - e^{-\alpha}).$$

WEYL DENOMINATOR

$M(\mu)$ VERMA MODULE

$$\mathrm{CH}\ M(\mu) = \underbrace{e^{\mu}}_{\text{CH. OF } \mu} \underbrace{\prod (1 - e^{-\alpha})^{-1}}_{\text{CH OF } U(\mathfrak{n}_-)}$$

FOLLOWS FROM $M(\mu) \cong \mu \otimes U(\mathfrak{n}_-)$
AS $\mathfrak{h}$-MODULES

SO FORMALLY

$$\mathrm{CH}\ L(\lambda) = \sum_{w \in W} (-1)^{\ell(w)} \underbrace{\mathrm{CH}\ M(w \cdot \lambda)}_{e^{w(\lambda+\rho)-\rho} \prod_{\alpha \in \Delta^+} (1-e^{-\alpha})^{-1}}$$

$$= \Delta^{-1} e^{w(\lambda+\rho)}$$

CATEGORY $\mathcal{O}$. LET $V$ BE A $\mathfrak{g}$-MODULE. TO BE IN CATEGORY $\mathcal{O}$.

(i) HAS A WEIGHT SPACE DECOMPOSITION

$$V = \bigoplus V_\mu \qquad V_\mu \text{ FINITE-DIM'L.}$$

(ii) THERE ARE A FINITE SET OF $\lambda_i \in \mathfrak{h}^*$ SUCH THAT $V_\mu = 0$ UNLESS $\mu \leq \lambda_i$ SOME $i$ I.E.

$$\lambda_i - \mu = \sum n_i \alpha_i \qquad n_i \in \mathbb{N}$$

THIS DEFINES AN ABELIAN CATEGORY THAT CONTAINS HIGHEST WEIGHT MODULES SUCH AS $M(\lambda)$, $L(\lambda)$

BGG PROVED THAT IF $\mathfrak{g}$ FINITE-DIMENSION SEMISIMPLE. $\Rightarrow$ $L(\lambda)$ HAS A BGG RESOLUTION:

$$\cdots \to M(\lambda) \to L(\lambda) \to 0$$

RESOLUTION OF $L(\lambda)$ BY VERMA MODULES

THE WCF IS A REFLECTION OF THIS.

DIFFERENCE BETWEEN FD CASE & GENERAL HM CASE:

IF $\mathfrak{g}$ IS SEMISIMPLE, F.D. $\Rightarrow$ $M(\lambda)$ HAS A COMPOSITION SERIES,
$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \overset{\shortparallel}{V}$

$$0 = V_0 \subset V_1 \subset \cdots \subset V_n = V$$

$V_{i+1}/V_i$ IRREDUCIBLE $\quad V_{i+1}/V_i \cong L(\lambda_i)$.

PROOF: TAKE A MAXIMAL PROPER SUBMODULE $V^1$
$V \supset V^1 \qquad V/V^1$ IRREDUCIBLE.

FIND A MAXIMAL PROPER SUBMODULE $V^2$ OF $V^1$
SO $V^1/V^2$ IRR. CONTINUED BUT ISSUE:
DOES THE CHAIN

$$V = V^0 \supset V^1 \supset V^2 \supset \cdots \quad \text{TERMINATE?}$$

EACH QUOTIENT $V^i/V^{i+1}$ IS A HIGHEST WEIGHT MODULE. LET $v^i$ BE A HIGHEST WEIGHT VECTOR.
A PREIMAGE IN $V$ IS A PRIMITIVE VECTOR.

$$\|\lambda_i+\rho\|^2 = \|\lambda+\rho\|^2$$ BY CASIMIR EIGENVALUE COMPARISON.

$\Omega$ ACTS BY $(\lambda|\lambda+2\rho) = \|\lambda+\rho\|^2 - \|\rho\|^2$
ON EACH CONSTITUENT.

IN THE FINITE DIM'L CASE THIS CONFINES $\lambda_i$ TO A COMPACT SUBSET OF $\mathfrak{h}^*$ BECAUSE $( , )$ IS POSITIVE DEFINITE.

$V_{\lambda_i}$ F.D. $\Rightarrow$ ONLY FINITELY MANY POSSIBLE PRIMITIVE VECTORS. //

IN KAC MOODY CASE $M(0)$ HAS A PRIMITIVE VECTOR OF WEIGHT $\rho - w(\rho)$ FOR EVERY $w \in W$.

(EXERCISE 10.3 OF KAC. BEGINNING OF KAZHDAN-LUSZTIG CONJECTURE PROVED BY BEILINSON-BERNSTEIN, BRYLINSKY-KASHIWARA.)

$$\rho - w(\rho) = \sum_{\substack{\alpha \in \Delta^+ \\ w^{-1}(\alpha) \in \Delta^-}} \alpha$$

PROVED TUESDAY THE CARDINALITY OF THIS SET IS $\ell(w)$.

$$\rho - s_i \rho = \alpha_i$$

$$s_i \rho = \rho - \underbrace{\langle \alpha_i^\vee, \rho \rangle}_{1} \alpha_i$$

SO PROVE

$$\boxed{\rho - w(\rho) = \sum_{\substack{\alpha \in \Delta^+ \\ w^{-1}(\alpha) \in \Delta^-}} \alpha}$$

ASSUME TRUE FOR $w$ AND PROVE FOR $w s_i > w$

BY INDUCTION

$$\rho - w s_i(\rho) = (\rho - w\rho) + w(\rho - s_i(\rho))$$

$$\left( \sum_{\substack{\alpha \in \Delta^+ \\ w^{-1}(\alpha) \in \Delta^-}} \alpha \right) + w(\alpha_i)$$

NEED TO KNOW

$$\{\alpha \in \Delta^+ \mid (w s_i)^{-1}(\alpha) \in \Delta^-\} = \{\alpha \in \Delta^+ \mid w^{-1}(\alpha) \in \Delta^-\} \cup \{w(\alpha_i)\}$$

WHICH I PROVED ON TUESDAY.

THE CLAIM $M(0)$ HAS A PRIMITIVE

VECTOR OF THIS WEIGHT. (ACTUALLY ITS NEGATIVE)

IF $v_{\omega(\rho)-\rho}$ IS A PV OF THIS WEIGHT,

CONSTRUCT A PV OF WEIGHT

$\Delta_i \omega(\rho) - \rho$. THE IDEA IS TO SHOW

$k = \langle \alpha_i^\vee \mid \omega(\rho) - \rho \rangle$ IS A NONNEG. INTEGER.

DEFINE $v_{\Delta_i \omega(\rho)-\rho} = f_i^{k+1} v_{\omega(\rho)-\rho}$.

THIS IS A P.V.

$e_i f_i^{k+1} v_{\omega(\rho)-\rho} = 0$ BY $SL(2)$ THEORY.


$e_j f_i^{k+1} v_{\dots} = 0$ IF $j \neq i$

SINCE $[e_j, f_i] = 0$

$[M(\lambda) : L(\mu)]$ = # OF TIMES $L(\mu)$ APPEARS IN A COMPOSITION SERIES (UNIQUE BY JORDAN-HÖLDER THEOREM

SURPRISE: IT IS POSSIBLE THAT

$$\|\mu+\rho\|^2 = \|\lambda+\rho\|^2 \text{ AND}$$

$$[M(0) : M(\mu)] > 1.$$

$\lambda = 0$

THEOREM (KL CONJ. PROVED BB, BK)

THE MULTIPLICITY OF $-\rho - w(\rho) = w w_0(\rho) - \rho$ IS $P$

$w_0$ = LONG ELT
SO $w_0(\rho) = -\rho$

BECAUSE OF THIS NUANCE LET'S FIRST RESTRICT TO SEMISIMPLE CASE.

$$0 \to V' \to V \to V'' \to 0 \qquad ch(V) = ch(V') + ch(V'')$$

$$\operatorname{ch} M(\lambda) = \sum_{\substack{\|\mu+\rho\|^2 = \|\lambda+\rho\|^2 \\ \mu \preceq \lambda}} [M(\lambda) : L(\mu)] \operatorname{ch} L(\mu)$$

$\lambda \in P$

$$= \sum_{\substack{\|\mu+\rho\|^2 = \|\lambda+\rho\|^2 \\ \mu \preceq \lambda}} d_{\lambda,\mu} \operatorname{ch} L(\mu)$$

WHAT WE KNOW $d_{\lambda,\lambda} = 1$ AND $d_{\lambda,\mu} = 0$ UNLESS $\mu \preceq \lambda$.

IT IS A UPPER TRIANGULAR MATRIX INDEXED BY PAIRS OF WEIGHTS SO IT IS INVERTIBLE.

$$\operatorname{ch} L(\lambda) = \sum_{\substack{\|\mu+\rho\|^2 = \|\lambda+\rho\|^2 \\ \mu \preceq \lambda}} c_{\lambda,\mu} \operatorname{ch} M(\mu)$$

$$\sum_{\substack{\|\mu+\rho\|^2 = \|\lambda+\rho\|^2 \\ \mu \preceq \lambda}} c_{\lambda,\mu} e^{\mu} \prod_{\alpha \in \Delta^+} (1 - e^{-\alpha})^{-1}$$

WANT $\sum_{\omega \in W} (-1)^{\ell(\omega)} e^{\omega(\lambda+\rho)} \underbrace{e^{-\rho} \prod (1-e^{-\alpha})^{-1}}_{\Delta^{\cdot}}$

WE NEED TO KNOW $c_{\lambda,\mu} = 0$ UNLESS
$\mu = \omega \cdot \lambda = \omega(\lambda+\rho) - \rho$ AND
$c_{\lambda, \omega\cdot\lambda} = (-1)^{\ell(\omega)}$

IF WE KNOW THIS

$$\mathrm{ch}\, L(\lambda) = \sum_{\omega \in W} (-1)^{\ell(\omega)} e^{\omega(\lambda+\rho)} \Delta^{-1}$$

AS REQUIRED.

$$\mu = \omega(\lambda+\rho) - \rho$$

WE RECALL FROM LECTURE 5 THAT
$L(\lambda)$ IS INTEGRABLE SO $\Delta_i\, \mathrm{ch}\, L(\lambda) = \mathrm{ch}\, L(\lambda)$.

$$\Delta_i \prod_{\alpha \in \Delta^+} (1-e^{-\alpha}) =$$

$$(1-e^{\alpha_i}) \prod_{\substack{\alpha \in \Delta^+ \\ \alpha \neq \alpha_i}} (1-e^{-\alpha}) =$$

$$e^{\alpha_i} \cdot (-1) \prod_{\alpha \in \Delta^+} (1 - e^{-\alpha_i}).$$

REMEMBERING $\Delta_i(\rho) = \rho - \alpha_i$ IT FOLLOWS

$$\Delta_i \Delta = \Delta_i \left( e^{\rho} \prod (1 - e^{-\alpha_i}) \right) = -e^{\rho} \prod (1 - e^{-\alpha_i})$$

$$\underbrace{e^{\rho - \alpha_i} (e^{\alpha_i} - 1)}_{-(1 - e^{-\alpha_i})} \times \prod_{\text{OTHERS.}}$$

$$\Delta_i \Delta = -\Delta$$

$$w(\Delta) = (-1)^{\ell(w)} \Delta.$$

$$\operatorname{ch} L(\lambda) = \sum_{\substack{\|\mu+\rho\|^2 = \|\lambda+\rho\|^2 \\ \mu \preceq \lambda}} c_{\lambda,\mu} \operatorname{ch} M(\mu)$$

$$\boxed{\sum_{\substack{\|\mu+\rho\|^2 = \|\lambda+\rho\|^2 \\ \mu \preceq \lambda}} c_{\lambda,\mu} e^{\mu} \prod_{\alpha \in \Delta^+} (1 - e^{-\alpha})^{-1}}$$

SINCE $\omega \operatorname{ch} L(\lambda) = \operatorname{ch} L(\lambda)$ AND

$$\operatorname{ch} L(\lambda) = \sum c_{\lambda,\mu} e^{\mu+\rho} \Delta^{-1} \checkmark$$

$$= \sum c_{\lambda,\mu} e^{\omega(\lambda+\rho)} (-1)^{\ell(\omega)} \Delta^{-1}$$

TO BE EQUAL

$$= \sum c_{\lambda,\omega\cdot\rho} e^{\omega\cdot\rho+\rho} \Delta^{-1}$$

MUST HAVE

$$c_{\lambda,\omega\cdot\lambda} = (-1)^{\ell(\omega)} c_{\lambda,\mu}$$

NEED TO SHOW $C_{\lambda,\mu} = 0$ UNLESS $\mu = \omega \cdot \lambda$ FOR SOME $\omega$. FIND $\omega$ SUCH THAT $\omega(\mu + \rho)$ IS DOMINANT

TO BE CONTINUED ...