Lecture 17: The Cauchy identity

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Recall that if $\lambda = (\lambda_1, \cdots, \lambda_n) \in \Lambda$ (the weight lattice for $U(n)$ or its complexification $GL(n, \mathbb{C})$) then $\lambda_1 \geq \cdots \geq \lambda_n$. If $\lambda_n \geq 0$ then $\lambda$ is a partition of length $\leq n$ and in this case the representation $\pi_\lambda$ is polynomial.

This means that the character $\chi_\lambda(g)$ is a polynomial $s_\lambda(z)$ in the eigenvalues $z = (z_1, \cdots, z_n)$. More generally, if $g = (g_{ij})$, all matrix coefficients of $\pi_\lambda$ are polynomials in the $g_{ij}$.

Special cases are the elementary symmetric polynomials $e_k = s_{(1^k)}$ with $k \leq n$ and the complete symmetric polynomials $h_k = s_{(k)}$ that are the characters of the exterior and symmetric powers of the standard representation.
Let $g \in \text{GL}(n, \mathbb{C})$ have eigenvalues $\gamma_1, \cdots, \gamma_n$. We will compute the character of $g$ in the modules $\wedge \mathbb{C}^n$ and $\vee \mathbb{C}^n$. Let $e_i$ be the standard basis of $\mathbb{C}^n$. The eigenvalue of $g$ on

$$e_{i_1} \wedge \cdots \wedge e_{i_k}$$

is $\alpha_{i_1} \cdots \alpha_{i_k}$ so the character is

$$\text{tr} \left( g \mid \wedge \mathbb{C}^n \right) = \sum_{k=0}^{n} \sum_{i_1 < \cdots < i_k} \alpha_{i_1} \cdots \alpha_{i_k} = \prod_{i=1}^{n} (1 + \alpha_i).$$
Symmetric powers

The computation for $\bigvee \mathbb{C}^n$ is the same, except now there are an infinite number of basis vectors and so we get a series

$$\text{tr} \left( g \mid \bigvee \mathbb{C}^n \right) = \sum_{k=0}^{n} \sum_{i_1 \leq \cdots \leq i_k} \alpha_{i_1} \cdots \alpha_{i_k} = \prod_{i=1}^{n} (1 - \alpha_i)^{-1}.$$ 

This is convergent if $|\alpha_i| < 1$. 
The Cauchy identity

Cauchy’s identity is a formal identity in combinatorics with many applications. It says

\[
\sum_{\lambda} s_\lambda(\alpha)s_\lambda(\beta) = \prod_{i=1}^{n} \prod_{j=1}^{m} (1 - \alpha_i \beta_j)^{-1}
\]

where \( \alpha = (\alpha_1, \cdots, \alpha_n) \) and \( \beta = (\beta_1, \cdots, \beta_m) \) are two sets of variables. The sum is over all partitions \( \lambda \) of length \( \leq \min(n, m) \). Alternatively, we may interpret \( s_\lambda(\alpha_1, \cdots, \alpha_n) = 0 \) if the length of \( \lambda \) is \( > n \), and then the sum may be taken over all partitions.

Both sides are convergent if \( |\alpha_i|, |\beta_i| < 1 \), or may be interpreted as a formal identity.
The symmetric algebra on $\mathbb{C}^n \otimes \mathbb{C}^m$

The Cauchy identity has an interpretation as describing the decomposition of the symmetric algebra over $\mathbb{C}^n \otimes \mathbb{C}^m$-modules into irreducible modules for $\text{GL}(n, \mathbb{C}) \times \text{GL}(m, \mathbb{C})$. The right hand side is formally the character of $g \otimes h$ on the symmetric algebra

$$\bigvee (\mathbb{C}^n \otimes \mathbb{C}^m)$$

is

$$\prod_{i=1}^{n} \prod_{j=1}^{m} (1 - \alpha_i \beta_j)^{-1},$$

where $\alpha_i$ and $\beta_j$ are the eigenvalues of $g \in \text{GL}(n, \mathbb{C})$ and $h \in \text{GL}(m, \mathbb{C})$. Indeed, this is obtained from our previous computation of the character of the symmetric algebra via the tensor product embedding $\text{GL}(n) \otimes \text{GL}(m) \rightarrow \text{GL}(nm)$. 
The Cauchy identity as a character formula

Thus the Cauchy identity is telling us that

\[ \bigvee (\mathbb{C}^n \otimes \mathbb{C}^m) = \bigoplus_{\lambda} \pi_{\lambda}^{\text{GL}(n)} \otimes \pi_{\lambda}^{\text{GL}(m)} \]

as a $\text{GL}(n) \otimes \text{GL}(m)$-module. The sum is over partitions $\lambda$ of length $\leq \min(n, m)$; or we make the convention that $\pi_{\lambda}^{\text{GL}(n)}$ is the zero representation if $\ell(\lambda) > n$.

Similarly there is a dual Cauchy identity

\[ \sum_{\lambda} s_{\lambda}(\alpha)s_{\lambda'}(\beta) = \prod_{i=1}^{n} \prod_{j=1}^{m} (1 + \alpha_i \beta_j). \]

Here $\lambda'$ is the conjugate partition. This means

\[ \bigwedge (\mathbb{C}^n \otimes \mathbb{C}^m) = \bigoplus_{\lambda} \pi_{\lambda}^{\text{GL}(n)} \otimes \pi_{\lambda'}^{\text{GL}(m)} \]
Using the involution of symmetric function theory, the Cauchy and dual Cauchy identities are equivalent. We recall the involution from Lecture 16. Let $\Lambda^{(n)}(\alpha) = \mathbb{Z}[\alpha_1, \cdots, \alpha_n]$ and let $\Lambda(\alpha)$ be the inverse limit under the homomorphisms $\Lambda^{(n+1)}(\alpha) \rightarrow \Lambda^{(n)}(\alpha)$ in which $\alpha_{n+1}$ is set to 0. The complete, elementary and Schur polynomials are compatible with this specialization and hence have images in the inverse limit. The ring

$$\Lambda = \mathbb{Z}[e_1, e_2, \cdots] = \mathbb{Z}[h_1, h_2, \cdots]$$

in terms of the elementary and complete symmetric functions. The Schur functions $s_\lambda$ that are the inverse limits of the Schur polynomials $s_\lambda$ are a $\mathbb{Z}$-basis of $\Lambda$. 
The involution

We saw in Lecture 16 that it has an involution that interchanges the $e_i$ and the $h_i$, and we deduced from Pieri’s formula that it sends $s_\lambda \rightarrow s_{\lambda'}$ where $\lambda'$ is the conjugate partition.

Assume now that the Cauchy identity is known. We will show how to derive the dual Cauchy identity. To do this rigorously, we regard the Cauchy identity as a statement in $\Lambda^{(n)}(\alpha) \otimes \Lambda^{(m)}(\beta)$:

$$\sum_{\lambda} s^{(n)}_\lambda(\alpha)s^{(m)}_\lambda(\beta) = \prod_{i=1}^{n} \sum_{k=0}^{\infty} h_k(\beta_1, \ldots, \beta_m) \alpha_i^k.$$

We wrote $s^{(m)}_\lambda(\beta)$ to remind us that this is a symmetric polynomial in $m$ variables. We used:

$$\sum_{k} h_k(\beta)t = \prod_{j=1}^{m} (1 - t\beta_j)^{-1}$$
Keep $n$ fixed but let $m \to \infty$

We want to use the involution in one of the symmetric functions (but not both). The easiest way to do this rigorously is to let $m \to \infty$ but keep $n$ fixed.

Thus let $m \to \infty$ and obtain

$$\sum_{\lambda} s_{\lambda}^{(n)}(\alpha)s_{\lambda}(\beta) = \prod_{i=1}^{n} \sum_{k=0}^{\infty} h_k(\beta)\alpha_i^k,$$

and identity in $\Lambda^{(n)}(\alpha) \otimes \Lambda(\beta)$. Here $s_{\lambda}(\beta)$ is the unique element of $\Lambda$ whose image under the projection $\Lambda(\beta) \to \Lambda^{(m)}(\beta)$ is $s_{\lambda}^{(m)}(\beta)$ for all $n$. 
Now apply the involution

Now we may apply the involution in $\Lambda(\beta)$ and obtain

$$
\sum_{\lambda} s_{\lambda}(\alpha)s_{\lambda'}(\beta) = \prod_{i=1}^{n} \sum_{k=0}^{\infty} e_k(\beta)\alpha_i^k. 
$$

We apply the projection homomorphism $\Lambda^{(m)}(\beta) \rightarrow \Lambda(\beta)$ which maps $e_k(\beta)$ to zero if $k > m$. Remembering that

$$
\sum_{k=0}^{n} e_k(\beta_1, \ldots , \beta_m)t^m = \prod (1 + \beta_j t)
$$

we obtain the dual Cauchy identity

$$
\sum_{\lambda} s_{\lambda}(\alpha)s_{\lambda'}(\beta) = \prod_{i,j} (1 + \alpha_i\beta_j).
$$
Introduction

We have proved that the Cauchy and dual Cauchy identities are equivalent. We will prove them both by different methods to demonstrate two different techniques. The method we use to prove the Cauchy identity is conceptually important but somewhat specific to this particular problem. The method we use to prove the dual Cauchy identity is widely useful, and other examples may be found in Chapter 26 and its exercises.

The following proof of the Cauchy identity was shown to me by Steve Rallis. Since this gives some real insight into why the identity is true it is worth understanding. Before we look at the details we summarize the idea.
If $G$ is any compact group then $G \times G$ acts on representative functions $\mathcal{M}$ by $(g, h)\phi(x) = \phi(h^{-1}xg)$. We have

$$\mathcal{M} = \bigoplus_{\pi} \hat{\pi} \otimes \pi$$

where the sum is over all irreducibles. For $U(n)$, so $\mathcal{M}$ can be identified with the ring $\mathcal{O}(GL(n, \mathbb{C})) = \mathbb{C}[g_{ij}, \det^{-1}]$ of regular functions on $G_{\mathbb{C}}$. We modify the action to $(g, h)\phi(x) = \phi(t^{h}xg)$.

$$\mathcal{O}(GL(n, \mathbb{C})) = \bigoplus_{\text{dominant weights } \lambda} \pi_{\lambda} \otimes \pi_{\lambda}.$$  

Discard weights that are not partitions, since these are singular on the determinant locus.

$$\mathcal{O}(\text{Mat}(n, \mathbb{C})) = \bigoplus_{\text{partitions } \lambda} \pi_{\lambda} \otimes \pi_{\lambda},$$
To prove the Cauchy identity we may assume \( n = m \). Otherwise if \( n > m \) we may obtain the \( \text{GL}(n) \times \text{GL}(m) \) Cauchy identity from the \( \text{GL}(n) \times \text{GL}(n) \) identity by setting \( \beta_{n+1}, \ldots, \beta_m \) to zero.

With \( n = m \) let \( \text{GL}(n, \mathbb{C}) \times \text{GL}(n, \mathbb{C}) \) act on the affine algebra \( \mathcal{O}(\text{GL}(n, \mathbb{C})) = \mathbb{C}[g_{ij}, \det^{-1}] \), where \( g_{ij} \) are the coordinate functions on \( g = (g_{ij}) \in \text{GL}(n, \mathbb{C}) \). We may identify this with the space of representative functions on the maximal compact subgroup \( U(n) \).

**Lemma**

*Every representative function on \( U(n) \) can be extended uniquely to an element of \( \mathcal{O}(\text{GL}(n, \mathbb{C})) = \mathbb{C}[g_{ij}, \det^{-1}] \).*
Proof

We know that every irreducible representation $\pi_\lambda$ can be extended to an analytic representation of $\text{GL}(n, \mathbb{C})$. Pick a $U(n)$-invariant bilinear form on the space $V_\lambda$ of the representation. Then we may extend a matrix coefficient

$$\langle \pi(g)v, w \rangle$$

by the same formula to an analytic function on $\text{GL}(n, \mathbb{C})$. It is an element of $\mathbb{C}[g_{ij}, \det^{-1}]$; indeed if $\lambda$ is a partition it is a polynomial in the $g_{ij}$ because the character is a polynomial, and the matrix coefficients can be obtained from the character by left and right translation, so they are polynomials too. If $\lambda$ is not a partition, then $\det^N \otimes \pi_\lambda$ is a polynomial for sufficiently large $N$, so we still know that the matrix coefficients of $\pi_\lambda$ are polynomials times $\det^{-N}$. 
Uniqueness

To prove that the extension is unique we must know that an analytic function $\phi$ on $\text{GL}(n, \mathbb{C})$ is zero if it vanishes on $U(n)$. The function $\phi \circ \exp$ would be an analytic function on $\text{Mat}_n(\mathbb{C})$ that vanishes on the real subspace $u(n)$ of real Hermitian matrices. Since $\text{Mat}_n(\mathbb{C}) = u(n) \oplus iu(n)$ this follows from the well-known fact that an analytic function on $\mathbb{C}^N$ is zero if it vanishes on $\mathbb{R}^N$. 
**Proposition**

Let $\text{GL}(n, \mathbb{C}) \times \text{GL}(n, \mathbb{C})$ act on the space

$$\mathcal{O}(\text{GL}(n, \mathbb{C})) = \mathbb{C}[g_{ij}, \det^{-1}]$$

$\text{GL}(n, \mathbb{C})$ by left and right translation:

$$(g, h)\phi(x) = \phi(h^{-1}xg).$$

Then this space is isomorphic to

$$\bigoplus_{\lambda} \hat{\pi}_\lambda \otimes \pi_\lambda.$$ 

*dominant weights $\lambda$*
Proof

As before, we identify $O(GL(n, \mathbb{C}))$ with the space $\mathcal{M}$ of representative functions on $U(n)$. If $\lambda$ is a dominant weight and $d_\lambda$ is the degree of $\pi_\lambda$, let $\mathcal{M}_\lambda$ be the $d_\lambda^2$ dimensional space of matrix coefficients of $\pi_\lambda$.

$$\mathcal{M}_\lambda \cong \hat{\pi}_\lambda \otimes \pi_\lambda$$

where $\hat{\pi}_\lambda$ is the contragredient representation. (See Exercise 2.3.) Since $\mathcal{M} = \bigoplus \mathcal{M}_\lambda$ the statement follows.
Use the involution $g \mapsto t^{g^{-1}}$

We actually want to change the action. It follows from the fact that every matrix of $\text{GL}(n, \mathbb{C})$ is conjugate to its transpose that the involution $g \mapsto t^{g^{-1}}$ turns any representation into its contragredient. Therefore with the modified action

$$(g, h)\phi(x) = \phi(t^{h}x^{g})$$

we have

$$\mathcal{O}(\text{GL}(n, \mathbb{C})) \cong \bigoplus_{\text{dominant weights } \lambda} \pi_{\lambda} \otimes \pi_{\lambda}.$$ 

The group $U(n) \times U(n)$ acts by

$$(g, h)\phi(x) = \phi(h^{-1}x^{g}).$$
Polynomial representations extend to $\text{Mat}_n(\mathbb{C})$

Now $\mathcal{O}(\text{GL}(n, \mathbb{C}))$ is the localization of the affine algebra $\mathcal{O}(\text{Mat}_n(\mathbb{C}))$ along the determinant locus. Hence $\mathcal{O}(\text{Mat}_n(\mathbb{C})) \subset \mathcal{O}(\text{GL}(n, \mathbb{C}))$ consists of those elements that are polynomials in the coordinate functions $g_{ij}$. This may be identified with the symmetric algebra on $\mathbb{C}^n \otimes \mathbb{C}^n$ by associating the coordinate function $g_{ij}$ with the generator $e_i \otimes e_j$.

We have already noted that an element of $\mathcal{M}_\lambda$ is polynomial if and only if $\lambda$ is a partition. Thus

$$\bigvee (\mathbb{C}^n \otimes \mathbb{C}^n) = \bigoplus_{\text{partitions } \lambda} \pi_\lambda \otimes \pi_\lambda.$$

The Cauchy identity follows.
We will prove the dual Cauchy identity by a different technique. This method is applicable to a variety of problems. See Chapter 26.

Let \( g \) be the Lie algebra of a compact Lie group \( G \) and \( g_C \) its complexification. Suppose we have a representation of \( G \) that we wish to decompose into irreducibles. If \( \alpha \) is a root let \( X_\alpha \in g_C \) generate the corresponding root space. Every irreducible representation has a unique (up to scalar) highest weight vector \( v \), which may be characterized by \( X_\alpha(v) = 0 \) for simple roots \( \alpha \Phi^+ \). If you can find these you know the decomposition into irreducibles.
Look for highest weight vectors

Specializing to $G = \text{GL}(n, \mathbb{C}) \otimes \text{GL}(m, \mathbb{C})$ we can seek the highest weight vectors in the action on $\bigwedge (\mathbb{C}^n \otimes \mathbb{C}^m)$. Let

$$I = \{1, 2, 3, \cdots, n\} \times \{1, 2, \cdots, m\}.$$

A basis of $\bigwedge (\mathbb{C}^n \otimes \mathbb{C}^m)$ is parametrized by subsets $S$ of $I$; given $S$ let

$$v_S = \bigwedge_{(i,j) \in S} (e_i \otimes e_j).$$

The wedge can be taken in any fixed order.

To test whether a vector $v$ is a highest weight vector we must show that $(X_{\alpha_i} \otimes I)v = 0$ for $1 \leq i < n$ and $(I \otimes X_{\alpha_j})v = 0$ for $1 \leq j < m$. 
Effect of $X_{\alpha_i}$ on basis vectors

Now $X_{\alpha_i} \otimes I$ is the endomorphism of $\mathbb{C}^n$ that sends $e_{i+1} \rightarrow e_i$ and annihilates all other basis vectors. Applying it to $v_S$ we see that $X_{\alpha_i}v_S$ may be computed as follows. If $(i + 1, j) \in S$ but $(i, j) \notin S$ let $L_{i,j}(S)$ be the set obtained from $S$ by replacing $(i + 1, j)$ by $(i, j)$. Then $X_{\alpha_i}(v_S)$ is a sum over such pairs $(i, j)$ of terms $\pm v_{L_{i,j}}(S)$.

We see that $X_{\alpha_i} \otimes I$ annihilates $v_S$ if whenever $(i + 1, j) \in S$ we have also $(i, j) \in S$. Similarly $I \otimes X_{\alpha_j}$ annihilates $v_S$ if whenever $(i, j + 1) \in S$ we also have $(i, j) \in S$. 
We may now find highest weight vectors among the $v_S$. Let $\lambda$ be a partition. Assume that the Young diagram of $\lambda$ fits in an $n \times m$ box. Then we may take

$$S_\lambda = \{(i,j)|1 \leq j \leq \lambda_i\} = \{(i,j)|1 \leq i \leq \lambda'_j\}.$$

Let $v_\lambda = v_{S_\lambda}$. As an example, suppose that $\lambda = (3, 1)$. Then $S_\lambda$ is the set of $(i,j)$ that fit in the Young diagram:

and

$$v_\lambda = (e_1 \otimes e_1) \wedge (e_1 \otimes e_2) \wedge (e_1 \otimes e_3) \wedge (e_2 \otimes e_1).$$
The weight

This vector

\[ v_\lambda = (e_1 \otimes e_1) \wedge (e_1 \otimes e_2) \wedge (e_1 \otimes e_3) \wedge (e_2 \otimes e_1). \]

is an eigenvector for the torus element

\[
\begin{pmatrix}
  t_1 \\
  t_2 \\
  \vdots \\
\end{pmatrix},
\begin{pmatrix}
  u_1 \\
  u_2 \\
  \vdots \\
\end{pmatrix}
\]

with eigenvalue \( t_1^3 t_2 \cdot u_1^2 u_2 u_3 = t^\lambda u^{\lambda'} \). More generally \( v_\lambda \) is a highest weight vector with torus eigenvalue \( t^\lambda u^{\lambda'} \) for \((t, u)\) in the product of the diagonal tori of \( GL(n) \) and \( GL(m) \).
Dual Cauchy identity

We have found a highest weight vector with highest weight \((\lambda, \lambda')\) for every partition \(\lambda\) whose Young diagram fits in the \(n \times m\) box. This vector generates a submodule of \(\bigwedge (\mathbb{C}^n \otimes \mathbb{C}^m)\) isomorphic to \(\pi_\lambda \otimes \pi_{\lambda'}\). Moreover, a bit more work will show there are no other highest weight vectors. This proves

\[
\bigwedge (\mathbb{C}^n \otimes \mathbb{C}^m) \cong \bigoplus_{\lambda} \pi_\lambda \otimes \pi_{\lambda'},
\]

the dual Cauchy identity.
Correspondences

The Cauchy identity gives what Howe called a correspondence, an important notion capturing earlier work in representation theory and automorphic forms, such as the theory of theta functions. This is a relationship between representations of different groups, in this case $\text{GL}(n, \mathbb{C})$ and $\text{GL}(m, \mathbb{C})$. We will not formalize this notion today, but we point out its relevance. Suffice it to say that the Cauchy identity is a reflection of a “correspondence” between representations of $\text{GL}(n)$ and $\text{GL}(m)$. (It is the simplest case of Howe duality.) We will call this the Cauchy correspondence without warranting that anyone else uses this terminology.
We defined the Littlewood-Richardson coefficients $c_{\lambda \mu}^{\nu}$ to be the structure coefficients of the ring of symmetric functions:

$$s_{\lambda}s_{\mu} = \sum_{\nu} c_{\lambda \mu}^{\nu} s_{\nu}.$$

This is defined for partitions $\lambda, \mu, \nu$ which today we will regard as dominant weights for $GL(n)$ if $n$ is sufficiently large. Since the Schur functions are essentially the characters of $GL(n, \mathbb{C})$, this means, provided that $n$ is sufficiently large

$$\pi_{\lambda}^{GL(n)} \otimes \pi_{\mu}^{GL(n)} \cong \bigoplus_{\nu} c_{\lambda \mu}^{\nu} \pi_{\nu}^{GL(n)}.$$
Another occurrence of Littlewood coefficients

It turns out that these coefficients arise in a seemingly different way as a **Levi branching rule**. Let $p, q$ be two sufficiently large integers. Then $GL(p) \times GL(q)$ is a Levi subgroup of $GL(p + q)$, and we may restrict $\pi^{GL(p+q)}_{\nu}$ to $GL(p) \times GL(q)$ and decompose it into irreducibles.

Since this is a different problem than the tensor product decomposition, it is surprising that the branching multiplicities are again Littlewood-Richardson coefficients.
Levi branching

We will prove:

**Proposition**

*We have*

\[
\pi^\text{GL}(p+q)\big|_{\text{GL}(p) \times \text{GL}(q)} \cong \sum_{\lambda,\mu} c_{\lambda,\mu}^{\nu} \pi^\text{GL}(p) \otimes \pi^\text{GL}(q).
\]

In terms of symmetric functions, this means

\[
s_{\nu}(\alpha, \beta) = \sum_{\lambda,\nu} c_{\lambda,\mu}^{\nu} s_{\lambda}(\alpha)s_{\mu}(\beta).
\]
The see-saw

We will deduce the fact that the tensor product rule for $\text{GL}(n)$ and the Levi branching rule for $\text{GL}(p + q) \rightarrow \text{GL}(p) \times \text{GL}(q)$ from the Cauchy identity. The following “see-saw” diagram (in Kudla’s terminology) is a schematic for this proof. Vertical lines are inclusions, and diagonal lines are instances of the Cauchy correspondence.
Proof

We let $d_{\lambda,\mu}^\nu$ be the coefficients in the
$\text{GL}(p+q) \rightarrow \text{GL}(p) \times \text{GL}(q)$ branching rule, so

$$\pi_{\nu}^{\text{GL}(p+q)}|_{\text{GL}(p) \times \text{GL}(q)} \cong \sum_{\lambda,\mu} d_{\lambda,\mu}^\nu \pi_{\lambda}^{\text{GL}(p)} \otimes \pi_{\mu}^{\text{GL}(q)},$$

or

$$s_{\nu}(\alpha, \beta) = \sum_{\lambda,\mu} d_{\lambda,\mu}^\nu s_{\lambda}(\alpha)s_{\mu}(\beta).$$

What we have to show is that $d_{\lambda,\mu}^\nu = c_{\lambda,\mu}^\nu$. We work with 3 sets of variables, $\gamma_1, \cdots, \gamma_n$ representing the eigenvalues of $g \in \text{GL}(n, \mathbb{C})$; $\alpha_1, \cdots, \alpha_p$ and $\beta_1, \cdots, \beta_q$ representing eigenvalues of elements of $\text{GL}(p, \mathbb{C})$ and $\text{GL}(q, \mathbb{C})$, respectively.
Proof (continued)

We have

\[
\prod_{i=1}^{n} \left( \prod_{j=1}^{p} (1 - \gamma_i \alpha_j)^{-1} \right) \prod_{k=1}^{q} (1 - \gamma_i b_k)^{-1} = \sum_{\nu} s_{\nu}(\gamma) s_{\nu}(\alpha, \beta)
\]

\[
= \sum_{\lambda, \mu, \nu} d_{\lambda \mu}^{\nu} s_{\nu}(\gamma) s_{\lambda}(\alpha) s_{\mu}(\beta).
\]

On the other hand, this also equals

\[
\left[ \prod_{i} \prod_{j} (1 - \gamma_i \alpha_j)^{-1} \right] \left[ \prod_{i} \prod_{k} (1 - \gamma_i b_k)^{-1} \right] =
\]

\[
\sum_{\lambda} s_{\lambda}(\gamma) s_{\lambda}(\alpha) \sum_{\mu} s_{\mu}(\gamma) s_{\mu}(\beta) = \sum_{\lambda, \mu, \nu} c_{\lambda \mu}^{\nu} s_{\nu}(\gamma) s_{\lambda}(\alpha) s_{\mu}(\beta).
\]

Comparing coefficients of \( s_{\nu}(\gamma) s_{\lambda}(\alpha) s_{\mu}(\beta) \) gives \( d_{\lambda \mu}^{\nu} = c_{\lambda \mu}^{\nu} \).
The \( \text{GL}(n) \Rightarrow \text{GL}(n - 1) \) branching rule

There is a special case where we know the Littlewood-Richardson coefficients, namely Pieri’s rule. This is enough to give us the branching rule from \( \text{GL}(n, \mathbb{C}) \) to \( \text{GL}(n - 1, \mathbb{C}) \).

The group \( \text{GL}(n - 1) \) is embedded in \( \text{GL}(n) \) via

\[
g \mapsto \begin{pmatrix} g & \vphantom{1} \\ 0 & 1 \end{pmatrix}.
\]

We will prove that the \( \text{GL}(n) \Rightarrow \text{GL}(n - 1) \) branching rule is multiplicity free. This means that \( \pi^\text{GL}(n)_\lambda \) when restricted to \( \text{GL}(n - 1, \mathbb{C}) \) decomposes into irreducibles \( \pi^\text{GL}(n-1)_\mu \), with not representation have a multiplicity \( > 1 \).
The branching rule

Let $\lambda = (\lambda_1, \cdots, \lambda_n)$ and $\mu = (\mu_1, \cdots, \mu_{n-1})$ be partitions of lengths $\leq n, n - 1$ respectively, or more generally dominant weights for $\text{GL}(n)$ and $\text{GL}(n - 1)$. We say they interleave if

$$\lambda_1 \geq \mu_1 \geq \lambda_2 \geq \mu_2 \geq \cdots \geq \lambda_n.$$ 

It is easy to see that this is equivalent to $\lambda/\mu$ being a skew shape that is a horizontal strip.

Theorem

Let $\lambda$ be a dominant weight for $\text{GL}(n)$. Then $\pi^{\text{GL}(n)}|_{\text{GL}(n-1)}$ decomposes into irreducibles with multiplicity one. $\pi^\mu_{\text{GL}(n-1)}$ is one of these if and only if $\lambda$ and $\mu$ interleave.
Proof

Tensoring with a power of the determinant, we may assume $\lambda$ is a partition and $\pi_\lambda$ is a polynomial representation.

We know the branching rule for $\text{GL}(n)$ to $\text{GL}(n - 1) \times \text{GL}(1)$. Dominant weights for $\text{GL}(1)$ are necessarily of the form $(k)$. Since Schur polynomials are homogeneous, if $\pi_{\mu}^{\text{GL}(n-1)}$ occurs in $\pi_{\lambda}^{\text{GL}(n)}|_{\text{GL}(n-1)}$ then $\pi_{\mu} \otimes \pi_{(k)}$ occurs in $\pi_{\lambda}^{\text{GL}(n)}|_{\text{GL}(n-1) \times \text{GL}(1)}$ for a unique $k$, namely $k = |\lambda| - |\mu|$. Therefore the multiplicity is $c_{\mu,(k)}^\lambda$, which is known by Pieri’s formula to be 1 if $\lambda/\mu$ is a horizontal strip, and 0 otherwise.