CHARACTERS OF AFFINE LIE ALGEBRAS
AND STRING FUNCTIONS AS MODULAR FORMS.
S-MATRIX $\rightsquigarrow$ FUSION CATEGORY

$$q = e^{2\pi i \tau} \qquad \tau \in \mathcal{H}$$

$$\updownarrow$$

$$e^{-\delta}$$

WHAT ABOUT e^{λ} $\lambda \in \hat{\mathfrak{h}}^*$ (OR $\lambda \in P$).

IF $\lambda \in \hat{\mathfrak{h}}^*$ REGARD λ AS A FUNCTION OF $v \in \mathfrak{h}$. IDENTIFY $\hat{\mathfrak{h}}, \hat{\mathfrak{h}}^*$ BY DUAL PAIRING $(\, | \,)$. CONSIDER e^{λ} TO BE THE FUNCTION

$$e^{\lambda}(v) = e^{(\lambda | v)} \quad \text{FOR } v \in \mathfrak{h}.$$

A SERIES LIKE

$$a_\lambda^\Lambda = \sum \dim L(\Lambda)_{\lambda - n\delta}\, q^{\delta}$$

IS CONVERGENT AS A FUNCTION OF $\mathcal{H} = \{\tau \mid \mathrm{Im}(\tau) > 0\}$.
(THEN $|q| < 1$)

IT MAY BE SHOWN THAT

$$\operatorname{ch} L(\Lambda) = \sum_{\lambda} (\dim V_\lambda) e^{\lambda}$$

$V = L(\Lambda)$

CONVERGES AS A FUNCTION ON THE SUBSET

$$Y = \{\alpha \in \mathfrak{h} \mid \operatorname{re}(\alpha \mid \delta) > 0\}.$$

"TITS CONE". THE CHARACTER IS A FUNCTION ON Y. WE SAW LAST TIME HOW TO EXPAND IT IN THETA FUNCTIONS.

$$\operatorname{ch} L(\Lambda) = \underset{\substack{\uparrow \\ \text{WEYL DENOM.}}}{\Delta^{-1}} \sum_{W} =$$

$W = \mathring{W} \ltimes T$

$\uparrow$ FINITE W.G.

$\uparrow$ GROUP TRANSLATIONS BY $\bar{Q}$

$\uparrow$ ROOT LATTICE OF THE FINITE SIMPLE L.A. $\mathfrak{g}$.

$$\Delta^{-1} \sum_{w \in \mathring{W}} (-1)^{\ell(w)} \underbrace{\sum_{\alpha \in \bar{Q}} e^{t_\alpha(w(\Lambda+\rho))}}_{\text{A THETA FUNCTION}}$$

(NOT QUITE NORMALIZED CORRECTLY.)

$\alpha \in Q \subset \mathring{\mathfrak{h}}^*$

$\mathring{\mathfrak{h}}^*$ = WHAT I DENOTED $\mathfrak{h}^*$

DUAL SPACE OF CARTAN ALGEBRA OF $\mathfrak{g}$

$t_\alpha : \mathfrak{h}^*_k \to \mathfrak{h}^*_k = \{\lambda \mid (\delta|\lambda) = k\}$ LEVEL k SUBSPACE OF $\mathfrak{h}^*$.

$\langle K, \lambda \rangle$

$t_\alpha(\lambda) =$

$\lambda + k\alpha - \left((\lambda|\alpha) + \frac{k}{2}|\alpha|^2\right)\delta$ LEVEL OF $\rho = h^\vee$.

$= \lambda + k\alpha - \frac{1}{2k}\left(|\lambda + k\alpha|^2 - |\lambda|^2\right)\delta$.

$\Theta_\lambda = e^{-\frac{|\lambda|^2}{2k}} \sum_{\alpha \in Q} e^{t_\alpha(\lambda)}$ FUNCTION ON Y.

$$\operatorname{ch} L(\Lambda) = \Delta^{-1} \sum_{w \in \mathring{W}} (-1)^{\ell(w)} \sum_{\alpha \in \bar{Q}} e^{t_\alpha(w(\Lambda+\rho))}$$

$$A_\lambda = \sum_{w \in \mathring{W}} (-1)^{\ell(w)} \Theta_{w(\lambda)}.$$

DENOMINATOR Δ RESEMBLES A_ρ

NUMERATOR RESEMBLES $A_{\Lambda+\rho}$.

DISCREPANCY HAS TO DO WITH

$e^{\frac{1}{2h}|\lambda|^2\delta}$ NEEDED FROM

COMPLETING THE SQUARE IN THETA FUNCTION.

$$A_\rho = e^{-\frac{|\rho|^2}{2h^\vee}\delta}\,\Delta$$

$$\Delta = \sum_{w\in W}(-1)^w e^{w(\rho)} =$$

$$e^\rho \prod_{\alpha\in Q}(1-e^{-\alpha})^{\mathrm{mult}(\alpha)}$$

$$= e^{\bar\rho + h^\vee\Lambda_0}\prod(1-e^{-\alpha})^{\mathrm{mult}(\alpha)}$$

$$\uparrow$$

$$\rho$$

$$A_\rho = e^{\bar\rho + h^\vee\Lambda_0 - \frac{|\rho|^2}{2h^\vee}\delta}\prod(\ldots)^{\mathrm{mult}(\alpha)}$$

$$\uparrow$$

NUMERATOR IN WCF ALSO HAS AN ADJUSTMENT BY $\frac{|\rho+\Lambda|^2}{2(k+h^\vee)}$ BECAUSE OF THE LEVEL OF $\Lambda+\rho$ BEING $k+h^\vee$.

$$m_\Lambda = \frac{|\Lambda+\rho|^2}{2(k+h^\vee)} - \frac{|\rho|^2}{2h^\vee}.$$

DEFINE THE NORMALIZED CHAR.

$$\chi_\Lambda = e^{-m_\Lambda \delta} \operatorname{ch} L(\Lambda) = \frac{A_{\Lambda+\rho}}{A_\rho}.$$

THIS WE CAN EXPECT TO BE A MODULAR FORM.

$$\chi_\Lambda = \frac{\sum_{w \in \mathring{W}} (-1)^{\ell(w)} \Theta_{w(\Lambda+\rho)}}{\sum_{w \in \mathring{W}} (-1)^{\ell(w)} \Theta_{w(\rho)}}.$$

KAC' DESCRIPTION OF MODULAR FORMS IN CH. 13 SECTIONS 1-6.

IN LANGUAGE OF AUTOMORPHIC FORMS THESE THETA FUNCTIONS ARE THETA LIFTS:

$$O(r,1) \rightsquigarrow SL(2,\mathbb{R}).$$

DUAL PAIRS $G, H \subset Sp(2(r+1))$ Ω

WEIL REP'N

$\Omega | G \times H$

DECOMPOSES INTO DIRECT SUM OR DIRECT INTEGRAL OF PIECES.

$$\begin{pmatrix} \text{IRR. OF} \\ G \end{pmatrix} \otimes \begin{pmatrix} \text{IRR. OF} \\ H \end{pmatrix}$$

USEFUL TO BRING IN THE HEISENBERG GROUP.

$$\mathring{\mathfrak{h}}_{\mathbb{R}} = \bigoplus_{i=1}^{r} \mathbb{R}\alpha_i^\vee.$$

$$\mathring{\mathfrak{h}} = \bigoplus \mathbb{C}\alpha_i^\vee.$$

$$N = \mathring{\mathfrak{h}}_{\mathbb{R}} \oplus \mathring{\mathfrak{h}}_{\mathbb{R}} \oplus i\mathbb{R}.$$

$\alpha \qquad \beta \qquad \mu$

2r+1 DIMENSIONAL UNIPOTENT GROUP.

$$(\alpha, \beta, \mu)(\alpha', \beta', \mu') =$$

$$(\alpha+\alpha', \beta+\beta' + \mu+\mu' + i\pi((\alpha|\beta') - (\alpha'|\beta))).$$

ACTS ON Y

OPERATORS ϕ_α, t_α

$$t_\alpha(v) = v + h\alpha - \frac{1}{2h}(|\lambda+h\alpha|^2 - |\lambda|^2)$$

$\phi_\alpha(v) = v + 2\pi i\alpha$ (ACTS BY TRANSLATION.)

THE HEISENBERG GROUP ACTS

$$(\alpha, \beta, \mu)(v) = t_\beta(v) + 2\pi i\alpha + (\mu - i\pi(\alpha|\beta))\delta$$

ϕ_α ACTION

ACTION ON $\mathcal{X}$.

$$N_{\mathbb{Z}} = \{(\alpha, \beta, n) \mid \alpha, \beta \in \overline{Q},\ n \in 2\pi i \mathbb{Z}\}.$$

DISCRETE SUBGROUP.

IN KAC INTERPRETATION (SIMILAR TO LITERATURE WEIL, MUMFORD'S BOOK ON THETA FUNCTIONS.

ADELIC.

VERGNE-LION; IGUSA'S BOOK ON THETA FUNCTIONS.

LEVEL n THETA FUNCTIONS IS THEN INTERPRETED AS THE SPACE $\widetilde{Th}_n$ OF HOLOMORPHIC FUNCTIONS ON $\mathcal{X}$ S.T.

$$F(v + a\delta) = e^{na} F(v) \qquad a \in \mathbb{C}.$$

CENTER OF N.

$$F(n(v)) = F(v) \quad \text{IF } n \in N_{\mathbb{Z}}.$$

THE FUNCTION $\Theta_\lambda = e^{\tau|\lambda|^2/2h\,\delta} \sum_{\alpha \in Q} e^{t_\alpha(\lambda)}$

IS IN $\widetilde{TH}_h$. $\qquad (\lambda|\delta) = h$.

THIS ENCODES ALL TRANSFORMATION PROPERTIES SINCE $SL(2,\mathbb{R})$ ACTS ON N WITH $SL(2,\mathbb{Z})$ FIXING $N_\mathbb{Z}$.

IDEA: N HAS A UNIQUE IRR. REP'N WITH CENTRAL CHARACTER

$(\alpha, \beta, a) \to e^{ha}$. (STONE-VON NEUMANN)

I HAVE A REALIZATION OF THIS IN THE ABOVE SPACE OF FUNCTIONS

$F(v + a\delta) = e^{ha}F(v) \quad a \in \mathbb{C}$.

$\uparrow$ CENTER OF N.

$F(n(v)) = F(v)$ IF $n \in N_\mathbb{Z}$.

$\mathrm{IND}_{N_\mathbb{Z} \cdot Z(N)}^{N} (e^{a\delta})$. $\qquad Z(N) = \{(0,0,*)\}$

IF I APPLY AN AUTOMORPHISM OF N THAT FIXES N_A AND $Z(N)$.

THEN THIS GIVES ANOTHER REALIZATION OF THE SAME REPRESENTATION.

SO THERE IS AN INTERTWINING MAP

$$\mathrm{IND}_{N_A \cdot Z(N)}^{N}(e^{a\delta}) \xrightarrow[\omega\left(\begin{smallmatrix}a&b\\c&d\end{smallmatrix}\right)]{} \mathrm{IND}_{N_A \cdot Z(N)}^{N}(e^{a\delta})$$

AUTOMORPHISM COME FROM $SL(2,\mathbb{Z})$

$$\begin{pmatrix}a&b\\c&d\end{pmatrix} \in SL(2,\mathbb{Z}) \circlearrowright N$$

$$\begin{pmatrix}a&b\\c&d\end{pmatrix}(\alpha,\beta,u) = (a\alpha+b\beta,\ c\alpha+d\beta,\ u)$$

THIS MAP $\omega\begin{pmatrix}a&b\\c&d\end{pmatrix}$ IS DETERMINED UP TO SCALAR (SCHUR'S LEMMA.)

$S : \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ $N \to N$

$(\alpha, \beta, m) \to (-\beta, \alpha, m)$.

ACTS ON Θ_λ (λ LEVEL h).

THIS ACTION IS RELATED TO AN ACTION ON Y.

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} : (\tau, z, u) = \left(\frac{a\tau + b}{c\tau + d}, \frac{z}{c\tau + d}, u - \frac{c|z|^2}{2(c\tau + d)} \right).$$

$\uparrow$ LOOKS ON $\mathfrak{h}$.

$$\sum z_i \alpha_i^\vee + \tau \Lambda_0 + u\delta \qquad z_i = (z_1, \dots, z_l)$$

$$\overset{\shortparallel}{(\tau, z, u)}$$

IF $\tau \in \mathcal{H}$ I.E. $\mathrm{Im}(\tau) > 0$ THIS IS IN Y.

THIS PREDICTS AUTOMORPHIC PROPERTIES THAT CAN ALSO BE PROVED BY

POISSON SUMMATION FORMULA.

$$\Theta_\lambda\left(-\frac{1}{\tau}, \frac{z}{\tau}, u - \frac{(z|z)}{2\tau}\right) = (-i\tau)^{r/2} \cdot \text{CONSTANT} \cdot \times \sum_{\mu} e^{-\frac{2\pi i}{k}(\bar\lambda|\bar\mu)} \Theta_\mu(\tau, z, u)$$

$$S = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

WEIGHT $= \frac{r}{2}$

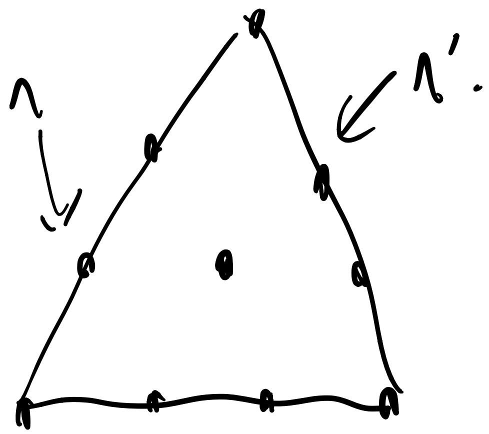
μ SUMMED OVER WEIGHTS OF LEVEL k FUNDAMENTAL ALCOVE MOD. $k\bar{Q}$

NOW THERE IS THIS TRANSFORMATION PROPERTY:

$$\chi_\Lambda\left(-\frac{1}{\tau}, \frac{z}{\tau}, u - \frac{(z|z)}{2\tau}\right) = \sum_{\Lambda'} S_{\Lambda,\Lambda'} \chi_{\Lambda'}(\tau, z, u).$$

$\sum$ $\Lambda' \in$ LEVEL k FUND. ALCOVE.

$S_{\Lambda,\Lambda'}$ CAN BE MADE EXPLICIT. THERE IS A CALCULATION INVOLVING Δ.



$k = 3$

$\mathfrak{g} = \mathfrak{sl}(3)$

THM 13.8

$$S_{\Lambda,\Lambda'} = (\text{const.}) \sum_{w \in W^0} (-1)^{\ell(w)} \times$$

$$e^{-2\pi i \left((\bar{\Lambda}+\bar{\rho}) | \Lambda'+\bar{\rho}\right)/(k+h^\vee)}.$$

SAGE 9.2 KNOWS THIS FORMULA.

FusionRing CLASS.

THE S-MATRIX APPEARS IN THE THEORY OF THE FUSION RING.

THERE IS A COMPOSITION LAW ON

THE χ_Λ ($\Lambda \in$ LEVEL k FUNDAMENTAL ALCOVE.)

FUSION

CONFORMAL FIELD THEORY.

$$\bigoplus_\Lambda L(\Lambda) \otimes L(\Lambda) \quad = \text{HILBERT SPACE.}$$

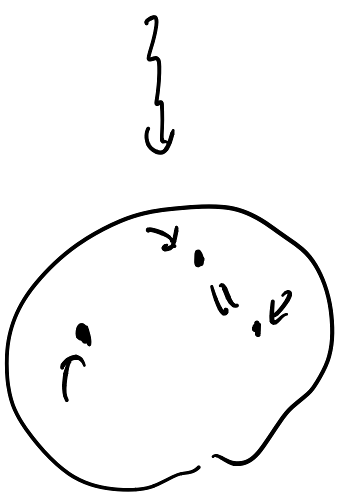
$\Lambda \in$ LEVEL k

OPERATOR PRODUCT EXPANSION

t

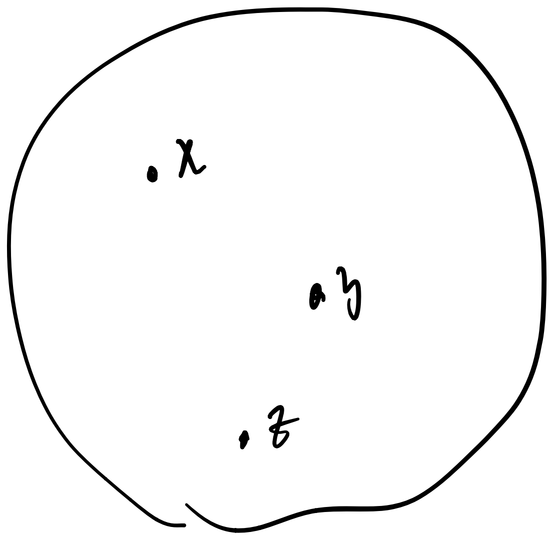
$$\hat{\mathfrak{g}}' = (\mathbb{C}[t, t^{-1}] \otimes \mathfrak{g}) \oplus \mathbb{C}K.$$

$v_t \qquad v \in L(\Lambda).$



S-MATRIX ENCODES THE FUSION STRUCTURE.

THURSDAY: DISCUSS VERLINDE FORMULA AND RELATED MATTERS.



$v \in L(\cancel{A})$

v_x ..

$$\int f(x) v_x \, dx .$$

OPERATION

$$\tau \to -\frac{1}{\tau} .$$

$$U_q(\mathfrak{g})$$

$\uparrow$

$e^{\pi i/(k+h^\vee)}$

$$U_q(\mathfrak{g}) \rightsquigarrow \text{SIX VERTEX MODEL ETC.}$$

$\uparrow$

q GENERAL $\mathbb{C}^\times$

ISING MODEL.

SPIN

ENERGY

$\longleftrightarrow$

CFT. BPZ

MINIMAL MODEL.

WZW $(\hat{A}_1)_1$ $\widehat{\mathfrak{sl}(2)}_1$.