MODULAR FORMS

THETA FUNCTIONS

CHARACTERS OF AFFINE LIE ALGEBRAS AS MODULAR FORMS

$SL(2,\mathbb{R}) \curvearrowright \mathcal{H} = \{\tau = x+iy \mid y>0\}$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} : \tau \to \frac{a\tau+b}{c\tau+d}.$$

$\Gamma(1) = SL(2,\mathbb{Z})$ DISCRETE SUBGROUP OF $SL(2,\mathbb{R})$

$\Gamma(1)\backslash\mathcal{H}$ IS NOT COMPACT

$SL(2,\mathbb{Z})\backslash SL(2,\mathbb{R})$ NOT COMPACT

THESE QUOTIENTS HAVE FINITE VOLUME IN INVARIANT MEASURE

$$\int_{\Gamma(1)\backslash\mathcal{H}} \frac{dx\wedge dy}{y^2} < \infty$$

SUBGROUPS OF FINITE INDEX IN $\Gamma(1)$

$\Gamma(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \bmod N \right\}$ NORMAL

$$\Gamma_0(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \,\middle|\, c \equiv 0 \ (N) \right\}$$

SORT OF UNIVERSAL FOR MANY PURPOSES.

A MODULAR FORM OF WT k FOR

Γ $(= \Gamma_0(N)$ OR $\dots)$

$f : \mathcal{H} \to \mathbb{C}$ $\qquad k > 0, \quad k = \frac{1}{2}, 1, \frac{3}{2}, 2, \dots$

$$f\left(\frac{a\tau + b}{c\tau + d}\right) = (*)\,(c\tau + d)^k f(\tau)$$

$(*)$ COULD BE A CHARACTER E.G.

IF $\Gamma = \Gamma(N)$, $k \in \mathbb{Z}$

$(*) = \chi(d)$ $\qquad \chi : \mathbb{Z}/N\mathbb{Z} \to \mathbb{C}$

A DIRICHLET CHARACTER

$\chi(nm) = \chi(n)\chi(m)$

$\chi(n) = 0 \iff \gcd(n, N) > 1.$

IF $k = \frac{1}{2}, \frac{3}{2}, \dots$ $\qquad (*)$ MIGHT BE MORE COMPLICATED TO DESCRIBE

$$f\left(\frac{a\tau+b}{c\tau+d}\right) = j(g,\tau)\, f(\tau)$$

$$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$$

$$j(g,\tau) = (*)\,(c\tau+d)^k$$

$$j(g_1 g_2,\tau) = j(g_1, g_2\tau)\, j(g_2,\tau).$$

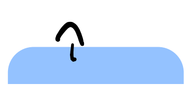
IF f IS HOLOMORPHIC AND BOUNDED NEAR CUSPS THEN f IS CALLED A MODULAR FORM OF WEIGHT k.

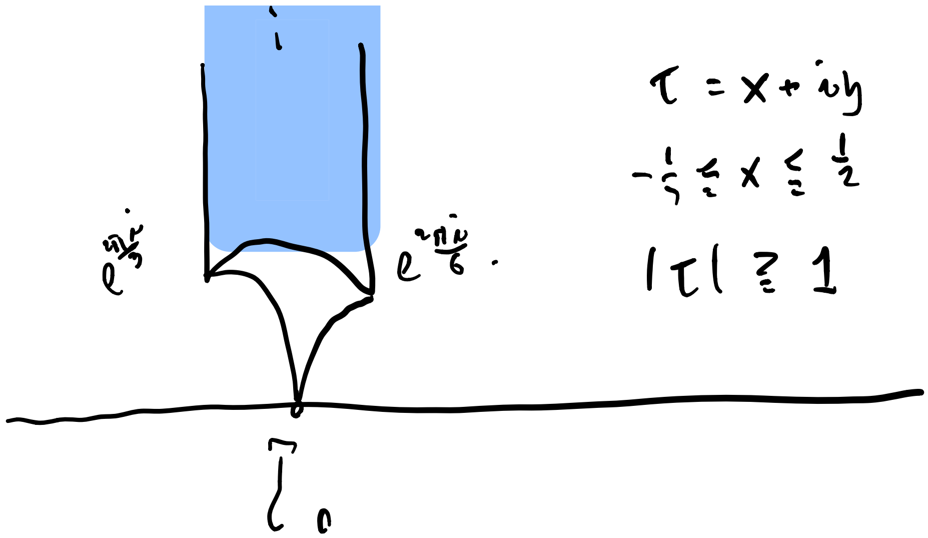
CUSPS ARE REPRESENTATIVES OF Γ ORBIT OF ACTION ON BOUNDARY POINTS.

$$\mathbb{Q} \cup \{\infty\}$$

FOR $SL(2,\mathbb{Z})$ THERE IS ONE CUSP, ∞.

THIS REFLECTS THE FUNDAMENTAL DOMAIN:



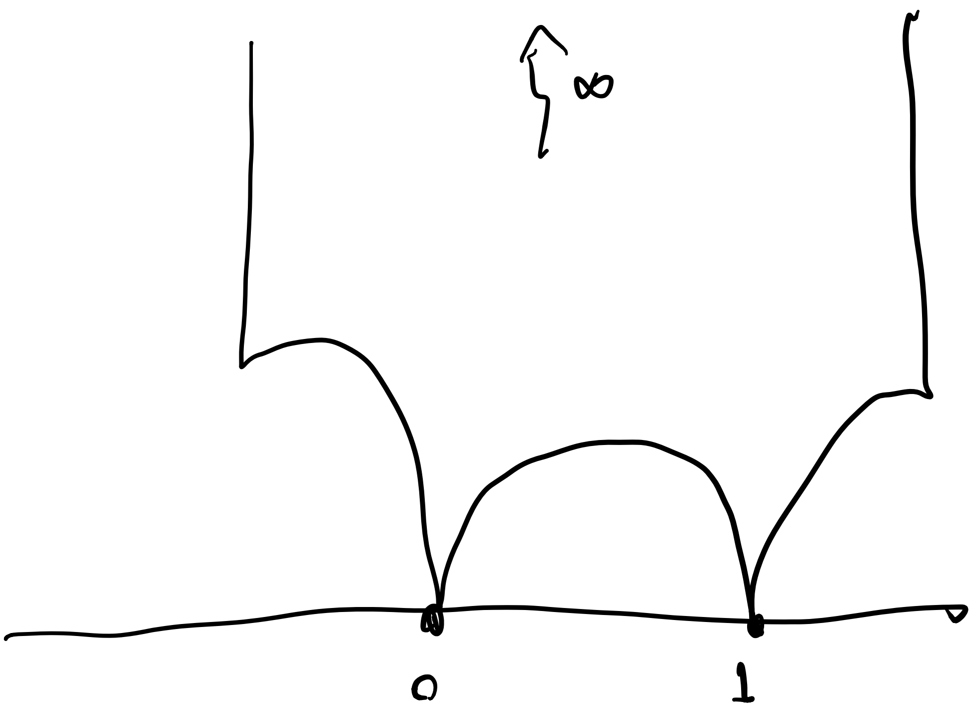


$\tau = x + iy$

$-\frac{1}{2} \leq x \leq \frac{1}{2}$

$|\tau| \geq 1$

0, ∞ ARE SAME Γ ORBIT SO THEN REPRESENT SAME CUSP.



SOME OTHER Γ. WITH 3 CUSPS.

AMONG MODULAR FORMS THOSE THAT VANISH AT CUSPS ARE CALLED CUSP FORMS.

RAMANUJAN'S CUSP FORM OF WEIGHT 12:

$$q\prod_{n=1}^{\infty}(1-q^n)^{24} = \Delta(\tau)$$

$$q = e^{2\pi i n \tau} \qquad \tau \in \mathcal{H} \Rightarrow |q| < 1.$$

$$\Delta\left(\frac{a\tau+b}{c\tau+d}\right) = (c\tau+d)^{12}\Delta(\tau)$$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{Z}).$$

AROSE FROM THEORY OF ELLIPTIC CURVES.

PRODUCT SHOWS $\Delta \neq 0$ ON $\mathcal{H}$

BUT $\Delta \to 0$ AS $\tau \to \infty$ $(q \to 0)$.

$$\Delta^{1/24} = q^{1/24} \prod_{n=1}^{\infty} (1-q^n) = \eta(\tau).$$

THIS IS DEDEKIND'S ETA

$$\eta(\tau) = \sum_{-\infty}^{\infty} (-1)^n q^{(6n+1)^2/24}.$$

THE $q^{1/24}$ IS NEEDED TO COMPLETE THE SQUARE IN THE QUADRATIC $(6n+1)^2/24$.

$$q^{1/24} \prod (1-q^n) = \sum (-1)^n q^{(6n+1)^2/24}$$

CAN BE DEDUCED FROM JTP (SEE LECTURE 0.)

THETA FUNCTIONS.

SUPPOSE $Q \in \mathrm{MAT}_n(\mathbb{Q})$

$Q = {}^tQ$ I WANT TO ASSUME

Q IS POSITIVE DEFINITE. THEORY

OF THETA FUNCTIONS FOR INDEFINITE

QUADRATIC FORMS IS VERY SUBTLE.

(C. L. SIEGEL AND OTHERS.)

$$\theta_Q(\tau) = \sum_{x \in \mathbb{Z}^n} e^{-\pi Q[x] \tau}$$

$$Q[x] = {}^t x Q x$$

$$Q = (a_{ij}) \qquad a_{ij} = a_{ji}$$

$$Q[x] = \sum_{i,j} a_{ij} x_i x_j .$$

FOR EXAMPLE $n=1$:

$$Q=(1)$$

$$\Theta_Q(\tau)=\sum_{n\in\mathbb{Z}} e^{-\pi n^2\tau}.$$

JACOBI THETA FUNCTION.

IF WE INSTEAD ALLOW SUMMATION OVER COSETS OF $\mathbb{Z}^n$ IN $\mathbb{Q}^n$ WE COULD GET

$$\sum_{\cdots} e^{-\pi(6n+1)^2\tau/24}$$

TAKE A LINEAR COMBINATION OF TWO OF THESE:

$$\sum (-1)^n e^{-\pi(6n+1)^2\tau/24} = \eta\left(\frac{\tau}{2}\right)$$

$$q = e^{2\pi i \tau}$$

$$\Theta_Q(\tau) = \sum_{x \in \mathbb{Z}^n} e^{\pi i Q[x] \tau}$$

NICEST CASE IF $Q \in \mathrm{MAT}_n(\mathbb{Z})$

$\det(Q) = 1$ AND Q HAS $a_{i,i} \equiv 0 \ (2)$.

THIS CAN ONLY HAPPEN IF $8 \mid n$.

IN THIS CASE

$\Theta_Q(\tau)$ IS MODULAR FOR

FOR $SL(2, \mathbb{Z})$.

(Q = E_8 ROOT LATTICE, LEECH LATTICE)
$n = 24$.

I WILL ARGUE THAT Θ_Q IS A MODULAR FORM WITHOUT TRYING TO MAKE THE GROUP EXPLICIT. BUT KAC CHAPTER 13 HE HAS RESULTS FOR EXPLICIT LEVEL (I.E. PARTICULAR GROUP Γ) PROP 13.6.

NOTICE THAT Q HAS POSSIBLE DENOMINATORS BUT IF N = GCD OF THESE

$$\Theta_Q(\tau) = \sum e^{-\pi Q[x] \tau}$$

$$\Theta_Q(\tau) = \Theta_Q(\tau + n) \quad \text{PROVIDED}$$

$2N \mid n$.
$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & n \\ & 1 \end{pmatrix}$$

I WILL SHOW THAT THERE IS A MODULAR RELATION BETWEEN

$$\theta_Q(\tau) \text{ AND } \theta_{Q'}\left(-\frac{1}{\tau}\right)$$

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

Q' WILL RUN THROUGH CERTAIN OTHER QUADRATIC FORMS.

POISSON SUMMATION FORMULA.

IF $f \in \mathcal{S}(\mathbb{R})$ SCHWARTZ SPACE

$$\sum_{-\infty}^{\infty} f(n) = \sum_{-\infty}^{\infty} \hat{f}(n)$$

$$\hat{f}(x) = \int_{-\infty}^{\infty} e^{2\pi i x y} f(y)\, dy.$$

SKETCH OF THE PROOF:

DEFINE $F(x) = \sum_{-\infty}^{\infty} f(x+n)$

SMOOTH PERIODIC

$$F(x) = \sum_{-\infty}^{\infty} a_n e^{2\pi i n x}$$

$$a_n = \int_0^1 F(x) e^{-2\pi i n x} \, dx$$

$$= \int_0^1 \sum_{-\infty}^{\infty} f(x+k) \, e^{-2\pi i n x} \, dx$$

$$= \int_{-\infty}^{\infty} f(x) e^{-2\pi i n x} \, dx = \hat{f}(-n)$$

$$\sum_{-\infty}^{\infty} f(x+n) = \sum_{n=-\infty}^{\infty} \hat{f}(n) e^{2\pi i n x}.$$

SET $x = 0$ GIVES

$$\sum f(n) = \sum \hat{f}(n).$$

$$f_t(x) = e^{-\pi t x^2}$$

$$\hat{f}_t(x) = \frac{1}{\sqrt{t}} f_{1/t}(x)$$

$$\hat{f}_t(x) = \int_{-\infty}^{\infty} e^{-\pi t y^2} e^{2\pi i x y} \, dy$$

$$= e^{-\pi x^2/t} \int_{-\infty}^{\infty} e^{-\pi \left(\frac{i}{\sqrt{t}} x - \sqrt{t} y\right)^2} dy.$$

USE CAUCHY'S THEOREM TO MOVE PATH OF INTEGRATION.

$$e^{-\pi x^2/t} \int_{-\infty}^{\infty} e^{-\pi t y^2} \, dt$$

$$= \frac{1}{\sqrt{t}} e^{-\pi x^2/t}$$

$$\sum_{-\infty}^{\infty} e^{-\pi n^2 t} = \frac{1}{\sqrt{t}} \sum_{-\infty}^{\infty} e^{-\pi n^2/t}$$

$$\theta(it) = \sum_{-\infty}^{\infty} q^{n^2}$$

now $q = e^{-\pi t}$ ← WANT $e^{2\pi i \tau}$

$$\theta\left(\frac{it}{2}\right) = \frac{1}{\sqrt{t}} \, \theta\left(\frac{i}{2t}\right).$$

REPLACING $\frac{in t}{2}$ BY τ

$$\theta(\tau) = \frac{1}{\sqrt{-2i\tau}}\,\theta\left(\frac{-1}{4\tau}\right).$$

COMBINE THIS WITH $\theta(\tau+1) = \theta(\tau)$

GIVES MODULARITY FOR THE GROUP

GEN'D BY

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 0 & 1/2 \\ -2 & 0 \end{pmatrix}.$$

THIS GROUP IS CONJUGATE IN

$SL(2,\mathbb{R})$ TO $\Gamma_0(4)$.

FOR MORE GENERAL Q.

FIND S A SYMMETRIC MATRIC

SUCH THAT $S^2 = Q$

$$\theta_Q = \sum_{x \in \mathbb{Z}^n} e^{-\pi S[x] \tau}$$

USE POISSON SUMMATION FORMULA

FOR $\mathbb{R}^n$. $\sum_{x \in \mathbb{Z}^n} f(x) = \sum_{x \in \mathbb{Z}^n} \hat{f}(x)$

THE FOURIER TRANSFORM MAY BE

COMPUTED AND

$$\theta_Q(\tau) = \frac{1}{(-i\tau)^{n/2}} \theta_{Q^{-1}}\left(-\frac{1}{\tau}\right)$$

$\begin{pmatrix} & -1 \\ 1 & \end{pmatrix}$ GIVES MODULARITY BUT

SENDS $Q \to Q^{-1}$

$\begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}$. SO θ_Q IS A MODULAR FORM. WE COULD GET MORE PRECISE FORMULAS FOR PARTICULAR Q.

GOING BACK TO THE CHARACTER OF AN AFFINE LIE ALGEBRA $\mathfrak{g}$.

$$\mathrm{CH}\, L(\Lambda) = \Delta^{-1} \sum_{w \in W} (-1)^{w} e^{w(\Lambda+\rho)}$$

$\Lambda \in P^{+}$

$$\Delta = e^{\rho} \prod_{\alpha \in \Delta^{+}} (1 - e^{-\rho}) = \sum_{w \in W} (-1)^{\ell(w)} e^{w(\rho) - \rho}$$

WE CAN MANIPULATE THIS BY USING SEMIDIRECT PRODUCT DECOMPOSITION OF W:

$$W = \underset{\substack{\text{FINITE}\\ \text{WEYL GROUP}\\ \text{OF } \mathfrak{g}.}}{\overset{\circ}{W}} \ltimes T$$

T IS THE GROUP OF "TRANSLATIONS" BY ELEMENTS OF Q (= ROOT LATTICE)

$\alpha \in Q \Rightarrow t_\alpha \in W$

$$t_\alpha(\lambda) = \lambda + \underline{k\alpha} + \underline{\overbrace{\left((\lambda|\alpha) + \tfrac{k}{2}|\alpha|^2\right)}^{\text{QUADRATIC.}}}\,\delta$$

$\langle K, \lambda \rangle = k$ "LEVEL" CONSTANT THROUGH OUT. (ALL WEIGHTS

OF $L(\Lambda)$ HAVE
LEVEL k.

$$P_k \subset \hat{\mathfrak{h}}^*_k \longrightarrow \hat{\mathfrak{h}}^*_k / \mathbb{C}\delta$$

WEIGHTS
OF LEVEL K

ON THIS PIECE
t_α ACTS
BY $\lambda \mapsto \lambda + k\alpha$
mod δ.

$$t_\alpha(\lambda) = \lambda + \underline{k\alpha} - \underline{\widetilde{\left((\lambda|\alpha) + \tfrac{k}{2}|\alpha|^2\right)}}\,\delta$$

$$\Delta^{-1} \sum_{w \in W} (-1)^{\ell(w)} e^{w(\Lambda+\rho)}$$

$$= \Delta^{-1} \sum_{\mu \in \mathring{W}} \sum_{\alpha \in Q} (-1)^{\ell(w)} e^{\mu t_{k\alpha}(\Lambda+\rho)}$$

$$t_\alpha(\lambda) = \lambda + h\alpha - \frac{1}{2h}\left((\lambda + h\alpha \mid \lambda + h\alpha) + \frac{1}{2h}|\lambda|^2 \right)\delta$$

SUMMATION OVER α PRODUCES

A THETA FUNCTION.

$$\Delta^{-1} \sum_{w \in \mathring{W}} (-1)^{\ell(w)} \sum_{\alpha \in Q} e^{w t_\alpha(\lambda)}$$

THE EXPLICIT FORMULA FOR $t_\alpha(\lambda)$

SHOWS THE INNER SUM IS A

THETA FUNCTION. $\left(q = \underline{\underline{e^{-\delta}}}\right)$

$$\Delta^{+} \sum_{w \in \mathring{W}} (-1)^{\ell(w)} \Theta_{w(\lambda)}$$

$$\Theta_{w(\lambda)} = \sum_{\alpha \in kQ} e^{t_{k\alpha}(\lambda)}$$

KAC "COMPLETES THE SQUARE"

MODIFIES THE CHARACTER BY

CONSIDERING

$$e^{-\frac{|\lambda|^2}{2k}\delta} \cdot \mathrm{ch}\, L(\Lambda) =$$

$$\Delta^{-1} e^{-\frac{|\lambda|^2}{2k}} \Theta_{w(\lambda)} .$$

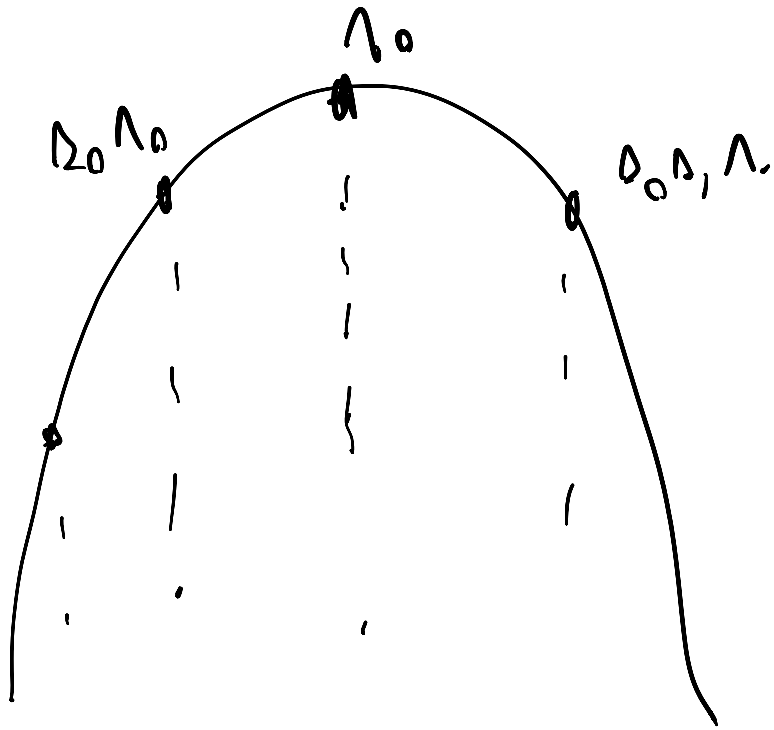
THREE SOURCES OF MODULARITY:

Δ IS MODULAR.

$\Theta_{w(\lambda)}$ IS MODULAR BUT IT

CAN BE FACTORED INTO A SERIES

OF STRING FUNCTIONS AND SUM OVER THE WEYL GROUP.



ON THURSDAY WE WILL LOOK AT THIS MORE CLOSELY.