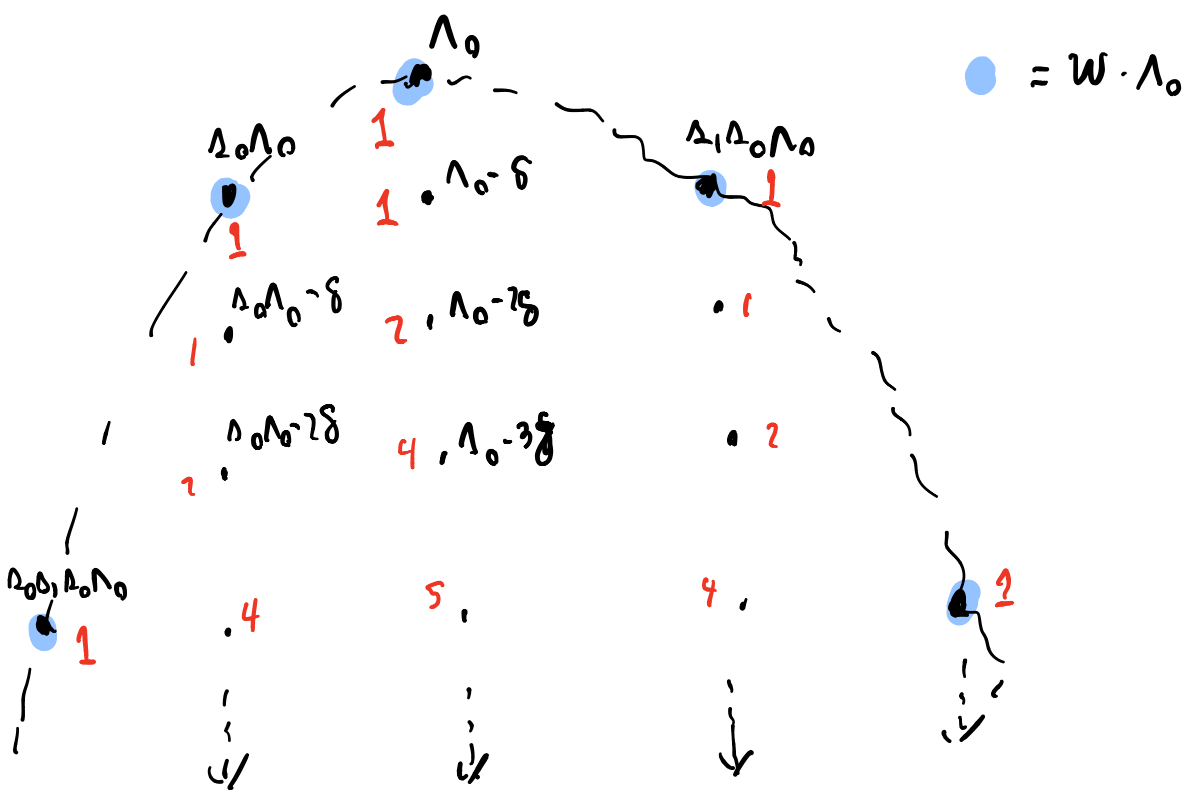
EXAMPLE FROM LAST THURSDAY

$\mathfrak{g} = \widehat{\mathfrak{sl}}(2)$ CARTAN TYPE $A_1^{(1)}$

$V = L(\Lambda_0)$ Λ_n : n-TH FUNDAMENTAL WEIGHT.



$-\delta$ DIRECTION.

t	0	1	2	3
$p(t)$	1	1	2	4

3
2, 1
1, 2
3

NOTICE IF $m(t) = \mathrm{MULT}(\mu - t\delta)$ FOR ANY WEIGHT μ IS MONOTONE INCREASING

AND 0 FOR $t \ll 0$.

<u>THEOREM</u>: IF Λ IS DOMINANT WEIGHT FOR AFFINE LIE ALGEBRA $\mathfrak{g}$, μ A WEIGHT OF $L(\Lambda) = V$ THEN DIM $V_{\mu - t\delta}$ IS A MONOTONE FUNCTION OF t.

PROOF:

$$\mathrm{ch}\, L(\Lambda) = \Delta^{-1} \sum_{w \in W} \mathrm{sgn}(w)\, e^{w(\Lambda+\rho)-\rho}$$

$$\Delta = \prod_{\alpha \in \Delta^+} (1 - e^{-\alpha})^{\mathrm{mult}(\alpha)} \qquad \begin{aligned} \mathrm{mult}(\alpha) &= r \text{ IF } \alpha = n\delta \\ &1 \text{ IF } \alpha \text{ REAL.} \end{aligned}$$

DIVIDE Δ INTO REAL AND IMAGINARY CONTRIBUTIONS:

$$\Delta = \prod_{n=1}^{\infty} (1 - e^{-n\delta})^r \prod_{\text{POS. REAL ROOTS}} (1 - e^{-\alpha})$$

$\mathfrak{g}$ IS OF TYPE $X_r^{(1)}$ UNTWISTED AFFINE

$X = A, D, E$ BECAUSE

WE WANTED TO SIMPLIFY THINGS AND CONSIDER SYMMETRIZABLE UNTWISTED CASE.

$$\mathrm{CH}\ L(\Lambda) = \prod (1-e^{-n\delta})^{-r} \times$$

$$\prod_{\alpha \in \Delta^+_{\mathrm{REAL}}} (1-e^{-\alpha})^{-1} \sum_{w \in W} e^{w(\Lambda+\rho)-\rho}$$

$$\sum_{t=0}^{\infty} c(t) e^{-t\delta} \cdot \sum_{\mu} d(\mu) e^{\mu} \cdot$$

$$\uparrow$$

$$\binom{r+1+t}{t}$$

OR SOMETHING.

BOTH $c(t)$ AND $d(\mu)$ OBVIOUSLY POSITIVE FROM $(1-x)^{-1} = (1+x+x^2+\ldots)$

AND $c(t)$ IS MONOTONE INCREASING.

COEFFICIENT

$$\mathrm{DIM}\ V_\mu = \sum_{t=0}^{\infty} c(t)\, d(\mu - \delta t)$$

FROM THIS $\mathrm{DIM}\ V_{\mu-\delta} = \sum c(t+1)\, d(\mu - \delta t)$

$$\geq \mathrm{DIM}\ V_\mu .$$

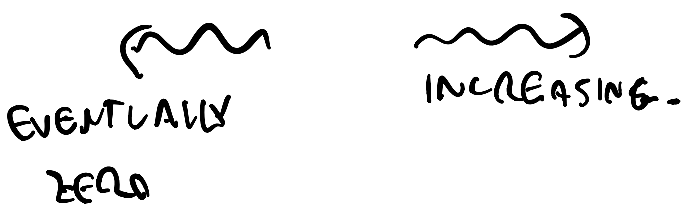
AND VANISHES IF $t \ll 0$.

THIS MOTIVATES THE INTRODUCTION OF STRING FUNCTIONS.

IN THE SEQUENCE

$$m(\mu) = \dim V_\mu$$

$$\cdots \quad m(\mu+\delta) \quad m(\mu) \quad m(\mu-\delta) \quad m(\mu-2\delta) \quad \cdots$$

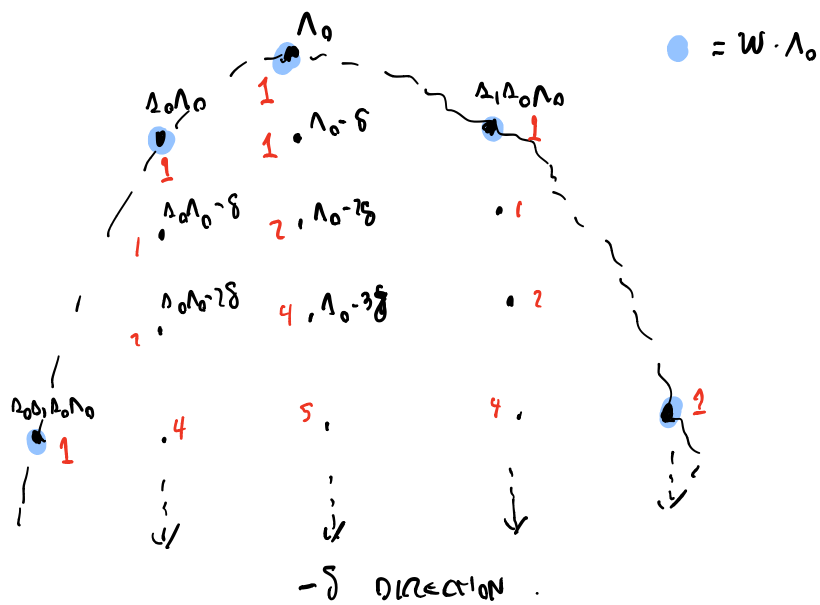


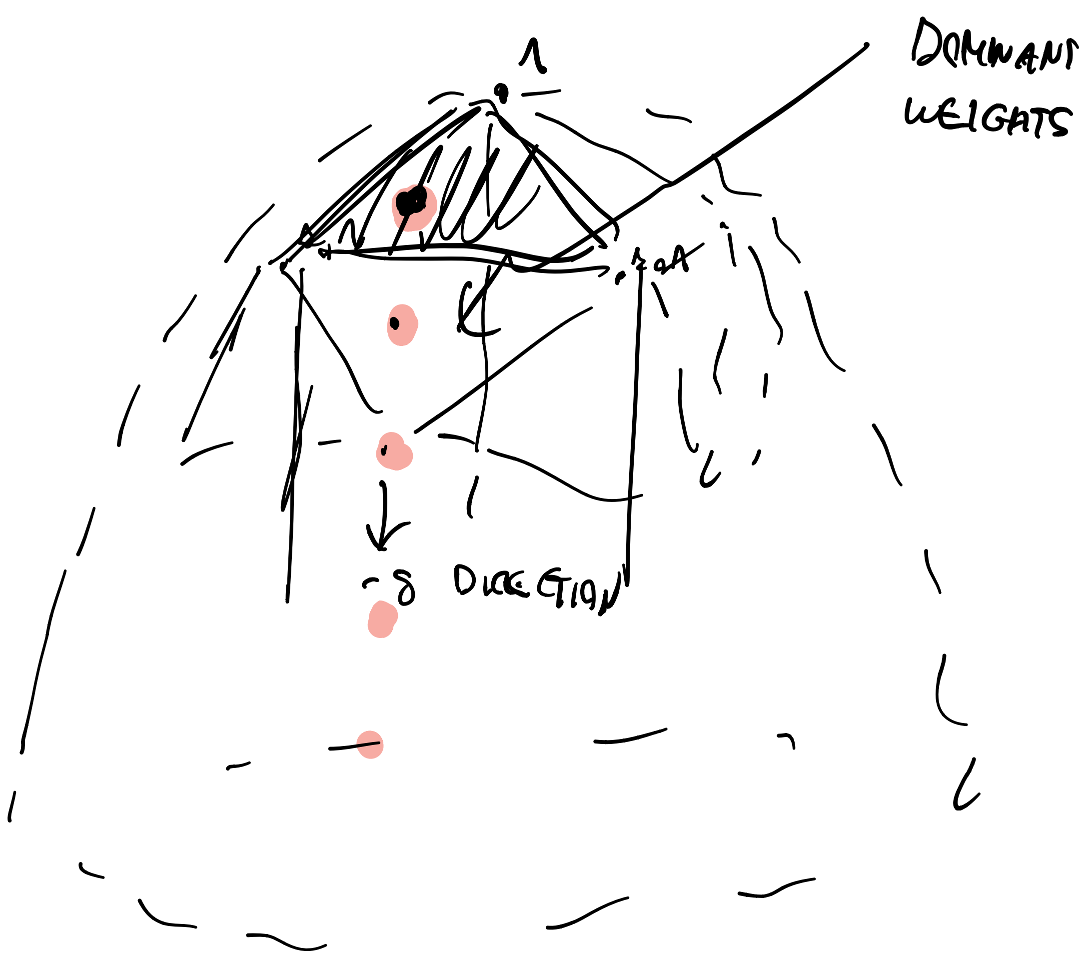
THERE IS A FIRST t SUCH THAT

$$m(\mu - t\delta) \neq 0.$$

IF $m(\mu - t\delta) \neq 0$ BUT $m(\mu-(t-1)\delta) = 0$ SAY μ IS MAXIMAL. IN EXAMPLE THE MAXIMAL WEIGHTS ARE

$$w(\Lambda_0) = \ldots, s_0\Lambda_0, \Lambda_0, s_1 s_0 \Lambda_0, \ldots$$





MAXIMAL WEIGHTS WILL LIE NEAR THE

PARABOLOID $\{\mu \mid \|\mu\|^2 = \|\Lambda\|^2\}$

(INDEFINITE METRIC).

THE WEYL ORBIT $W \cdot \Lambda$ WILL LIE ON THIS PARABOLOID, THERE MAY BE A FINITE NUMBER OF OTHER ORBITS.

EVERY W ORBIT INTERSECTS THE DOMINANT WEIGHTS $\{\mu \mid \langle \alpha_i^\vee, \mu \rangle \geq 0,\ i = 0, \ldots, r\}$.

THERE ARE FINITELY MANY DOMINANT MAXIMAL WEIGHTS.

AND THE ROOT STRING

$$\sum_{t=0}^{\infty} m(\mu - t\delta)\, q^t$$

IS ASSOCIATED WITH THIS MAXIMAL DOMINANT WEIGHT.

CENTRAL ELEMENT

$$K = \sum_{i=0}^{r} a_i^\vee \alpha_i^\vee$$

WEYL GROUP ACTION

IF μ IS ANY WEIGHT $k = \langle K, \mu \rangle$

IS CALLED THE LEVEL. EXPLAINED LAST TIME HOW TO VISUALIZE THE W-ACTION ON LEVEL k WEIGHTS. SINCE

$$\text{LEVEL } w(\mu) = \text{LEVEL}(\mu)$$

$$\langle K, w(\mu) \rangle = \langle K, \mu \rangle$$

CHECK FOR $w = s_i$ $(i = 0, 1, \dots, r)$.

$$\langle K, s_i(\mu) \rangle = \langle K, \mu - \langle \alpha_i^\vee, \mu \rangle \alpha_i \rangle$$

$$= \langle K, \mu \rangle \text{ SINCE}$$

$\alpha_i(K) = 0$. (DUE TO THE FACT K IS CENTRAL IN $\mathfrak{g}' = \langle e_i, f_i \rangle$.

SO LEVEL k PART OF $\hat{\mathfrak{h}}^*$ IS INVARIANT UNDER W. CALL IT $\hat{\mathfrak{h}}^*_k$.

$$\hat{\mathfrak{h}}^*_k = \hat{\mathfrak{h}}^*_0 + k\Lambda_0$$

Λ_0 IS THE AFFINE FUNDAMENTAL WEIGHT.

$\alpha \mapsto \alpha + k\Lambda_0$ ← COSET OF $\hat{\mathfrak{h}}_0^* \supset \mathbb{C}\delta$

$$\begin{array}{ccccc}
\mathfrak{h}^* & \longrightarrow & \hat{\mathfrak{h}}_k^* & \longrightarrow & \hat{\mathfrak{h}}_k^*/\mathbb{C}\delta \\
\downarrow w^{(k)} & & \downarrow w & & \downarrow w \\
\mathfrak{h}^* & \longrightarrow & \hat{\mathfrak{h}}_k^* & \longrightarrow & \hat{\mathfrak{h}}_k^*/\mathbb{C}\delta
\end{array}$$

IF $\lambda \in \mathfrak{h}^*$ EXTEND TO $\hat{\mathfrak{h}}^*$ BY ZERO ON K, d.

Λ_0 HAS LEVEL 1.

ρ HAS LEVEL $h^\vee = \sum_{i=0}^{r} a_i^\vee$.

Λ_i HAS LEVEL $a_i^\vee$ SO

$\rho = \sum_{i=0}^{r} \Lambda_i$ HAS LEVEL $h^\vee$.

$h^\vee$ IS CALLED THE DUAL COXETER NUMBER.

$w(\delta) = \delta$ FOR $w \in W$.

$\delta \in \hat{\mathfrak{h}}_0^*$ (IN FACT ALL ROOTS HAVE LEVEL 0.)

WE HAVE AN INDUCED WEYL ACTION ON

$\hat{\mathfrak{h}}^*_k$ WHICH WE DISCUSSED LAST TIME.

$s_i^{(k)} = s_i$ ON $\mathfrak{h}^*$ FOR $i = 1, \dots, r$.

$s_0^{(k)}$ IS MODIFIED.

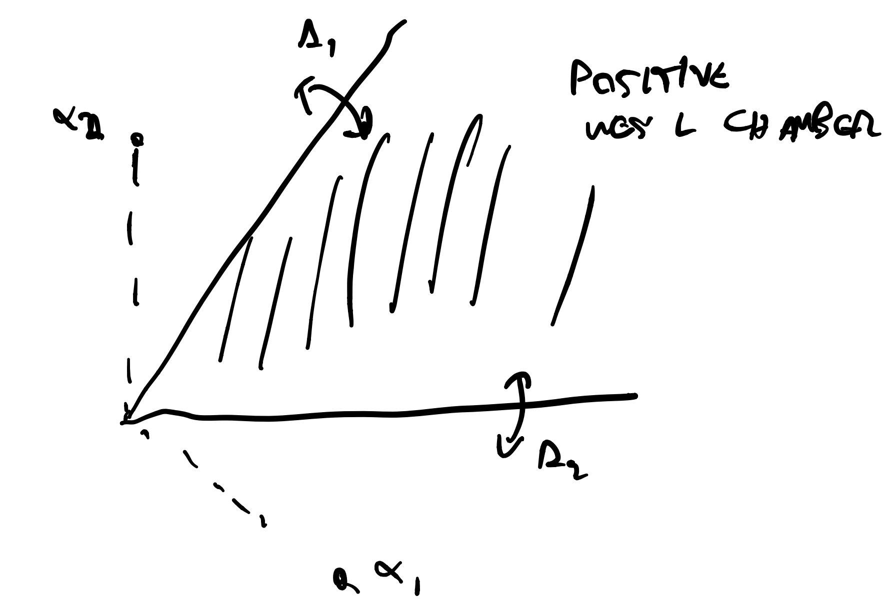
$$s_0^{(k)}(x) = r_\theta(x) + k\theta$$

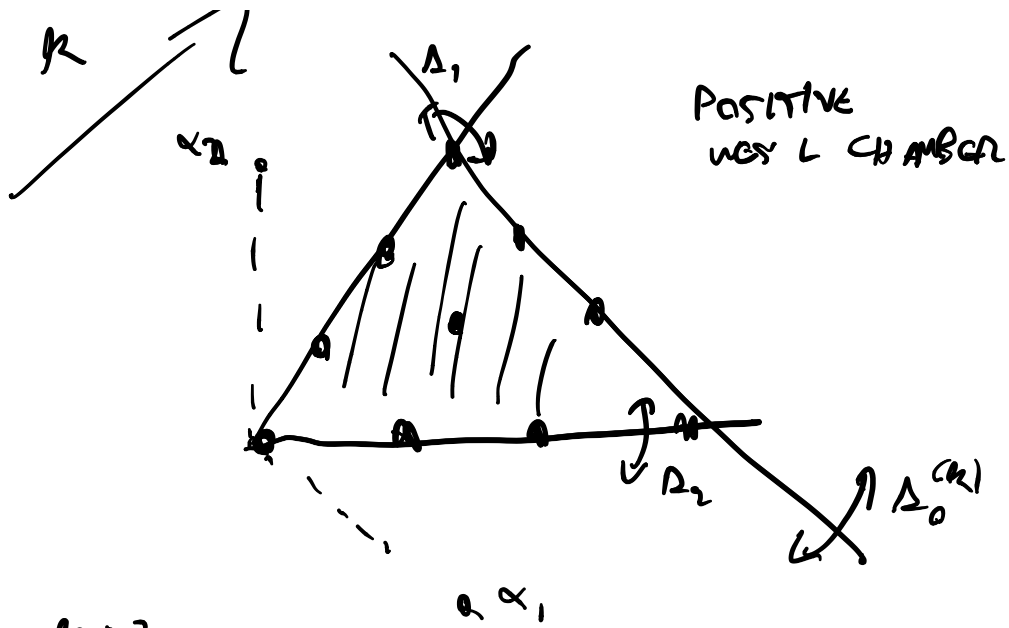
θ = LONGEST ROOT FOR $\mathring{\mathfrak{g}}$ ← FINITE DIM'L LIE ALGEBRA

r_θ = REFLECTION ORTHOGONAL TO θ

$$r_\theta(x) = x - \langle \theta^\vee, x \rangle \theta .$$

$\widehat{sl}(3)$





$k = 3$

LEVEL k

FUNDAMENTAL ALCOVE.

$$\mathfrak{J}_k = \{ x \in \mathfrak{h}^* \mid (\alpha_i \mid x) \geq 0 , \ (\theta \mid x) \leq k \} .$$

IS A FUNDAMENTAL DOMAIN FOR THIS ACTION. $W = \langle \Delta_0, \Delta_1, \cdots, \Delta_r \rangle$.

THEOREM: W IS THE SEMIDIRECT PRODUCT

$$W = \mathring{W} \cdot Q^{(k)}$$

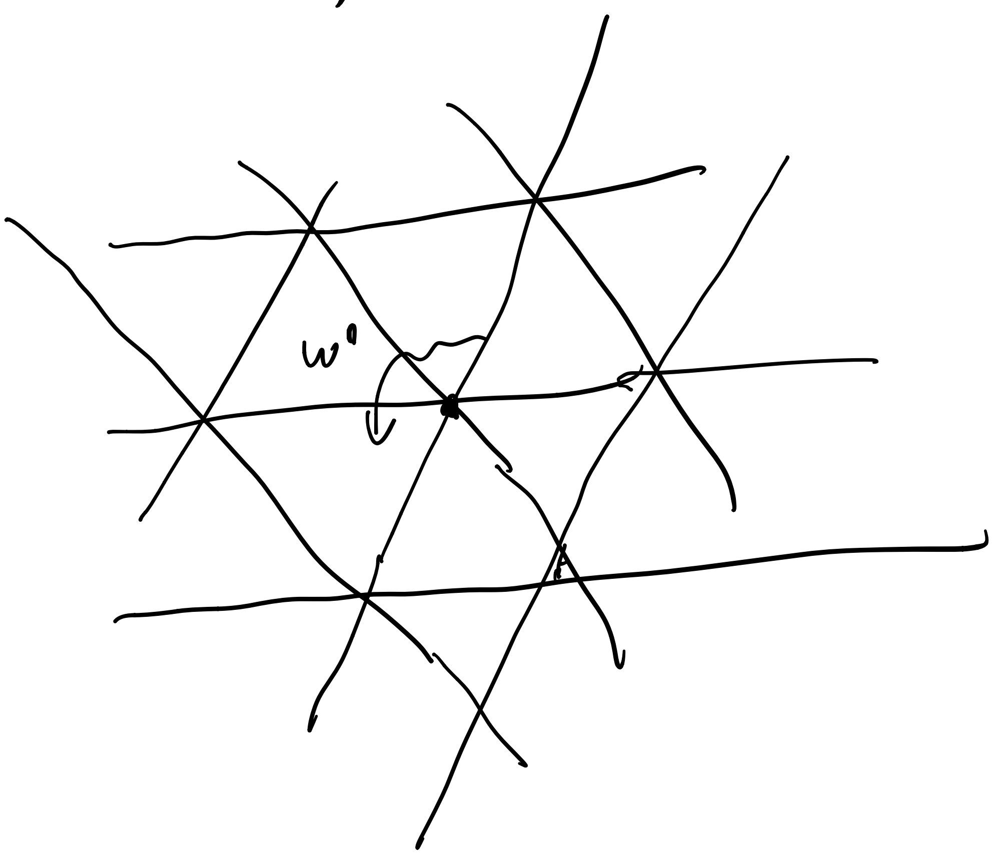
$\mathring{W}$: FINITE WEYL GROUP

$Q^{(k)}$ = GROUP OF TRANSLATIONS BY ELEMENTS OF kQ Q = ROOT LATTICE.

ALSO $W = \langle \Delta_0, \Delta_1, \cdots, \Delta_r \rangle$

IT IS CLEAR THAT THE GROUPS $\hat{W}$, $Q^{(k)}$ PRESERVE THE FAMILY OF HYPERPLANES $\{(\alpha | x) = m\} = H_{\alpha, m}$

WHERE $k | m$, $m \in \mathbb{Z}$.



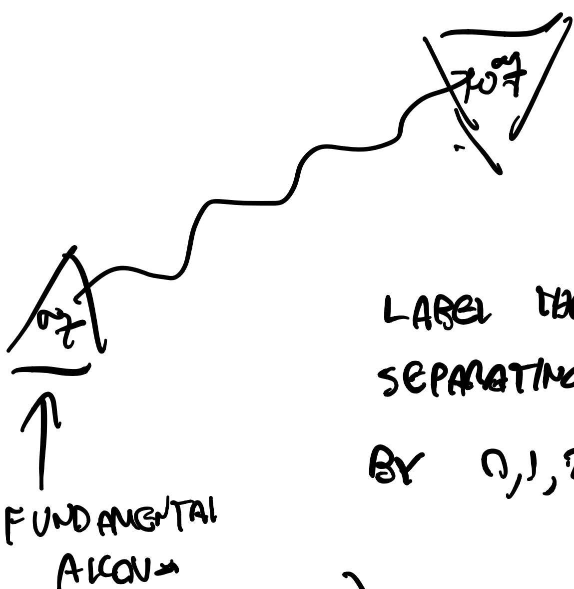
BECAUSE OBVIOUS. WE ALLOW TRANSFORMATIONS IN $\hat{W}$, TRANSLATIONS IN kQ.

LET US NOTE $\Delta_0, \Delta_1, \cdots, \Delta_r$ LIE IN THIS GROUP $\mathring{W} \cdot \Omega^{(h)}$

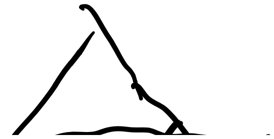
$\Delta_1, \cdots, \Delta_r \in W^a$ AND $\Delta_0 = \gamma_\theta \cdot$ TRANSLATION BY $-h\theta$.

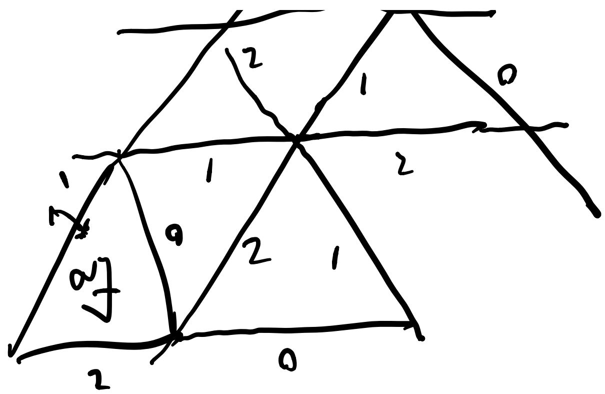
$\langle \Delta_0, \Delta_1, \cdots, \Delta_r \rangle \subset W^a \cdot \Omega^{(h)}$

FOR OTHER DIRECTION IF w IS ANY TRANSFORMATION STABILIZING THIS CONFIGURATION OF HYPERPLANES WE CAN ARGUE $w \in \langle \Delta_0, \Delta_1, \cdots, \Delta_r \rangle$ AS FOLLOWS.



LABEL THE HYPERPLANES SEPARATING a_f AND wa_f BY $0, 1, 2, \cdots, r$.

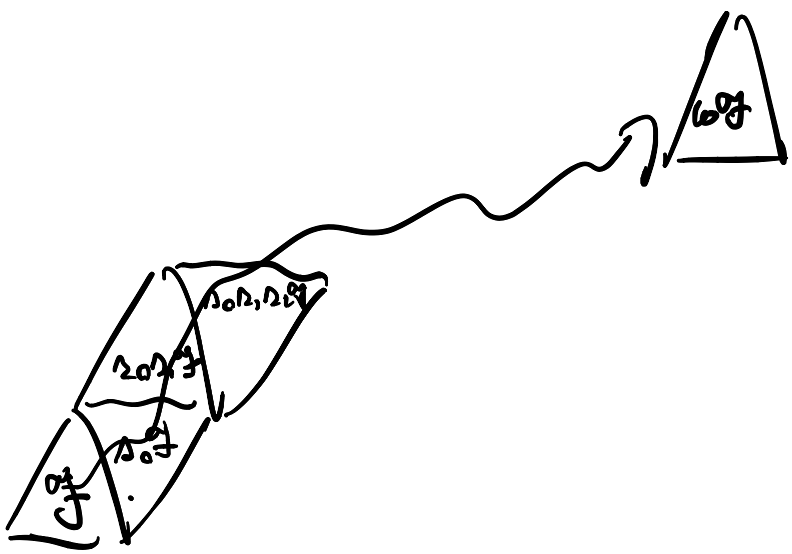




STARTING WITH $\mathfrak{a}_f$, BOUNDING HYPERPLANE IS REFLECTION IN s_i AND LABEL IT BY i

AND ASK THE LABELS BE INVARIANT UNDER W.

CHOOSE A PATH $\mathfrak{a}_f \rightsquigarrow w\mathfrak{a}_f$.



READING OFF THE LABELS EXPRESSES

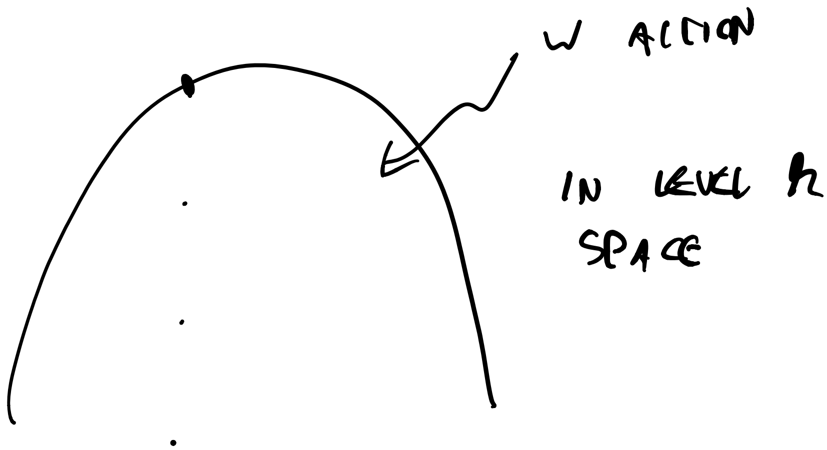
$$w\mathfrak{a}_f = s_{i_1} s_{i_2} \cdots s_{i_m} \mathfrak{a}_f \quad \Rightarrow$$

$$w = s_{i_1} \cdots s_{i_m} \quad \text{so}$$

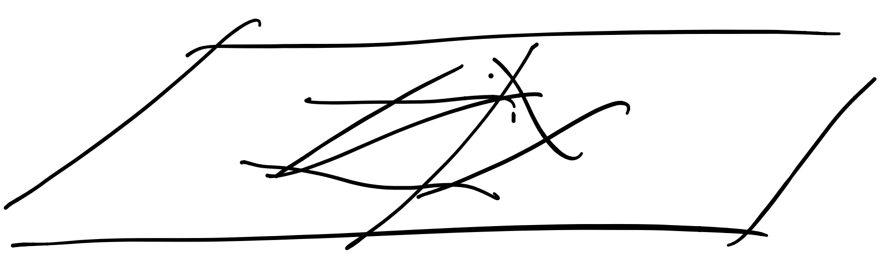
CONCLUSION: THE GROUP PRESERVING ALL THESE HYPERPLANES IS

$$\langle s_0, s_1, \dots, s_r \rangle.$$

THIS IS $W^{(k)}$ LEVEL k AFFINE WEYL GROUP.



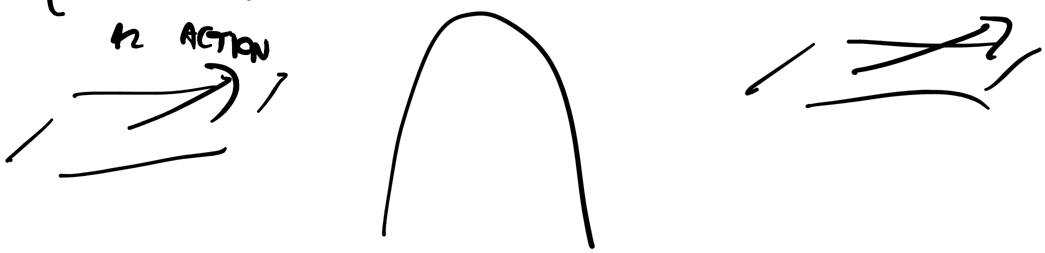
$\downarrow$ DIVIDE BY $\mathbb{C}\delta$.



HOW DO THE TRANSLATIONS LIFT FROM

$$\mathfrak{g}^* \longrightarrow \hat{\mathfrak{g}}^*_{(n)} \longrightarrow \hat{\mathfrak{g}}^*_{(n)}$$

(WITH LEVEL
n ACTION



t_α : TRANSLATION IN $n\alpha \in nQ$

$$t_\alpha(x) = x + \underbrace{\langle K, x\rangle}_{n}\alpha + (?)\delta$$

ANSWER:

$$t_\alpha(x) = x + \langle K, x\rangle\alpha - \Big(\overbrace{(x|\alpha) + \tfrac{1}{2}|\alpha|^2}^{\xi}\Big)\delta$$

TO CHECK THIS WE USE

$$\|t_\alpha(x)\|^2 = \|x\|^2$$

I AM OUT OF TIME SO WE'LL CHECK THIS NEXT TIME.

CH 12, 13 of KAC.