Lecture 2: Schur Orthogonality

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In this lecture we will talk about results that are valid for compact groups, whether Lie groups or other. Compact groups include:

- Compact Lie groups such as $U(n)$ or $O(n)$;
- Finite groups;
- Profinite groups such as $GL(n, \mathbb{Z}_p)$ where $\mathbb{Z}_p$ is the ring of $p$-adic integers, or (possibly infinite) Galois groups.

For compact groups, the representation theory is very similar to finite groups. According to the Peter-Weyl theorem, every irreducible representation is finite-dimensional and unitary, and every unitary representation is a direct sum of irreducible representations. These properties fail for noncompact groups such as $GL(n, \mathbb{R})$. 
Let $G$ be a locally compact topological space with a Borel measure $\mu$. The measure $\mu$ is called regular if for every measurable subset $X$ we have

$$\mu(X) = \inf \{\mu(U) | U \supseteq X, U \text{ open}\} = \sup \{\mu(K) | K \subseteq X, K \text{ compact}\}.$$ 

A theorem that we will take for granted is that if $G$ is a locally compact topological group, then $G$ admits a regular Borel measure $d\mu_L$ that is invariant under left translation. Compact sets have finite volume in this left Haar measure.

Similarly there is a right Haar measure $d\mu_R$. 

*Haar measure*
Left and right Haar measures are the same in many important cases, but for an example where they are different, let

\[ G = \left\{ \begin{pmatrix} y & x \\ 1 & 1 \end{pmatrix} \mid x, y \in \mathbb{R}, y > 0 \right\}. \]

Then

\[ d\mu_L = y^{-2} \, dx \, dy, \quad d\mu_R = y^{-1} \, dx \, dy. \]

On the other hand, left and right Haar measures are equal for compact groups, and for many other important groups such as \( \text{GL}(n, \mathbb{R}) \) or \( \text{SL}(n, \mathbb{R}) \), \( O(n) \), \( \text{Sp}(2n, \mathbb{R}) \). The group \( G \) is \textbf{unimodular} if \( \mu_L = \mu_R \).
The modular quasicharacter

If $G$ is a locally compact group a **quasicharacter** $\chi$ is a continuous homomorphism $\chi : G \rightarrow \mathbb{C}^\times$. It is a **character** (or is **unitary**) if $|\chi(g)| = 1$.

**Proposition**

Let $G$ be a locally compact group. There is a quasicharacter $\delta : G \rightarrow \mathbb{R}_+^\times$ such that

$$d\mu_L = \delta(g) \, d\mu_R.$$ 

This $\delta$ is called the **modular quasicharacter**
Proof

Indeed, conjugation is an automorphism of $G$, so it takes left Haar measure to another left Haar measure. Thus there exists a constant $\delta(g) > 0$ such that

$$\int_G f(g^{-1}hg) \, d\mu_L(h) = \delta(g) \int_G f(h) \, d\mu_L(h).$$

From this formula it is easy to check that $\delta(g_1g_2) = \delta(g_1)\delta(g_2)$. Since $d\mu_L$ is left invariant this means

$$\int_G f(hg) \, d\mu_L(h) = \delta(g) \int_G f(h) \, d\mu_L(h)$$

so $\delta(g)\mu_L$ is a right Haar measure.
If $G$ is compact, it is natural to normalize the measure so that $\mu_L(G) = 1$.

**Proposition**

*A compact group is unimodular.*

To prove that left and right Haar measures are equal on a compact group $G$ consider note that $\delta(G)$ is a compact subgroup of $R^\times_+$. The only such compact subgroup is $\{1\}$, so $\mu_L$ is trivial and therefore $G$ is unimodular.
Viewpoint

The representation theory of a compact group is so similar to the representation theory of a finite group that we may say that the main difference is that the general compact group will have infinitely many irreducible representations.

Schur orthogonality has two aspects.

- Matrix coefficients of irreducible representations give an orthonormal basis of $L^2(G)$;
- Characters of irreducible representations give an orthonormal basis of class functions in $L^2(G)$. 
Maschke’s Theorem

Let \((\pi, V)\) be a finite-dimensional representation of the compact group \(G\). By an inner product on a vector space we mean a positive definite Hermitian form \(\langle x, y \rangle\).

**Theorem**

Then \(V\) has an inner product \(\langle , \rangle\) that is invariant under the group action, i.e.

\[
\langle \pi(g)x, \pi(g)y \rangle = \langle x, y \rangle.
\]

The representation \((\pi, V)\) is completely reducible, meaning that \(V\) is a direct sum of irreducible submodules.
Proof

To prove this, we start with an arbitrary inner product $\langle \langle , \rangle \rangle$ on $V$. We make it invariant by averaging:

$$\langle x, y \rangle = \int_G \langle \langle \pi(g)x, \pi(g)y \rangle \rangle dg.$$

It is easy to check that this is an inner product, and by averaging we have made it invariant, i.e. $\langle \pi(g)x, \pi(g)y \rangle = \langle x, y \rangle$.

Now let $W$ be a nonzero invariant subspace of $V$ of minimal dimension. Obviously $W$ is irreducible. The orthogonal complement $W^\perp$ of $W$ is also invariant and by induction it is completely reducible. Hence $V = W \oplus W^\perp$ is completely reducible.
Matrix coefficients

Fix a representation (typically irreducible) \((\pi, V)\). Let \(v \in V\) and \(L \in V^*\). The function \(\phi(g) = L(\pi(g)v)\) is called a matrix coefficient of \(\pi\). This terminology is natural, because if we choose a basis \(e_1, \ldots, e_n\), of \(V\), we can identify \(V\) with \(\mathbb{C}^n\) and represent \(g\) by matrices:

\[
\pi(g)v = \begin{pmatrix} \pi_{11}(g) & \cdots & \pi_{1n}(g) \\ \vdots & \ddots & \vdots \\ \pi_{n1}(g) & \cdots & \pi_{nn}(g) \end{pmatrix} \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix}, \quad v = \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix} = \sum_{j=1}^n a_j e_j.
\]

Then each of the \(n^2\) functions \(\pi_{ij}\) is a matrix coefficient. Indeed

\[
\pi_{ij}(g) = L_i(\pi(g)e_j),
\]

where \(L_i(\sum_j a_j e_j) = a_i\).
Representative functions

A finite linear combination of matrix coefficients (of different representations) is called a representative function.

**Proposition**

Representative functions on $G$ are continuous functions. The pointwise sum or product of two representative functions is a representative function, so they form a ring.

Representative functions are continuous since representations are continuous. If $f_1$ and $f_2$ are matrix coefficients of $(\pi_1, V_1)$ and $(\pi_2, V_2)$ then $c_1f_1 + c_2f_2$ is a matrix coefficient of $V_1 \oplus V_2$ and $f_1f_2$ is a matrix coefficient of $V_1 \otimes V_2$, from which it follows that the space of representative functions generated by matrix coefficients is closed under linear combinations and multiplication.
Representative functions (continued)

We have actions of $G$ on the space of functions on $G$ by left and right translation. Thus if $f$ is a function and $g \in G$, the left and right translates are

\[
(\lambda(g)f)(x) = f(g^{-1}x), \quad (\rho(g)f)(x) = f(xg).
\]

**Theorem**

Let $f$ be a function on $G$. The following are equivalent.

(i) The functions $\lambda(g)f$ span a finite-dimensional vector space.
(ii) The functions $\rho(g)f$ span a finite-dimensional vector space.
(iii) The function $f$ is a representative function.
Proofs

It is easy to check that if \( f \) is a matrix coefficient of a particular representation \( V \), then so are \( \lambda(g)f \) and \( \rho(g)f \) for any \( g \in G \). Since \( V \) is finite-dimensional, its matrix coefficients span a finite-dimensional vector space, Thus, (iii) implies (i) and (ii).

We show (ii) \( \implies \) (iii); (i) is similar. Suppose that the functions \( \rho(g)f \) span a finite-dimensional vector space \( V \). Then \( (\rho, V) \) is a finite-dimensional representation of \( G \), and we claim that \( f \) is a matrix coefficient of this representation. Indeed, define a functional \( L : V \rightarrow \mathbb{C} \) by \( L(\phi) = \phi(1) \). Clearly, \( L(\rho(g)f) = f(g) \), so \( f \) is a matrix coefficient, as required.
Intertwining operators

If \((\pi_1, V_1)\) and \((\pi_2, V_2)\) are representations, an intertwining operator, also known as a \(G\)-equivariant map \(T : V_1 \rightarrow V_2\) or (since \(V_1\) and \(V_2\) are sometimes called \(G\)-modules) a \(G\)-module homomorphism, is a linear transformation \(T : V_1 \rightarrow V_2\) such that

\[ T \circ \pi_1(g) = \pi_2(g) \circ T \]

for \(g \in G\). We will denote by \(\text{Hom}_\mathbb{C}(V_1, V_2)\) the space of all linear transformations \(V_1 \rightarrow V_2\) and by \(\text{Hom}_G(V_1, V_2)\) the subspace of those that are intertwining maps.


**Theorem (Schur’s lemma)**

(i) Let \((\pi_1, V_1)\) and \((\pi_2, V_2)\) be irreducible representations, and let \(T : V_1 \rightarrow V_2\) be an intertwining operator. Then either \(T\) is zero or else it is an isomorphism.

(ii) Suppose that \((\pi, V)\) is an irreducible representation of \(G\) and \(T : V \rightarrow V\) is an intertwining operator. Then there exists a scalar \(\lambda \in \mathbb{C}\) such that \(T(v) = \lambda v\) for all \(v \in V\).

This is proved the same way as for representations of finite groups.
We will consider the space $L^2(G)$ of functions on $G$ that are square-integrable with respect to the Haar measure. This is a Hilbert space with the inner product

$$
\langle f_1, f_2 \rangle_{L^2} = \int_G f_1(g) \overline{f_2(g)} \, dg.
$$

Schur orthogonality will give an orthonormal basis for $L^2(G)$ of matrix coefficients of irreducible representations.

If $(\pi, V)$ is a representation and $\langle \ , \ \rangle$ is an invariant inner product on $V$, then every linear functional is of the form $x \mapsto \langle x, v \rangle$ for some $v \in V$. Thus a matrix coefficient of $\pi$ may be written in the form $g \mapsto \langle \pi(g)w, v \rangle$. 
Averaging to obtain an intertwining operator

Lemma

Suppose that $(\pi_1, V_1)$ and $(\pi_2, V_2)$ are complex representations of the compact group $G$. Let $\langle \cdot, \cdot \rangle$ be any inner product on $V_1$. If $v_i, w_i \in V_i$, then the map $T : V_1 \longrightarrow V_2$ defined by

$$T(w) = \int_G \langle \pi_1(g)w, v_1 \rangle \pi_2(g^{-1})v_2 \, dg$$

is $G$-equivariant.

$$T(\pi_1(h)w) = \int_G \langle \pi_1(gh)w, v_1 \rangle \pi_2(g^{-1})v_2 \, dg.$$ 

The variable change $g \longrightarrow gh^{-1}$ shows that this equals $\pi_2(h)T(w)$, as required.
Schur Orthogonality

**Theorem (Schur orthogonality)**

Suppose that $(\pi_1, V_1)$ and $(\pi_2, V_2)$ are irreducible representations of the compact group $G$. Either every matrix coefficient of $\pi_1$ is orthogonal in $L^2(G)$ to every matrix coefficient of $\pi_2$, or the representations are isomorphic.

Suppose matrix coefficients $f_i : G \rightarrow \mathbb{C}$ of $\pi_i$ are not orthogonal. Define $T : V_1 \rightarrow V_2$ as above. Then

$$\langle T(w_1), w_2 \rangle \neq 0$$

equals

$$\int_G \langle \pi_1(g)w_1, v_1 \rangle \langle \pi_2(g^{-1})v_2, w_2 \rangle \, dg = \int_G \langle \pi_1(g)w_1, v_1 \rangle \overline{\langle \pi_2(g)w_2, v_2 \rangle} \, dg$$

By assumption this is nonzero so $T \neq 0$. It is an isomorphism by Schur’s lemma.
Schur Orthogonality (continued)

So matrix coefficients for nonisomorphic irreducibles are orthogonal. For two matrix coefficients of the same representation:

**Theorem (Schur orthogonality)**

Let \((\pi, V)\) be an irreducible representation of the compact group \(G\), with invariant inner product \(\langle \ , \ \rangle\). Then

\[
\int_G \langle \pi(g)w_1, v_1 \rangle \overline{\langle \pi(g)w_2, v_2 \rangle} \, dg = \frac{1}{\dim(V)} \langle w_1, w_2 \rangle \langle v_2, v_1 \rangle. \tag{2}
\]

See Theorem 2.4 in our book for the proof. It uses the same method as the first statement of Schur orthogonality, but is slightly longer; particularly showing that the proportionality constant is \(1/\dim(V)\) requires some work.
An orthonormal system

Let \((\pi_\lambda, V_\lambda)\) be representatives of the isomorphism classes of irreducible representations. Let \(d_\lambda = \dim(V_\lambda)\). For each \(V_\lambda\) choose an inner product and an orthonormal basis \(v^\lambda_i\) \((i = 1, \cdots, d_\lambda)\). Then it follows from the last two theorems that

\[
f_{i,j}^\lambda(g) = \sqrt{d_\lambda} \langle \pi_\lambda(g)v^\lambda_i, v^\lambda_j \rangle
\]

are an orthonormal system, consisting of representative functions.

**Theorem (Peter-Weyl)**

This orthonormal set is complete. That is, it spans \(L^2(G)\) as a Hilbert space.

This is proved in Chapter 4 of the book. We won’t take the time to do it in the course.
Class functions

To repeat: Schur orthogonality has two aspects.

- Matrix coefficients of irreducible representations give an orthonormal basis of $L^2(G)$;
- Characters of irreducible representations give an orthonormal basis of class functions in $L^2(G)$.

A class function $f(g)$ is one that is constant on conjugacy classes. The character of a representation is a class function.

The character $\chi$ of a representation $(\pi, V)$ is a sum of matrix coefficients of $V$, so it is a representative function.
**Orthonormality of characters**

Let \((\pi_1, V_1)\) and \((\pi_2, V_2)\) be the characters of two irreducible representations, and let \(\chi_i\) be their characters.

**Theorem**

\[
\int_G \chi_1(g) \overline{\chi_2(g)} \, dg = \begin{cases} 
1 & \text{if } \pi_1 \cong \pi_2; \\
0 & \text{otherwise}
\end{cases}
\]

If \(\pi_1\) and \(\pi_2\) are not isomorphic, then \(\chi_i\) are (sums of) matrix coefficients of nonisomorphic representations, and therefore the characters are orthogonal. If the representations are isomorphic, then some work is required to determine that

\[
\int |\chi_i(g)|^2 \, dg = 1.
\]

See Theorem 2.5 in the book.