KAC-WEYL CHARACTER FORMULA

PROTOTYPE FOR KAC' PROOF WAS BY BERNSTEIN-GELFAND-GELFAND (1960's) WHO PROVED THE CLASSICAL WCF FOR SEMISIMPLE LIE ALGEBRAS BY CATEGORY $\mathcal{O}$ METHODS:

FIND PRIMITIVE VECTORS IN $M(\lambda)$ $\lambda \in P^+$ (DOMINANT WEIGHTS).

"DOT" ACTION OF W ON P (WEIGHT LATTICE)

$$P = \{\lambda \in \mathfrak{h}^* \mid \langle \alpha_i^\vee, \lambda \rangle \in \mathbb{Z}\}$$

$$P^+ = \{ \cdots \mid \cdots \in \mathbb{N}\} \qquad \mathbb{N} = \{0, 1, 2, \dots\}$$

$$w \cdot \lambda = w(\lambda + \rho) - \rho$$

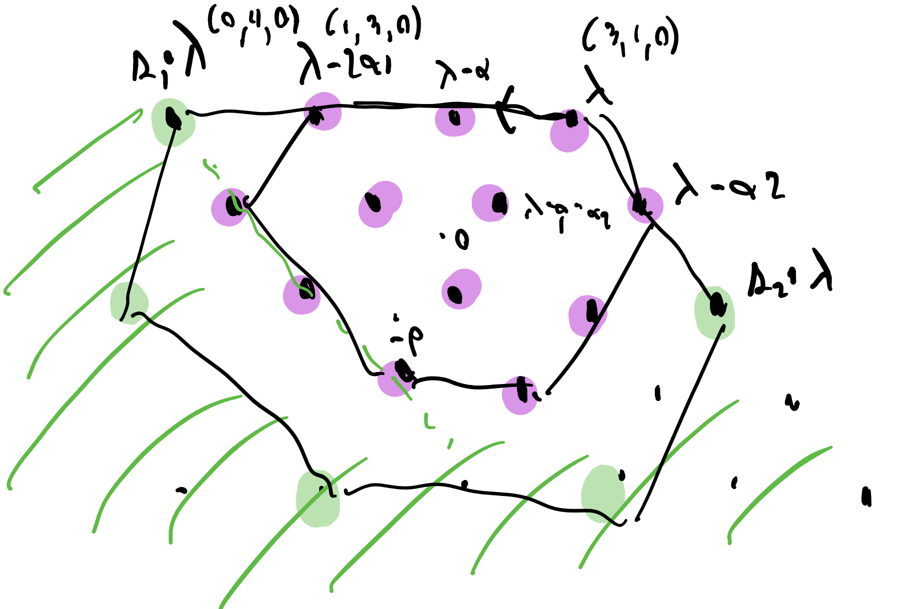
SO THE FIXED POINT OF THE ACTION IS MOVED TO $-\rho$.

THE KEY THING TO SHOW IS THAT THE WEIGHTS OF THE PRIMITIVE VECTORS ARE

$$w \cdot \lambda \quad (w \in W).$$

$$s_1 \cdot \lambda = s_1(\lambda + \rho) - \rho = s_1\lambda + \overbrace{s_1\rho - \rho}^{-\alpha_1}$$
$$= (1,3,0) - (1,-1,0) = (0,4,0)$$

$\lambda = (3,1,0)$ FOR $\mathfrak{sl}(3)$ $\qquad \rho = \alpha_1 + \alpha_2$



= WEIGHTS OF $L(\lambda)$

. = VERMA WEIGHT.

= WEIGHTS OF P.V. (EXCLUDE λ)

$L(\lambda) = M(\lambda) /$ SUBMODULE GEN'D BY P.V. WITH WEIGHTS $w \cdot \lambda$ $(w \neq 1.$

MAX'L PROPER SUBMODULE

WE HAVE SOME INFORMATION:

PROPOSITION: IF μ IS A PRIMITIVE WEIGHT THEN $|\mu+\rho| = |\lambda+\rho|$ WHERE $|\mu| = (\mu|\mu)^{1/2}$.

$(\ ,\ )$ = INVARIANT INNER PRODUCT ON $\mathfrak{h}^*$.

PROOF: Ω = CASIMIR ELEMENT ACT BY A SCALAR ON $M(\lambda)$:

$$(\lambda \mid \lambda + 2\rho)$$

$\gamma : \mathfrak{h} \to \mathfrak{h}^*, \rho$

$$\Omega = \sum_{\alpha \in \Delta^+} \sum_{j=1}^{\dim \mathfrak{X}_\alpha} X^j_{-\alpha} X^j_\alpha + \gamma^{-1}(\rho) + \sum H_i H^i .$$

X^j BASIS OF $\mathfrak{X}_\alpha$.

H_i BASIS OF $\mathfrak{h}$, H^i DUAL BASIS

APPLYING Ω TO v_λ, FIRST PART $\sum X^j_{-\alpha} X^j_\alpha$ ANNIHILATES IT. CALCULATED

$\sum H_i H^i$ PRODUCES $(\lambda \mid \lambda)$

$\gamma^{-1}(\rho)$ PRODUCES $(\lambda \mid 2\rho)$.

$M(\lambda)$ HAS A SUBQUOTIENT THAT IS A HW MODULE WITH HIGHEST WEIGHT μ.

$$(\lambda \mid \lambda + 2\rho) = (\mu \mid \mu + 2\rho)$$

$$(\lambda \mid \lambda + 2\rho) = |\lambda + \rho|^2 - |\rho|^2 \text{ SO}$$

$$|\lambda + \rho|^2 - |\rho|^2 = |\mu + \rho|^2 - |\rho|^2 .$$

NOTE $|w\cdot\lambda+\rho|^2 = |\lambda+\rho|^2$

$w\cdot\lambda = w(\lambda+\rho)-\rho$

SINCE W ACTION IS ISOMETRIC.

THIS FACT IS CONSISTENT WITH OUR CLAIM.

A CLOSER LOOK AT W ACTION ON ROOTS.

PROPOSITION: LET α_i BE A SIMPLE ROOT, s_i SIMPLE REFLECTION. THEN IF α IS A POSITIVE ROOT $\alpha\neq\alpha_i$ THEN $s_i(\alpha)\in\Delta^+$.

PROVED THIS LAST WEEK. REASON IS EASY:

α APPEARS IN n_+ OR n_-. AND

n_+ IS GENERATED BY $e_1,\dots,e_r$.

$n_+\hookrightarrow U(n_+)$ THE POSSIBLE WEIGHTS ARE ALL OF THE FORM $\alpha_1^{n_1}\cdots\alpha_r^{n_r}$

$U(n_+)$ HAS A PBW BASIS $e_1^{n_1}\cdots e_r^{n_r}$.

SO $\alpha=\sum n_i\alpha_i$. IF $\alpha\neq\alpha_i$ SOME $n_j\neq 0$ WITH $j\neq i$. (STEMS FROM $[e_i,e_i]=0$.)

AND ALL $n_j\geq 0$

SIMILARLY WEIGHTS OF $\mathfrak{n}_-$ HAVE ALL

$n_j \le 0$. SO

$$s_i(\alpha) = \alpha - \langle \alpha_i^\vee, \alpha \rangle \alpha_i$$

HAS ALL n_j SAME AS α EXCEPT n_i.

ONE OF THESE WEIGHTS IS POSITIVE

SO THEY ARE ALL NONNEGATIVE.

$s_i(\alpha)$ IS A WEIGHT OF $\mathfrak{n}_+$ NOT $\mathfrak{h}$ OR $\mathfrak{n}_-$.

$s_i(\rho) = \rho - \alpha_i$ THIS WOULD BE

CLEAR IF $\mathfrak{g}$ IS FINITE DIM'L. SINCE

$$\rho = \tfrac{1}{2}\alpha_i + \underbrace{\tfrac{1}{2}\sum_{\substack{\alpha\in\Delta^+ \\ \alpha\neq\alpha_i}} \alpha}_{s_i \text{ PERMUTES}} \xrightarrow{s_i} -\tfrac{1}{2}\alpha_i + \tfrac{1}{2}\sum \cdots$$

$$= \rho - \alpha_i.$$

IN KM CASE ρ IS DEFINED BY

$\langle \alpha_i^\vee, \rho \rangle = 1$ SO $s_i(\rho) = \rho - \langle \alpha_i^\vee, \rho \rangle \alpha_i$

$$= \rho - \alpha_i.$$

LENGTH FUNCTION ON W.

IF $w \in W$ DEFINE
$$\ell(w) = \left|\{\alpha \in \Delta^+ \mid w(\alpha) \in \Delta^-\}\right|.$$

NOTE $\alpha \to -w(\alpha)$ IS A BIJECTION
$$\{\alpha \in \Delta^+ \mid w(\alpha) \in \Delta^-\} \to \{\alpha \in \Delta^+ \mid w^{-1}(\alpha) \in \Delta^-\}$$
SO $\ell(w) = \ell(w^{-1})$.

GOAL:
IF $w = s_{i_1} \cdots s_{i_k}$, THIS IS CALLED REDUCED EXPRESSION IF k IS MINIMAL.

THEOREM: THE LENGTH k IN A REDUCED EXPRESSION IS $\ell(w)$.

PROPOSITION: LET $s = s_i$ BE A SIMPLE REFLECTION
$$\ell(ws) = \ell(w)+1 \text{ IF } w(\alpha_i) \in \Delta^+$$
$$= \ell(w)-1 \text{ IF } w(\alpha_i) \in \Delta^-.$$

ASSUME $w(\alpha) \in \Delta^+$. THEN I CLAIM

✓ $\{\alpha \in \Delta^+ \mid (ws)^{-1}(\alpha) \in \Delta^-\} =$ ✓

✓ $\{\alpha \in \Delta^+ \mid w^{-1}(\alpha) \in \Delta^-\} \cup w(\alpha_i)$.

LET ME SHOW LHS $\subseteq$ RHS. OTHER INCLUSION EASY.

IF $\alpha \in \Delta^+$ AND $(ws)^{-1}(\alpha) \in \Delta^-$ AND $w^{-1}(\alpha) \in \Delta^+$ THEN I CLAIM $w^{-1}(\alpha) = \alpha_i$.

INDEED $(ws)^{-1}(\alpha) = s_i\big(\underbrace{w^{-1}(\alpha)}_{\text{POS}}\big) \in \Delta^-$.

AND α_i IS THE ONLY POSITIVE ROOT IT COULD BE $\Rightarrow$ LHS $\subseteq$ RHS.

$w(\alpha) \in \Delta^+ \Rightarrow \ell(ws) = \ell(w) + 1$.

DEDUCE SECOND PART SO ASSUME $w(\alpha) \in \Delta^-$.

LET $w' = ws$ THEN $w'(\alpha_i) = w(s\alpha_i)$

$= -w(\alpha_i) \in \Delta^+$

WE CAN APPLY FIRST PART TO w' AND SEE $\ell(w) = \ell(w's) = \ell(w') + 1$, THIS CASE IS DONE TOO.

PROPOSITION: LET $s = s_i$ BE A SIMPLE REFLECTION

$\ell(ws) = \ell(w)+1$ IF $w(\alpha) \in \Delta^+$

$= \ell(w)-1$ IF $w(\alpha) \in \Delta^-$.

THEOREM: THE LENGTH k IN A REDUCED EXPRESSION IS $\ell(w)$.

PROP. IMPLIES IF $w = s_{i_1} \cdots s_{i_k}$

$\ell(w) \leq k$. TRUE BY INDUCTION.

CONVERSE IF $s_{i_1} \cdots s_{i_k}$ IS REDUCED.

LET US PROVE

PROPOSITION: IF $\ell(\overbrace{s_{i_1} \cdots s_{i_k}}^{w}) < k$

THEN

$$w = s_{i_1} \cdots \hat{s}_{i_m} \cdots \hat{s}_{i_n} \cdots s_{i_k}$$

OMIT TWO FACTORS

LET m BE THE FIRST INTEGER

SUCH THAT

$\ell(s_{i_n}\cdots \underbrace{s_{i_m}}) = \ell(s_{i_m}\cdots s_{i_n}) < n-m+1$

$\ell(s_{i_{m+1}}\cdots s_{i_n}) = n-m$

BY THE PROPOSITION

$(s_{i_{m+1}}\cdots s_{i_n})^{-1}(\alpha_{i_m}) \in \Delta^-$

$y^{-1}(\alpha_{i_m})$

PROPOSITION: LET $s = s_i$ BE A SIMPLE REFLECTION

$\ell(ws) = \ell(w)+1$ IF $w(\alpha_i) \in \Delta^+$

$= \ell(w)-1$ IF $w(\alpha_i) \in \Delta^-$.

β

THEN THERE IS A SMALLEST INTEGER u

SUCH THAT $(s_{i_{m+1}}\cdots s_{i_{u-1}})(\alpha_{i_m}) \in \Delta^+$

AND $(s_{i_{m+1}}\cdots s_{i_u})^{-1}(\alpha_{i_m}) \in \Delta^-$.

$''$

$s_{i_u}(\beta)$

SO $\beta \in \Delta^+$, $s_{i_u}(\beta) \in \Delta^- \Rightarrow \beta = \alpha_{i_u}$

RECAP: $(s_{i_{m+1}} \cdots s_{i_n})^{-1}(\alpha_{i_m}) = \alpha_{i_n}$.

LEMMA: IF α_i, α_j ARE SIMPLE AND $w(\alpha_i) = \alpha_j$ THEN $w s_i w^{-1} = s_j$.

INDEED

$$(w s_i w^{-1})(x) = w s_i\left(w^{-1}(x)\right)$$

$$= w\left(w^{-1}(x) - \langle \alpha_i^\vee, \overset{\swarrow \text{WT}}{w^{-1}(x)} \rangle \alpha_i\right)$$

$$= w\left(w^{-1}(x) - \langle w^{-1}(\alpha_i^\vee), x \rangle \alpha_i\right)$$

$$= x - \langle \alpha_j^\vee, x \rangle \alpha_j \qquad s_i(\alpha_i) = \alpha_j.$$

SO $(s_{i_{m+1}} \cdots s_{i_n})\, s_{i_n} (s_{i_{m+1}} \cdots s_{i_n})^{-1} = s_{i_m}$.

THIS IMPLIES

$$s_{i_m} \cdots s_{i_n} = \hat{s}_{i_m} s_{i_{m+1}} \cdots \hat{s}_{i_{n-1}} s_{i_n}.$$

SO $s_{i_1} \cdots s_{i_k} = s_{i_1} \cdots \hat{s}_{i_m} \cdots \hat{s}_{i_n} \cdots s_{i_k}$.

THIS MEANS IF $\ell(s_{i_1} \cdots s_{i_k}) < k$

THE EXPRESSION IS NOT REDUCED AND

IF I DISCARD TERMS IN PAIRS UNTIL

I OBTAIN A REDUCED EXPRESSION I GET

THE OTHER INEQUALITY SO

$$\ell(w) = \text{LENGTH OF A REDUCED EXPRESSION.}$$

<u>THEOREM</u> (EXERCISE 10.3 IN KAC)

$M(0)$ HAS PRIMITIVE VECTORS OF

WEIGHT $-w \cdot \rho = \rho - w(\rho)$ FOR ALL

$w \in W$.

WE SEE IN KM CASE $M(0)$ CAN HAVE

INFINITELY MANY PRIMITIVE.

(VERMA MODULES CAN HAVE ∞ LENGTH!)

$$\mathfrak{h} = \mathbb{R}^3 / \mathbb{R}(1,1,1)$$

$$(3, 1, 0) = (3, 1, 0) - \left(\frac{4}{3}, \frac{4}{3}, \frac{4}{3}\right)$$

$$\left(\frac{2}{3}, -\frac{1}{3}, -\frac{4}{3}\right)$$

$$-p = (-1, 0, 1)$$

$(2, 2, 0)$ $(3, 1, 0)$

$(2, 1, 1)$

$(1, 2, 1)$ $(1, 1, 1)$