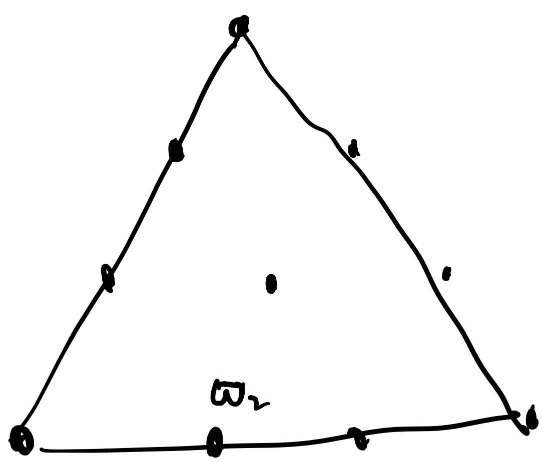
LET $\mathfrak{g}$ BE A SIMPLE LIE ALGEBRA, SIMPLY LACED $\hat{\mathfrak{g}}$ = UNTWISTED AFFINE LIE ALGEBRA.

DOMINANT WEIGHTS OF $\tilde{\mathfrak{g}}$ OF LEVEL k ARE IN BIJECTION WITH THE WEIGHTS OF $\mathfrak{g}$ IN THE FUNDAMENTAL ALCOVE OF LEVEL k.

$$\left\{ x \in \hat{\mathfrak{h}}^* \;\middle|\; \begin{array}{l} (\alpha_i \mid x) \geq 0 \quad i = 1, \dots, r \\ (\theta \mid x) \leq k \end{array} \right\}$$



$k = 2$

$\mathfrak{g} = sl(3)$

DOMINANT WEIGHTS OF $\mathfrak{g}$ INSIDE THIS ALCOVE.

FOR EXAMPLE
$\Lambda = \varpi_2 + 3\Lambda_0$ LEVEL 3 DOMINANT WEIGHT

OF $\hat{\mathfrak{g}}$. $L(\Lambda)$

IN THIS CASE THERE ARE 10 REPS

OF THIS TYPE.

THERE IS A BILINEAR OPERATION "FUSION"

ON THE IRREDUCIBLES

$L(\Lambda)$, $\Lambda \in$ LEVEL k DOMINANT.

IN A PREVIOUS TALK, I SHOWED YOU A

PAPER OF KAZHDAN-LUSZTIG WHERE THIS

FUSION OPERATION IS DESCRIBED.

THERE IS A CONVENIENT ALGORITHM THE

KAC-WALTON FORMULA FOR COMPUTING THIS.

THIS IS A VARIANT (RACAH-SPEISER, BRAUER-KLIMYK)

OF A TENSOR PRODUCT RULE FOR $\mathfrak{g}$.

IF λ, μ ARE DOMINANT WEIGHTS OF $\mathfrak{g}$.

$$L(\lambda) \otimes L(\nu) = \bigoplus_{\gamma} N^{\mu}_{\lambda\nu} L(\mu)$$

FINITE-DIMENSIONAL

PROBLEM: COMPUTE $N^{\gamma}_{\lambda\mu}$.

DECOMPOSE $L(\lambda)$ INTO WEIGHTS.

$$\mathrm{ch}\, L(\lambda) = \sum_{\mu} c_{\lambda,\mu} e^{\mu}$$

WEIGHT MULTIPLICITIES OR KOSTKA NUMBERS.

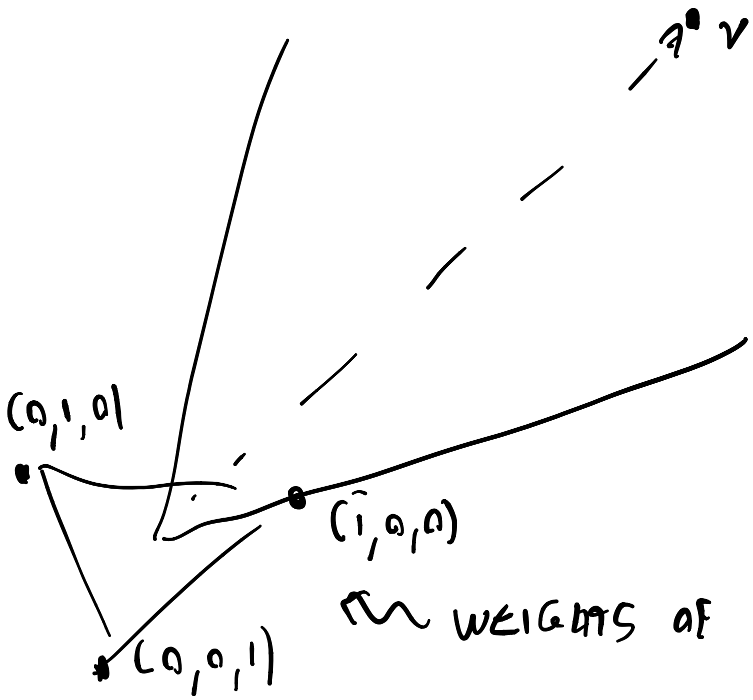
SUPPOSE γ IS SUCH THAT $\gamma + \mu$ IS DOMINANT FOR ALL μ SUCH THAT $c_{\lambda,\mu} \neq 0$.

THEN

$$\chi_{\lambda} \chi_{\nu} = \sum_{\mu} c_{\lambda,\mu} \chi_{\mu+\nu}$$

SO $N^{\mu}_{\lambda\mu} = c_{\lambda,\mu-\nu}$

$\lambda = (1,0,0)$ FOR $SL(3)$



~ WEIGHTS OF χ_λ.

$$\chi_\lambda \chi_\nu = \chi_{\nu+(1,0,0)} + \chi_{\nu+(0,1,0)} + \chi_{(0,0,1)}$$

PROOF IS EASY CONSEQUENCE OF WCF.

MORE GENERALLY EXTEND THE NOTATION

χ_ν TO INCLUDE NON-DOMINANT ν.

$$\chi_\nu = \Delta^{-1} \sum_{w \in W} (-1)^{\ell(w)} e^{w(\nu+\rho)}.$$

IF ν IS DOMINANT THIS THE CHAR

OF AN IRREDUCIBLE.

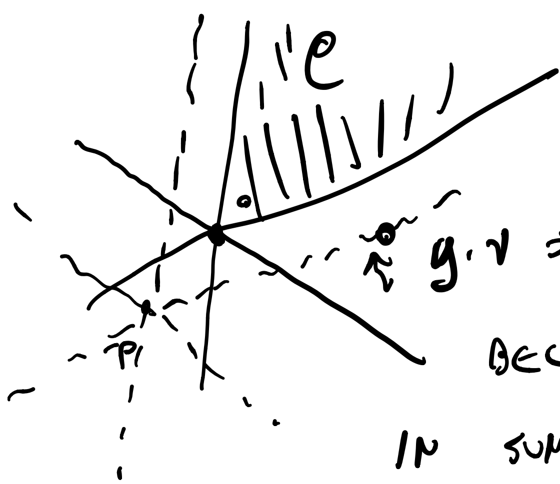
IF ν IS ARBITRARY FIND $y \in W$ SUCH THAT $y(\lambda+\rho)$ IS DOMINANT. IT MAY OR MAY NOT BE TRUE THAT

$$y \cdot w = y(\lambda+\rho)-\rho$$

IS DOMINANT. IF IT IS, A CHANGE OF VARIABLES SHOWS

$$\chi_\nu = (-1)^{\ell(w)} \chi_{w\cdot\nu}.$$

IF $y(\lambda+\rho)-\rho$ IS NOT DOMINANT, THEN $\chi_\nu = 0$. THIS HAPPENS WHEN ν LIES ON A TRANSLATE BY $-\rho$ OF A WALL OF THE POSITIVE W.C.



$g\cdot\nu \Rightarrow \chi_\nu = 0.$

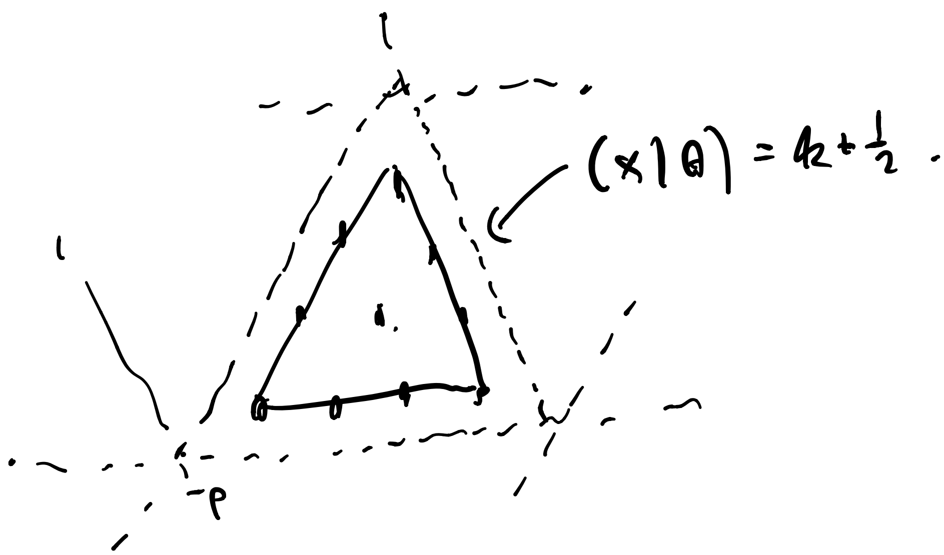
BECAUSE OF CANCELLATIONS IN SUM $\sum_{w\in W} (-1)^{\ell(w)} e^{w(\nu+\rho)}$.

WITH THIS MODIFICATION

$$\chi_\lambda \chi_\nu = \sum_\mu c_{\lambda,\mu} \cdot \chi_{\nu+\mu}$$

MIGHT BE NEG.
OF AN IRR OR
ZERO.

THE KAC-WALTON FORMULA FOR FUSION OF LEVEL k REPS OF $\hat{\mathfrak{g}}$ PARAMETRIZED BY REPS OF $\mathfrak{g}$ IN THE LEVEL k ALCOVE USES A VERSION OF DOT ACTION.



THE NEW DOT ACTION IS REFLECTION IN HYPERPLANES

$(x+\rho \mid \alpha) = n(k+h^\vee)$

$x \in$ ROOTS, $n \in \mathbb{Z}$, INCLUDES THE INDICATED HYPERPLANES. IF ν IS ON THE WALL OF ONE OF THE REFLECTING HYPERPLANES, INTERPRET

$$\chi_\nu = 0$$

OTHERWISE WE REFLECT IT INTO THE LEVEL k ALCOVE, MULTIPLY BY $(-1)^{\#\text{ of reflections}}$ AND INTERPRET THIS AS $\pm \chi_{w \cdot \nu}$

AFFINE DOT ACTION.

$$\chi_\lambda * \chi_\nu = \sum c_{\lambda,\mu} \chi_{w \cdot (\mu+\nu)}$$

FUSION

CONVENIENT ALGORITHM FOR COMPUTING FUSION. USE FusionRing CLASS

OF SAGE TO COMPUTE THESE DECOMPOSITIONS.

THE REPS OF LEVEL k ARE IN A CATEGORY CALLED LEVEL k FUSION CATEGORY. (WZW CFT) THERE IS AN ACTION OF $SL(2,\mathbb{Z})$ ON THIS CATEGORY THAT I WANT TO DESCRIBE. THE MODULARITY IS RELATED TO THIS ACTION.

THE FUSION CATEGORY IS SEMISIMPLE CATEGORY: EVERY OBJECT IS A DIRECT SUM OF SIMPLE OBJECTS.

$$\Pi_\lambda \qquad (\lambda \in \text{LEVEL } k \text{ FUNDAMENTAL ALCOVE.})$$

THE CATEGORY IS MONOIDAL MEANING IT HAS A COMPOSITION WHICH IS THE ABOVE FUSION RULE

$$\Pi_\lambda * \Pi_\gamma = \sum \underset{\substack{\uparrow \\ \text{FUSION COEFS.}}}{N_{\lambda\gamma}^{\mu}} \Pi_\mu$$

THIS IS A RIBBON CATEGORY.
IN PRACTICAL TERMS THIS MEANS THERE ARE APPLICATIONS TO KNOT THEORY (JONES POLYNOMIAL) AND POTENTIALLY QUANTUM COMPUTING.

A RIBBON CATEGORY IS A RIGID BRAIDED MONOIDAL CATEGORY WITH RIBBON STRUCTURE.

RIGID: OBJECTS HAVE DUALS.

$$\Pi_\lambda^* = \Pi_{-w_0 \lambda}$$

$w_0 \in$ (FINITE) WEYL GROUP IS THE LONG ELEMENT.

IF V IS ANY OBJECT THERE ARE MORPHISMS

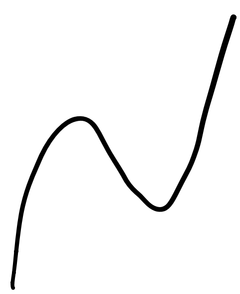
$$\mathbb{C} \xrightarrow[\text{coev}]{} V \otimes V^*$$

IDENTITY OBJECT IN THE MONOIDAL CATEGOR

$$V^* \otimes V \xrightarrow[\text{evaluation}]{} \mathbb{C}.$$

I (INSTEAD OF $\mathbb{C}$)

$$V \xrightarrow{I_V \otimes \mathrm{coev}_V} V \otimes V^* \otimes V \xrightarrow{\mathrm{ev}_V \otimes I_V} V.$$



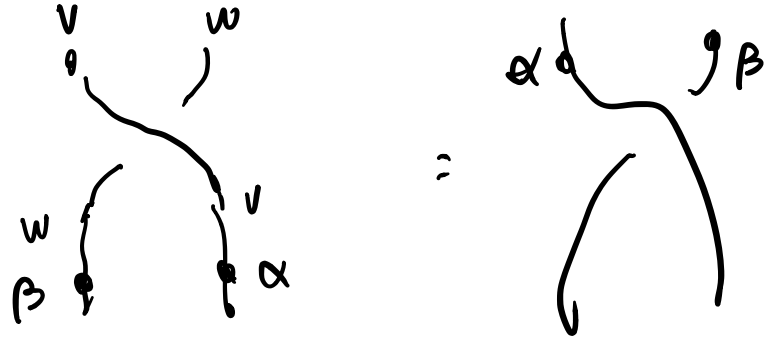
RIGID MONOIDAL CATEGORY.

BRAIDED: GIVEN OBJECTS V, W,

$$C_{V,W} : V \otimes W \to W \otimes V$$

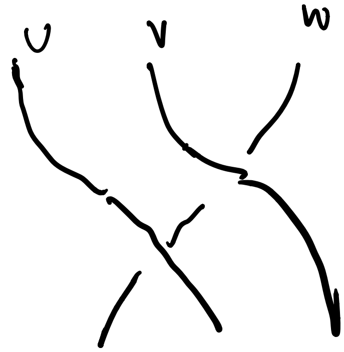
"THE R-MATRIX".

NATURALITY: $\alpha : V \to V'$, $\beta : V \to W'$



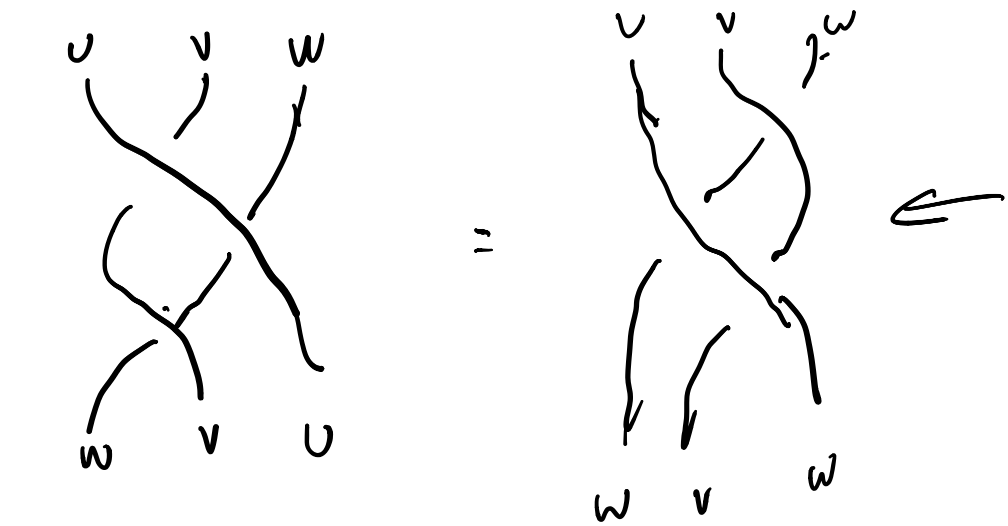
$$(\beta \otimes \alpha)\, C_{V,W} = C_{V,W}(\alpha \otimes \beta)$$

NATURALITY.



$$(C_{V,W} \otimes I_V)(I_U \otimes C_{V,W}) = C_{U\otimes V, W}.$$

ANOTHER SIMILAR.



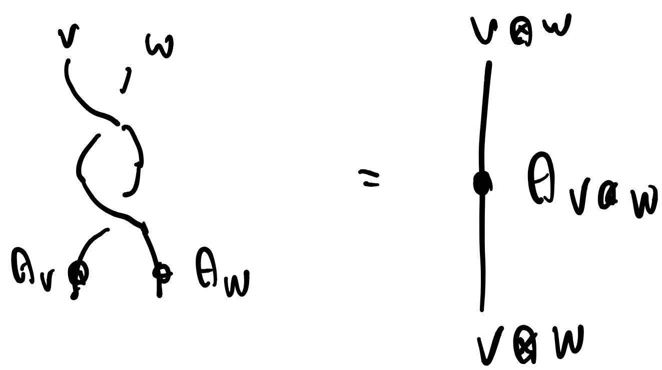
"YANG-BAXTER EQUATION".

RIGID BRAIDED CATEGORY.

RIBBON ELEMENT

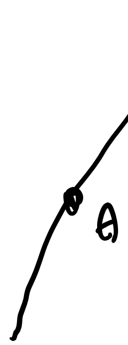
$$\theta_V : V \to V$$

HAS THE PROPERTY



$$(\theta_V \otimes \theta_W) c_{W,V}\, c_{V,W} = c_{W,V}\, c_{V,W} (\theta_V \otimes \theta_W)$$

$$= \theta_{V \otimes W}.$$



IN THE FUSION CATEGORY

IF $V = \pi_\lambda$

$$\theta_V = e^{i\pi (\lambda + 2\rho | \lambda)}$$

EIGENVALUE OF CASIMIR ELEMENT FOR $\mathfrak{g}$.

$$e^{i\pi (|\lambda+\rho|^2 - |\lambda|^2)}$$

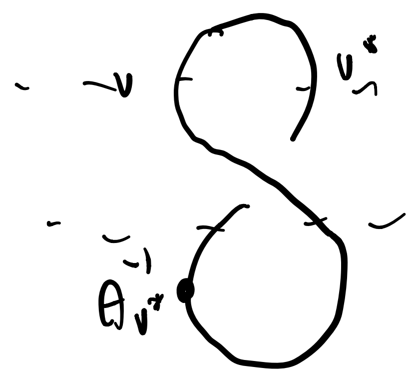
STUFF THAT APPEARS IN THE THETA FUNCTIONS.

WHAT THE RIBBON ELEMENT IS GOOD FOR:

IN A RIBBON CATEGORY

$f: V \to V$ IS A MORPHISM

WE CAN DEFINE THE (QUANTUM) TRACE:



$$I \xrightarrow[\text{coev}]{} V \otimes V^* \xrightarrow[c_{V,V^*}]{} V^* \otimes V \xrightarrow{\theta_V^{-1} \otimes I_V} V^* \otimes V \to I.$$

THIS IS SCALAR THIS IS THE QUANTUM TRACE.

IN CATEGORY OF F.D.V.S. $\theta_V = 1$

THIS IS THE TRACE OF AN ENDOMORPHISM.

$$tr(f \otimes g) = tr(f)\, tr(g).$$

RIBBON CATEGORY DUE TO TURAEV (+R ?)

THE FUSION CATEGORY IS A

MODULAR TENSOR TENSOR CATEGORY.

(I) A SEMISIMPLE RIBBON CATEGORY WITH FINITELY MANY IRREDUCIBLES.

(II) INVERTIBLE S-MATRIX.

WE DEFINE NORMALIZED CHARACTERS

$\chi^{\lambda}\ (\tau, z, u)$ $\quad \tau \in \mathbb{C}\ (\mathrm{Im}(\tau) > 0)$

$u \in \mathbb{C}$ IF WE WANT CONVERGENCE

$z = z_1, \dots, z_r$

$$(\tau, z, u) = \sum z_i \alpha_i^{\vee} + \tau \Lambda_0 + u \delta.$$

$z_1, \dots, z_r \in \mathbb{R}.$

$$e^{-m_\Lambda \delta} \operatorname{ch} L(\Lambda) = \chi_\lambda .$$

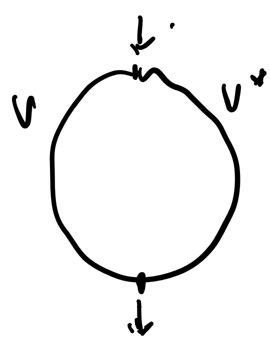
$$m_\Lambda = \frac{|\Lambda+\rho|^2}{2(k+h^\vee)} - \frac{|\rho|^2}{2h^\vee} .$$

$$\chi_\Lambda\left(-\frac{1}{\tau}, \frac{z}{\tau}, u - \frac{|z|^2}{2\tau}\right) = \sum S_{\Lambda,\Lambda'} \chi_{\Lambda'}(\tau, z, u)$$

$S_{\Lambda,\Lambda'}$ = UNITARY S-MATRIX.

IN RIBBON CATEGORY WE CAN

IDENTIFY $V^{**} = V$

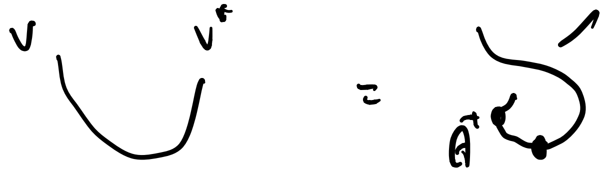


MEANS

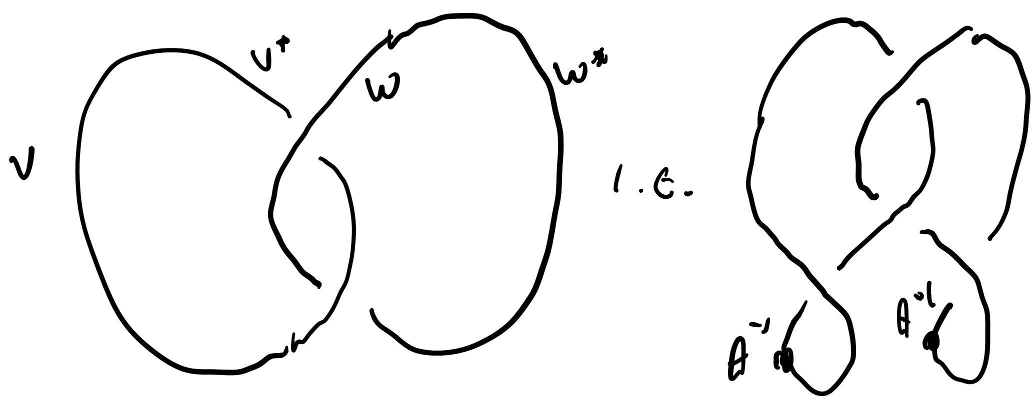
$ev : V^* \otimes V \to \mathbb{C}$

NOT $V \otimes V^* \to \mathbb{C}$.

IDENTIFY $V = V^{**}$



$ev : V \otimes V^* \to \mathbb{C}$ IS DEFINED USING θ^{-1}



$$\mathbb{C} \xrightarrow{\text{coev} \otimes \text{coev.}} V \otimes V^* \otimes W \otimes W^*$$

$$\big\downarrow c_{V^*, W}$$

$$V \otimes W \otimes V^* \otimes W^*$$

$$\big\downarrow c_{W, V^*}$$

$$V \otimes V^* \otimes W \otimes W^*$$

$$\big\downarrow \text{coev (using ribbon elt.}$$

$$\mathbb{C}$$

THIS IS A SCALAR. THIS IS (UP TO A CONSTANT.)

THIS IS THE S-MATRIX.

RECOMMEND BOOKS OF BAKALOV KIRILLOV, TURAEV.

CONTINUATION; SEE

http://sporadic.stanford.edu/quantum/lecture~~13~~.pdf.
14.

$V \otimes W$ REDUCIBLE

$U \subset V \otimes W$, $W \otimes V$

$$\begin{tikzcd}
V \otimes W \arrow[r] & W \otimes V \\
U \arrow[u, hook, "\Phi_{V,W,U}"'] \arrow[r, dashed] & U \arrow[u, hook, "\Phi_{W,V,U}"']
\end{tikzcd}$$

$$U \xrightarrow[\theta^*_U]{} U$$