$V = L(\Lambda)$ $\Lambda \in P^+_k$ (i.e. Λ is a dominant weight of level $k = \langle K, \lambda \rangle$)

Given λ proved

$\dim V_{\lambda - n\delta}$ is monotone

zero if $n << 0$

So there is a smallest n in the sequence

$$\ldots, \lambda + \delta, \lambda, \lambda - \delta, \ldots, \lambda - n\delta, \ldots$$

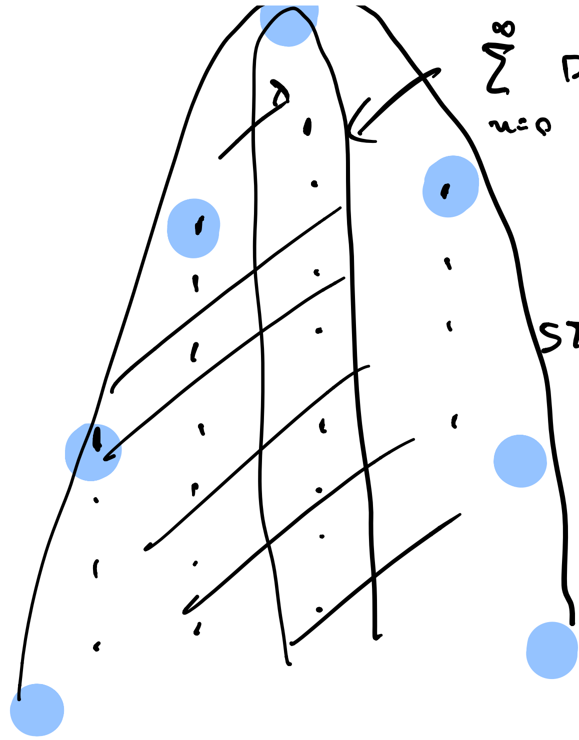
such that $\dim V_{\lambda - n\delta} \neq 0$.

For such n we say $\lambda - n\delta$ is a maximal weight.

$$\operatorname{ch} L(\Lambda) = \sum_{\lambda} \dim V_\lambda \cdot e^{\lambda}$$

$$= \sum_{\lambda \text{ maximal}} e^{-\lambda} \sum_{n=0}^{\infty} \dim(V_{\lambda - n\delta})\, q^n$$

$q = e^{-\delta}$.



$$\sum_{n=0}^{\infty} \dim V_{\lambda-n\delta} \cdot q^{n}$$

(UNNORMALIZED) STRING FUNCTION a^{Λ}_{λ}.

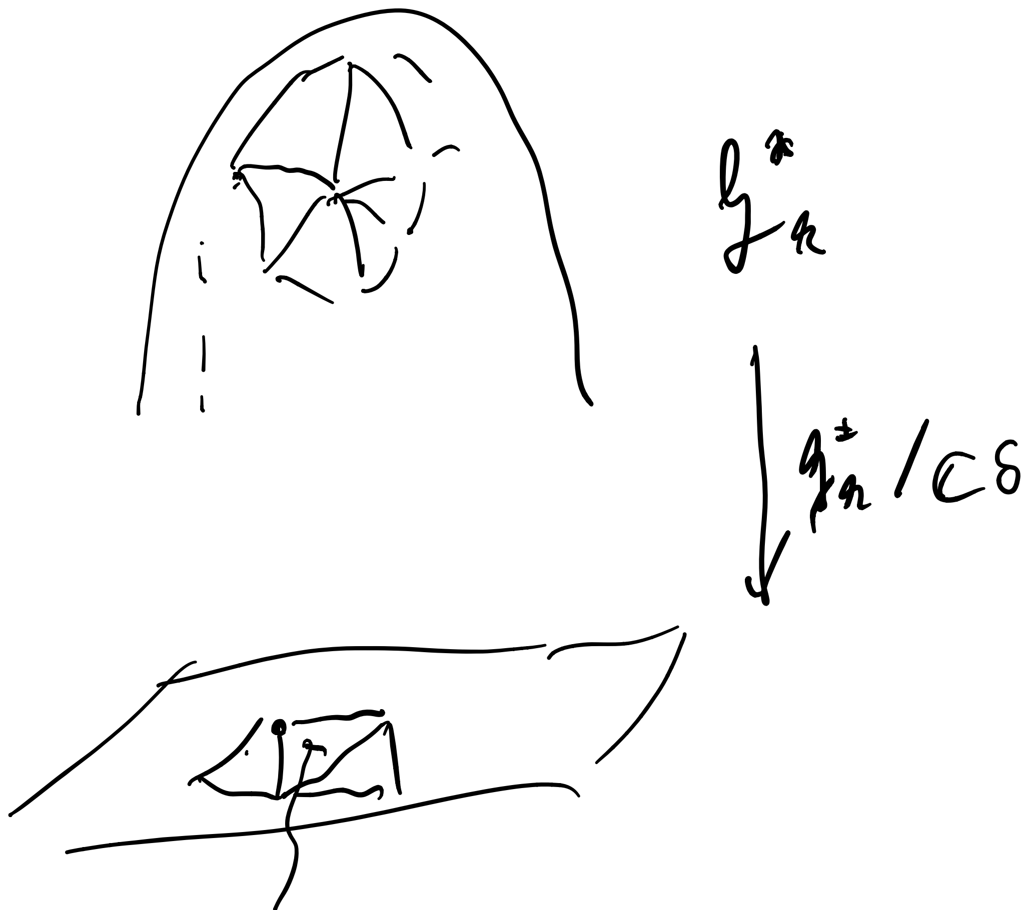
$$\mathrm{ch}\, L(\Lambda) = \sum_{\lambda \text{ max'l}} e^{\lambda} a^{\Lambda}_{\lambda}.$$

OBSERVE $\dim V_{\lambda-n\delta} = \dim V_{w(\lambda-n\delta)} =$

(SINCE $w\delta = \delta$) $\dim V_{w\lambda-n\delta}$.

WE CAN COLLECT THE TERMS IN W-ORBITS.

THERE ONLY A FINITE # OF MAXIMAL WEIGHTS THAT ARE DOMINANT.



$\mathring{W}$ COPIES OF FUNDAMENTAL ALCOVE

$\mathring{W} = \langle s_1, \dots, s_r \rangle$

WEYL GROUP OF $\mathfrak{g}$

(F.D. SIMPLY-LACED SIMPLE LIE ALGEBRA)

$W = \langle s_0, s_1, \dots, s_r \rangle$

AFFINE WEYL GROUP

$= \mathring{W} \cdot T$

T GROUP OF TRANSLATIONS $\cong Q$ (ROOT LATTICE).

THE FACT THAT THERE ARE ONLY FINITELY

MANY MAXIMAL DOMINANT WEIGHTS FOLLOWS FROM THE FACT THERE ARE ONLY FINITELY MANY WEIGHTS $\bar{\lambda}$ (PROJECTION OF λ INSIDE THE $\hat{\mathfrak{h}}^* \to \mathfrak{h}^*$)
LEVEL k FUNDAMENTAL ALCOVE, WHICH IS COMPACT.

$$\sum a_\lambda^\wedge \sum_{t \in T} e^{t\lambda}$$

~~λ~~ MAX'L DOMINANT WEIGHT

"THETA FUNCTION"

FINITE SUM.

FOR $k = 1$ THERE ARE 3 WEIGHTS IN LEVEL 1 FUNDAMENTAL ALCOVE

$$\bar{\lambda} \in \{\Lambda_0, \Lambda_1, \Lambda_2\}$$

$|$

WEIGHT FOR

$\mathfrak{sl}_3$

$$\hat{\mathfrak{h}}^*_k \longrightarrow \hat{\mathfrak{h}}^*_k / \mathbb{C}\delta \cong \mathfrak{h}^*$$

$$\bar{\lambda} + k\Lambda_0 \qquad\qquad \bar{\lambda}$$

SAY $\Lambda = \Lambda_0$ THERE IS ONLY ONE W ORBIT OF MAXIMAL DOMINANT WEIGHTS.

MORE GENERALLY, $t_\alpha : \lambda \to \lambda + k\alpha$ mod

$T = \{ t_\alpha \mid \alpha \in Q \}$

$$t_\alpha(\lambda) = \lambda + k\alpha - \underbrace{\left((\lambda|\alpha) + \frac{k}{2}|\alpha|^2 \right)}\delta$$

THE DOMINANT MAXIMAL WEIGHTS CORRESPOND TO THE ELEMENTS OF THE COSET $\Lambda + kQ$ IN $\mathcal{F}_k$ = LEVEL k FUND. ALCOVE.

Examples for $\widehat{sl}(3)$.

$k=2$ The level 2 dominant weights

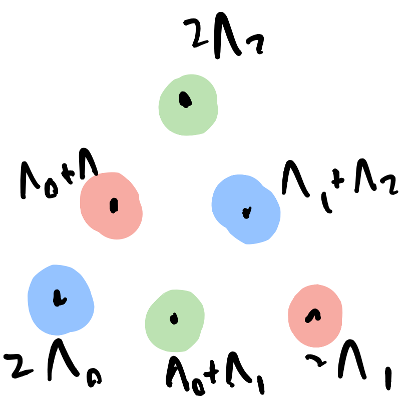
(up to a shift by a multiple of δ)

$$2\Lambda_0,\ 2\Lambda_1,\ 2\Lambda_2,\ \Lambda_0+\Lambda_1,\ \Lambda_0+\Lambda_2,\ \Lambda_1+\Lambda_2$$

In each case there are 2 elements of coset $\bar{\Lambda}+kQ$ in ${}^{\alpha}J_2$

$\bar{\Lambda}_0 = 0$

$\bar{\Lambda}_1 ; \bar{\Lambda}_2$ = fundamental weights for $sl(3)$



$$|\mathring{P}/\mathring{Q}| = 3$$

$\uparrow$

weight lattice for $sl(3)$

Elements with same color are in same coset.

$$\overline{2\Lambda_0-(\Lambda_1+\Lambda_2)} = -\bar{\Lambda}_1-\bar{\Lambda}_2 = -\theta$$ highest root

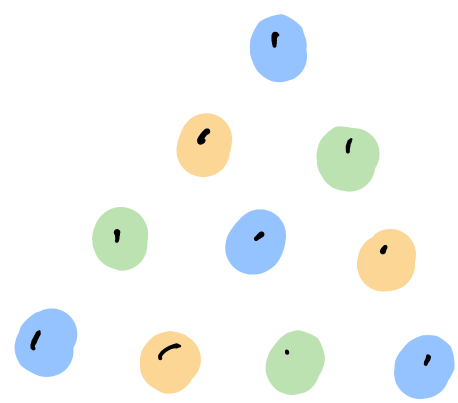
$\alpha_1+\alpha_2$

IF $k=2$, $L(\Lambda)$ HAS TWO DOMINANT MAXIMAL WEIGHTS.

LEVEL 3 DOMINANT WEIGHTS.

$$\hat{\mathfrak{h}}^*_k \longrightarrow \hat{\mathfrak{h}}^*_k/\mathbb{C}\delta \cong \mathfrak{h}^*$$

$$\bar{\lambda}+k\Lambda_0 \qquad \bar{\lambda}$$



$L(3\Lambda_0)$ HAS 4 MAXIMAL DOMINANT WEIGHTS AS DOES $L(\Lambda_0+\Lambda_1+\Lambda_2)$, $L(3\Lambda_1)$, $L(3\Lambda_2)$

(BLUE COSET).

THE OTHER DOMINANT WEIGHTS HAVE ONLY 3.

$$q^{1/24} \prod (1 - q^n) = \eta(\tau)$$

CUSP FORM OF WT 1/2.

$$\sum_{-\infty}^{\infty} q^{(6n+1)^2/24} (-1)^n ,$$

$$\mathrm{ch}\ L(\Lambda) = \sum_{\substack{\lambda \in \mathrm{MAX}(\Lambda) \\ \lambda \in P^+}} \left(\sum_{t \in T} e^{t(\lambda)} \right) a^{\Lambda}_{\lambda} .$$

(DOMINANT MAX'L WEIGHTS)

$$a^{\Lambda_0 + \Lambda_1 + \Lambda_2}_{2\Lambda_0 + \Lambda_1} = q + q^4 + q^{16} + q^{50} + \ldots$$

$$q = e^{-\delta}$$

$q^{11/18}\, a^{\Lambda_0 + \Lambda_1 + \Lambda_2}_{2\Lambda_0 + \Lambda_1}$ IS A MODULAR FORM.

$m^{\Lambda}_{\lambda} = \frac{11}{18}$ IS THE MODULAR CHARACTERISTIC (TO BE EXPLAINED)

$$t_\alpha(\lambda) = \lambda + h\alpha - \frac{1}{2h}\left(|\lambda + h\alpha|^2 - |\lambda|^2\right)\delta$$

$$(\lambda|\alpha) + \frac{h}{2}|\alpha|^2 .$$

$$\Theta_\lambda = e^{-\frac{|\lambda|^2}{2h}\delta} \sum_{\alpha \in Q} e^{t_\alpha(\lambda)}$$

$$e^{\lambda + h\alpha} e^{-\frac{1}{2h}|\lambda + h\alpha|^2 \delta}$$

NEEDED TO MAKE Θ_λ A THETA FUNCTION.

THE WEYL CHARACTER FORMULA HAS FORM

$$\operatorname{ch} L(\Lambda) = \Delta^{-1} \sum_{w \in W} (-1)^{\ell(w)} e^{w(\Lambda+\rho)-\rho}$$

$$= A_\rho^{-1} A_{\Lambda+\rho}$$

$$A_\lambda = \sum_{w \in W} (-1)^{\ell(w)} e^{w(\lambda)}$$

$$t_\alpha(\lambda)$$

$$\lambda = \bar{\lambda}$$

$\bar{\lambda}$ = PROJECTION ON $\mathfrak{h}^*$ $\lambda \in \mathfrak{h}_k^*$.

(?) $\lambda = \bar{\lambda} + k\Lambda_0 + \left(\frac{|\bar{\lambda}|^2 - |\lambda|^2}{2k}\right)\delta$

$$t_\alpha(\lambda) = k\Lambda_0 + (\bar{\lambda} + k\alpha) + \frac{1}{2k}\left(|\lambda|^2 - |\bar{\lambda} + k\alpha|^2\right)\delta$$

$$\sum_{w \in W} (-1)^{\ell(w)} e^{w(\rho) - \rho} =$$

$W = \mathring{W} \cdot T$

$T = \{t_\alpha \mid \alpha \in Q\}$

$$e^{-\rho} \sum_{w \in \mathring{W}} (-1)^{\ell(w)} \sum_{\alpha \in Q} e^{t_\alpha w(\rho)}$$

$$e^{-\rho + \frac{|\rho|^2}{2h^\vee}\delta} A_\rho.$$

ρ HAS LEVEL $h^\vee$ = DUAL COXETER NUMBER

$$m_\Lambda = \frac{|\Lambda + \rho|^2}{2(h^\vee + k)} - \frac{|\rho|^2}{2h^\vee}$$

$h^\vee + k$ = LEVEL OF $\Lambda + \rho$

$$m_{\Lambda,\lambda} = m_\Lambda - \frac{|\lambda|^2}{2k}$$

k = LEVEL OF Λ, λ.

$$\Theta_\lambda = e^{k\Lambda_0} \sum_{\gamma \in Q + \frac{1}{k}\bar{\lambda}} e^{-\frac{1}{2}k|\gamma|^2 \delta + k\gamma}$$

$$\Theta_\lambda = e^{-\frac{|\lambda|^2}{2k}\delta} \sum_{\alpha \in Q} e^{t_\alpha(\lambda)}$$

$$t_\alpha(\lambda) = k\Lambda_0 + (\bar{\lambda} + k\alpha) + \frac{1}{2k}\left(|\lambda|^2 - |\bar{\lambda} + k\alpha|^2\right)\delta$$

$$\gamma = \frac{1}{k}(\bar{\lambda} + k\alpha)$$

$$\chi_\lambda = e^{-m_\Lambda \delta} \operatorname{ch} L(\lambda)$$

$$c^\Lambda_\lambda = e^{-m_{\Lambda,\lambda}} a^\Lambda_\lambda$$

NORMALIZED STRING FUNCTIONS. MODULAR FORMS

$$\chi_\Lambda = \sum_{\text{max'l dominant weights}} c^\Lambda_\lambda \Theta_\lambda .$$

Θ_λ IS A MODULAR FORM. USING POISSON SUMMATION OR KAC CH. 13.

COMPARING THIS TO THE FORMULA

$$\chi_\Lambda = \frac{A_{\lambda+\rho}}{A_\rho}$$ WILL PRODUCE

INFORMATION ABOUT C_λ^Λ. (Ch. 13, NEXT WEEK.)

THE MODULE $L(\Lambda)$ IS ALSO A MODULE FOR VIRASORO ALGEBRA.

$$\bigoplus_{n=1}^{\infty} \mathbb{C} d_n + \mathbb{C} \cdot c$$

$$d_n = -t^{n-1} \frac{d}{dt} \circlearrowleft \mathbb{C}[t, t^{-1}] \otimes \mathfrak{g}.$$

$$\cup$$

EXCEPT $d_0 = -d \in \mathfrak{g}_t$ $\mathfrak{g}_t$.

IF WE MAKE CENTRAL EXTENSIONS

$$\mathrm{Vir} = \bigoplus \mathbb{C} d_n \oplus \mathbb{C} c$$

$$[d_n, d_m] = (n-m) d_{n+m} + \frac{n^3 - n}{12} \delta_{n,-m} \cdot c$$

FITS INTO A S.D.P. WITH $L(\Lambda)$.

(L(Λ) IS INVARIANT SUMMAND.)

IT IS POSSIBLE TO FIND COMPATIBLE ACTIONS OF VIR AND $\hat{\mathfrak{g}}$ ON $V(\Lambda)$.
SUGUWARA CONSTRUCTION IN CH, 12 OF KAC.

WESS ZUMINO WITTEN C.F.

$$\mathcal{H} = \bigoplus L(\Lambda) \otimes L(\Lambda)$$

HILBERT SPACE.

WEIGHTS IN LEVEL n ALCOVE.

E.G. $\Lambda_0, \Lambda_1, \Lambda_2$ IF $h = 1$

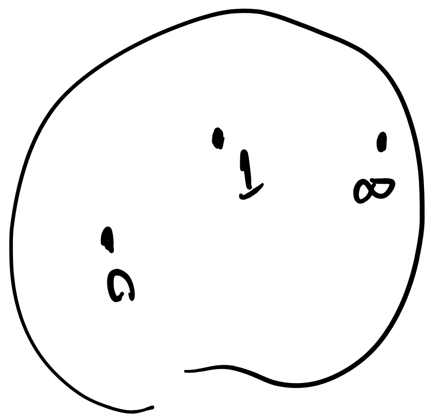
THE t WHICH APPEARED IN $\mathbb{C}[t] \otimes \mathfrak{g}$.
CAN BE THOUGHT OF AS A PARAMETER ON RIEMANN SPHERE $\mathbb{P}^1$

GIVEN A POINT $x \in \mathbb{P}^1$ AND

$v \in L(\Lambda)$ $\quad \Lambda$ LEVEL k.

THERE IS A "FIELD" OF OPERATORS

ON $\mathcal{H}$. WE CAN ITERATE THESE.



TAKE FIELD AT 0, COMBINE IT WITH

FIELD AT 1, DECOMPOSE IT

("OPERATOR PRODUCT EXPANSION") AT ∞.

THE 3 POINTS ARE UNIMPORTANT

BECAUSE $SL(2,\mathbb{C})$ IS 3-TRANSITIVE.

THIS GIVES A KIND OF TENSOR STRUCTURE

ON THE REPS $L(\Lambda)$ (Λ LEVEL k)