ACTION OF TRANSLATIONS ON LEVEL k WEIGHTS.

MACDONALD IDENTITIES.

24

$$q^{1/24} \prod (1-q^n) = q^{1/24} \phi(q) = \eta(\tau)$$

DEDEKIND

EULER

$q = e^{2\pi i \tau}$ A MODULAR FORM OF WEIGHT 1/2

$$\eta\left(\frac{az+b}{cz+d}\right) = (*)\,(cz+d)^{1/2}\,\eta(z)$$

$z \in \mathcal{H} = \{x+iy \mid y>0\}$ SO $|q|<1$

AND $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{Z})$ $(*) =$ SOME 24 ROOT OF UNITY.

WHY $q^{1/24}$?

ANOTHER EXAMPLE: IF $\mathfrak{g}$ IS A F.D. SIMPLE L.A. $(\alpha|\alpha) = 1$ α LONG ROOT

FREUDENTHAL, DE VRIES

$$\frac{\operatorname{Dim}(\mathfrak{g})}{24} = \frac{|\rho|^2}{2h^\vee}$$

$h^\vee$ = DUAL COXETER NUMBER.

A SORT OF EXPLANATION FOR WHY 1/24: DEDUCE FROM JTP (LECTURE 0)

$$q^{1/24}\prod(1-q^n) = \sum_{-\infty}^{\infty}(-1)^n q^{(6n+1)^2/24}$$

THE FACTOR $\frac{1}{24}$ IS NEEDED TO COMPLETE THE SQUARE.

SOON WE WILL SEE SIMILAR QUADRATIC EXPRESSIONS AND RATIONAL POWERS OF q SUCH AS $q^{1/24}$ WILL APPEAR AND THESE CAN BE UNDERSTOOD IN A SIMILAR WAY.

CONSIDER ACTION OF TRANSLATIONS ON LEVEL h SPACE WE WANT A FORMULA FOR t_α. LET $\alpha \in \mathfrak{h}^*$

($\mathfrak{g}$ = LIE ALGEBRA OF F.D. SIMPLE $\mathfrak{g}$
$\hat{\mathfrak{g}}$ = AFFINE LA.)

$\mathfrak{h}^* \to \hat{\mathfrak{h}}^*$ EXTEND LINEAR FNL BY 0 ON K, d

$$\hat{\mathfrak{h}} = \underbrace{\mathfrak{h} \oplus \mathbb{C} \cdot K}_{\hat{\mathfrak{h}}'} \oplus \underset{\substack{\uparrow \\ \text{DERIVATION}}}{\mathbb{C} \cdot d}$$

K IS CENTRAL IN $\hat{\mathfrak{g}}'$ (DERIVED LIE ALGEBRA.

IF $\lambda \in \hat{\mathfrak{h}}^*$ THE LEVEL IS $\langle K, \lambda \rangle = k$

UNCHANGED WEYL GROUP.

Λ_0 FUNDAMENTAL WEIGHT.

$$\langle \alpha_i^\vee, \Lambda_0 \rangle = \delta_{i0} \qquad \langle d, \Lambda_0 \rangle = 0$$

$$\underset{\substack{\uparrow \\ \text{LEVEL } k}}{\mathfrak{h}_k^*} = \mathfrak{h}_0^* + k\Lambda_0 \qquad \mathfrak{h}_0^* \ni \mathbb{C}\delta$$

$$\delta = \sum_{i=0}^{r} a_i^\vee \alpha_i^\vee = \alpha_0^\vee + \sum_{i=1}^{r} a_i^\vee \alpha_i^\vee$$

NULL ROOT, THE "FIRST" IMAGINARY ROOT.

$w(\delta) = \delta$ FOR $w \in W$.

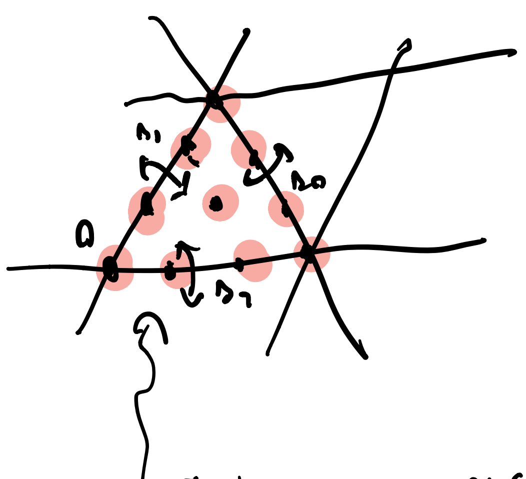
$\mathfrak{h}$ = L.A. OF $\mathfrak{g}$

$\downarrow$

$$\begin{tikzcd}
\mathfrak{h}^* \arrow[r, "\lambda + k\Lambda_0"'] \arrow[d, "w^{(k)}"] \arrow[dr, phantom, "\text{USUAL ACTION}"] & \hat{\mathfrak{h}}^*_k \arrow[r] \arrow[d, "w"] & \hat{\mathfrak{h}}^*_k / \mathbb{C}\delta . \arrow[d, "w"] \\
\mathfrak{h}^* \arrow[r] \arrow[rr, bend right, "\text{BIJECTION.}"'] & \mathfrak{h}^*_k \arrow[r] & \hat{\mathfrak{h}}^*_k / \mathbb{C}\delta
\end{tikzcd}$$

$\mathfrak{h}^* \subset \hat{\mathfrak{h}}^*$

WE SAW LAST TIME THE EFFECT OF w IN $W^{(k)}$ ACTION IS THE LEVEL k AFFINIZATION



A_2

FUNDAMENTAL ALCOVE IS A FUNDAMENTAL DOMAIN FOR $W^{(k)}$.

$W = \hat{W} \cdot Q^{(k)}$ (HERE $Q^{(k)}$ IS NORMAL. SDP.)

LET Q BE THE ROOT LATTICE ACTS ON

$\mathfrak{h}^*$ BY $\alpha : x \to x + k\alpha$.

$\hat{Q}$

W^0 = WEYL GROUP OF $\mathfrak{g}$ (F.D.)

UNDERSTAND HOW Q ACTS NOT JUST ON

$$\hat{\mathfrak{h}}^*_k / \mathbb{C}\delta$$

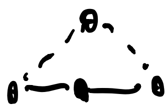
BUT MORE PRECISELY ON $\hat{\mathfrak{h}}^*_k$.

THIS ACTION CAN BE THOUGHT OF AS THE RESTRICTION TO $Q \subset \mathfrak{h}^*$ OF THE ACTION BY TRANSLATIONS BY $\alpha \in \mathfrak{h}^*$ (NOT JUST IN Q.)

WEYL GROUP W PRESERVES $(\mid)$ ON $\hat{\mathfrak{h}}^*$ = $r+2$ DIM'L SPACE.

THIS Q.F. HAS SIGNATURE $(r+1, 1)$

FOR $A_3^{(1)}$ CARTAN MATRIX IS



$$\begin{pmatrix} 2 & -1 & 0 & -1 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ -1 & 0 & -1 & 2 \end{pmatrix}$$

THIS IS THE MATRIX $a_{ij} = (\alpha_i | \alpha_j)$

(SINCE SIMPLY-LACED).

$\alpha_0, \alpha_1, \dots, \alpha_r$ CAN BE COMPLETED TO

A BASIS OF $\hat{\mathfrak{h}}^*$ BY THROWING IN Λ_0

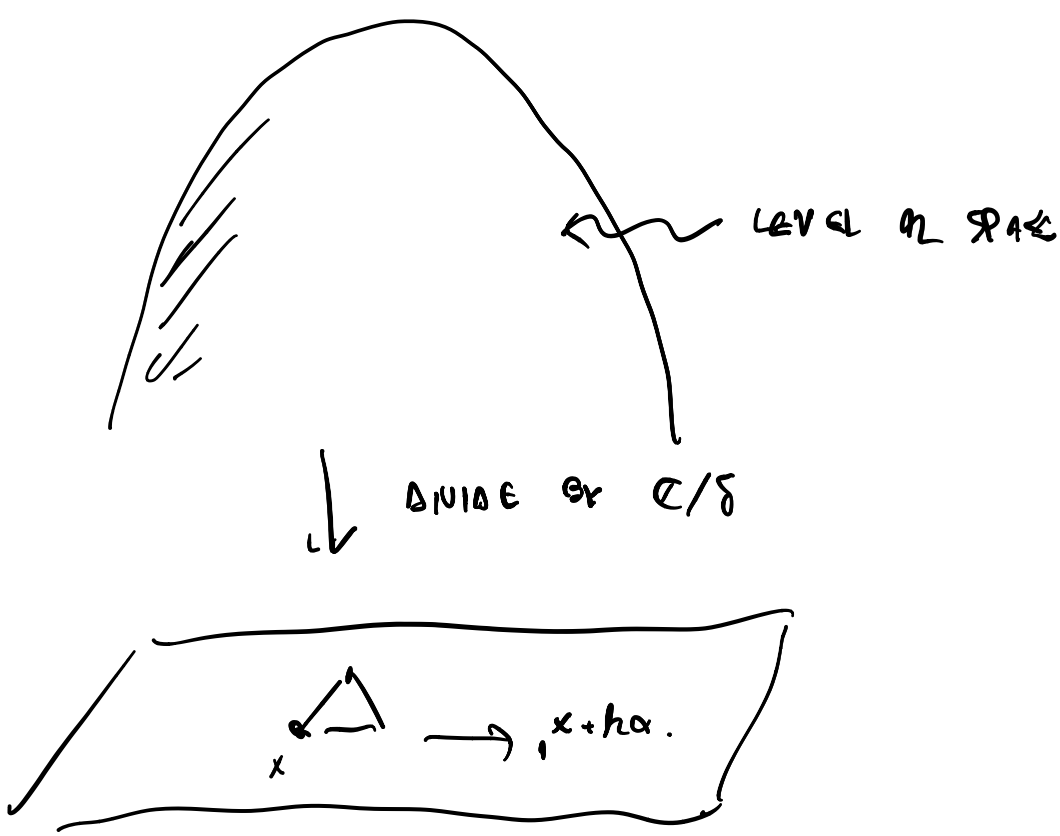
$$\begin{pmatrix} 2 & -1 & 0 & -1 & 1 \\ -1 & 2 & -1 & 0 & 1 \\ 0 & -1 & 2 & -1 & 1 \\ -1 & 0 & -1 & 2 & 1 \\ 1 & 1 & 1 & 1 & 0 \end{pmatrix}$$ SIGNATURE $(4,1)$.

$(\alpha_i | \Lambda_0) = 1$, $(\Lambda_0 | \Lambda_0) = 0$

SO $W \subset SO(r+1, 1)$

$$\begin{pmatrix} \mathfrak{so}(r+1) & \begin{matrix}*\\ \vdots \\ *\end{matrix} \\ * \cdots * & 0\end{pmatrix} \supset \begin{pmatrix} I_{(r+1)} & \begin{matrix}0\\ * \\ \vdots\\ *\end{matrix} \\ & 1\end{pmatrix}$$

UNIPOTENT SUBGROUP $\cong \mathbb{R}^r$
SHOULD ACT AS A GROUP OF "TRANSLATIONS"
ON $\hat{\mathfrak{h}}^*_{\mathbb{R}}/\mathbb{C}\delta$ WHICH WE WANT TO
IMPROVE TO AN ACTION ON $\hat{\mathfrak{g}}^*$



$\alpha \in \mathfrak{h}^*$

PROPOSITION: ACTION OF α ON $\hat{\mathfrak{h}}^*$

IS $\quad t_\alpha(\lambda) = \lambda + k\alpha - \left((\lambda|\alpha) + \frac{1}{2}k|\alpha|^2\right)\delta$

A PRIORI,

$$t_\alpha(\lambda) = \lambda + k\alpha - c\delta$$

WANT TO COMPUTE c.

AT LEAST IF $\alpha \in Q$ WE KNOW t_α SHOULD BE ORTHOGONAL;

$$|t_\alpha(\lambda)|^2 = |\lambda|^2$$

(THE ACTION WILL BE ORTHOGONAL FOR ANY $\alpha \in \mathfrak{h}^*$.)

$$0 = (\lambda + k\alpha - c\delta \mid \lambda + k\alpha - c\delta) - (\lambda|\lambda)$$

$(\delta|\delta) = 0$

$(\lambda|\delta) = 0$

δ IS ORTHOGONAL TO $\alpha_0, \alpha_1, \ldots, \alpha_r$.

$(\delta|\Lambda_0) = 1$.

$$0 = 2k(\lambda|\alpha) + k^2|\alpha|^2 + \cancel{c^2(\delta|\delta)} - 2c\underbrace{(\lambda|\delta)}_{k}$$
$$- \cancel{2kc(\alpha|\delta)}$$

$(\alpha|\delta) = 0$

$(\lambda|\delta) =$
$\langle \nu(\delta), \lambda\rangle = \langle K, \lambda\rangle = k$

DIVIDE BY $2k$

$$c = (\lambda|\alpha) + \tfrac{1}{2}k|\alpha|^2.$$

$\mathfrak{h}$

$$t_\alpha(\lambda) = \lambda + k\alpha - \left((\lambda|\alpha) + \tfrac{1}{2}k|\alpha|^2\right)\delta.$$

ACTION OF TRANSLATIONS BY Q ON $\mathfrak{h}^*$ ON LEVEL k SPACE.

$$\boxed{t_\alpha t_\beta = t_{\alpha+\beta}.}$$

MACDONALD FORMULAS (CH. 4 OF KAC)

IF $\lambda \in \hat{\mathfrak{h}}^*$ LET $\bar{\lambda} \in \mathfrak{h}^*$ BE ORTHOGONAL PROJECTION

$$\hat{\mathfrak{h}}^* = \mathfrak{h}^* \oplus (\mathbb{C}\Lambda_0 \oplus \mathbb{C}\delta)$$

ORTHOGONAL.

$(\Lambda_0|\Lambda_0) =$

$(\delta|\delta) = 0$

$(\Lambda_0|\delta) = 1.$

$$\lambda = \bar{\lambda} + b_1 \Lambda_0 + b_2 \delta$$

CAN COMPUTE b_1 BY TAKING PAIRING PRODUCT WITH K SO LET $k = \langle K, \lambda \rangle$

$= \langle K, \bar{\lambda} \rangle.$

$b_1 = k$. b_2 IS COMPUTED BY

$$|\lambda|^2 = |\bar{\lambda}|^2 + 2b_1 b_2 \quad \checkmark$$

$$b_2 = \frac{|\lambda|^2 - |\bar{\lambda}|^2}{2k}$$

$$\lambda = \bar{\lambda} + k\Lambda_0 + \frac{1}{2k}(|\lambda|^2 - |\bar{\lambda}|^2).$$

$$\langle \alpha_i^\vee, \rho \rangle = 1 \qquad \langle d, \rho \rangle = 0$$

$\rho = \bar{\rho} + h^\vee \Lambda_0$ (THERE IS NO δ CONTRIBUTION)

$$\sum (-1)^{\ell(w)} e^{w(\rho) - \rho} = \prod_{\alpha \in \Delta^+} (1 - e^{-\alpha})^{\mathrm{mult}(\alpha)}$$

$$= \prod_{\alpha \in \Delta_+^0} (1 - e^{\alpha}) \prod_{n=1}^{\infty} (1 - q^n)^r \prod_{\alpha \in \Delta_+^0} (1 - q^n e^{-\alpha})$$

$$q = e^{-\delta}$$

TO ANALYZE $e^{w(\rho) - \rho}$ WRITE

$$w = u t_\alpha$$

$\alpha \in Q$ $\quad u \in \mathring{W}$ $\qquad \bar{\rho}$ = RHO FOR $\mathfrak{g}$

ρ = RHO FOR $\hat{\mathfrak{g}}$.

$$u(\rho) - \rho$$

USING $\rho = \bar{\rho} + \hat{h} \Lambda_0$

$\Lambda_0 = u(\Lambda_0)$ BECAUSE $s_i(\Lambda_0) = \Lambda_0 - \langle \alpha_i^\vee, \Lambda_0 \rangle \alpha_0 = \Lambda_0$

ZERO IF $i \neq 0$

$$u t_\alpha(\rho) - \rho = u(\bar{\rho} + h^\vee \alpha) - \bar{\rho}$$

$$+ \frac{1}{2h^\vee} \left(|\bar{\rho}|^2 - |\bar{\rho} + h^\vee \alpha|^2 \right) \delta$$

FROM PREVIOUS FORMULAS. $\qquad q = e^{-\delta}$

KAC (12.1.4) OBTAINS MACDONALD IDENTITIES

IN FORM,

$$q^{|\bar{\rho}|^2/2h^\vee} \boxed{\prod_{\alpha\in\mathring{\Delta}^+} e^{\bar{\rho}}(1-e^{-\alpha})} \prod_{n=1}^{\infty}(1-q^n)^{\ell} \prod_{\alpha\in\mathring{\Delta}}(1-e^{-\alpha}q^n)$$

$$= \boxed{\sum_{\alpha\in Q}\left(\sum_{w\in\mathring{W}}(-1)^{\ell(w)}\, e^{w(\bar{\rho}+h^\vee\alpha)-\bar{\rho}}\right)} q^{|\bar{\rho}+h^\vee\alpha|^2/2h^\vee}.$$

PORTION IN RED BOX IS $\pm$ THE CHARACTER OF IRR REP'N OF $\mathfrak{g}$ WITH HIGHEST WEIGHT $w\cdot(h^\vee\alpha)$ WHERE w IS CHOSEN TO MAKE $h^\vee\alpha$ DOMINANT.

STRANGE FORMULA

TO REWRITE ANOTHER WAY.