Lie Subgroups of $\text{GL}(n, \mathbb{C})$

Exercises

Exercise 5.1. Show that $\text{O}(n, m)$ is the group of $g \in \text{GL}(n + m, \mathbb{R})$ such that $g J_1 g^t = J_1$, where

$$J_1 = \begin{pmatrix} I_n & -I_m \\ 0 & 0 \end{pmatrix}.$$

Exercise 5.2. If $F = \mathbb{R}$ or $\mathbb{C}$, let $\text{O}_J(F)$ be the group of all $g \in \text{GL}(N, F)$ such that $g J^t g = J$, where $J$ is the $N \times N$ matrix

$$J = \begin{pmatrix} \cdots & 1 \\ \cdot & \cdots \\ 1 & \cdots \end{pmatrix}. \quad (5.1)$$

Show that $\text{O}_J(\mathbb{R})$ is conjugate in $\text{GL}(N, \mathbb{R})$ to $\text{O}(n, n)$ if $N = 2n$ and to $\text{O}(n + 1, n)$ if $N = 2n + 1$. [Hint: Find a matrix $\sigma \in \text{GL}(N, \mathbb{R})$ such that $\sigma J^t \sigma = J_1$, where $J$ is as in the previous exercise.]

Exercise 5.3. Let $J$ be as in the previous exercise, and let

$$\sigma = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & \cdots & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & \cdots & -\frac{1}{\sqrt{2}} \\ \vdots & \vdots & \ddots & \vdots \\ -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & \cdots & \frac{1}{\sqrt{2}} \end{pmatrix},$$

with all entries not on one of the two diagonals equal to zero. If $N$ is odd, the middle element of this matrix is 1.

(i) Check that $\sigma J \sigma^{-1} = J$, with $J$ as in (5.1). With $\text{O}_J(F)$ as in Example ??, deduce that $\sigma^{-1} \text{O}_J(\mathbb{C}) \sigma = \text{O}(N, \mathbb{C})$. Why does the same argument not prove that $\sigma^{-1} \text{O}_J(\mathbb{R}) \sigma = \text{O}(n, \mathbb{R})$?

(ii) Check that $\sigma$ is unitary. Show that if $g \in \text{O}_J(\mathbb{C})$ and $h = \sigma^{-1} g \sigma$, then $h$ is real if and only if $g$ is unitary.
(iii) Show that the group $O_J(N, \mathbb{C}) \cap U(N)$ is conjugate in $GL(N, \mathbb{C})$ to $O(N)$.

**Exercise 5.4.** Let $V_1$ and $V_2$ be vector spaces over a field $F$, and let $q_i$ be a quadratic form on $V_i$ for $i = 1, 2$. The quadratic spaces are called equivalent if there exists an isomorphism $l : V_1 \rightarrow V_2$ such that $q_1 = q_2 \circ l$.

(i) Show that over a field of characteristic not equal to 2, any quadratic form is equivalent to $\sum a_i x_i^2$ for some constants $a_i$.

(ii) Show that, if $F = \mathbb{C}$, then any quadratic space of dimension $n$ is equivalent to $\mathbb{C}^n$ with the quadratic form $x_1^2 + \cdots + x_n^2$.

(iii) Show that, if $F = \mathbb{R}$, then any quadratic space of dimension $n$ is equivalent to $\mathbb{R}^n$ with the quadratic form $x_1^2 + \cdots + x_r^2 - x_{r+1}^2 - \cdots - x_n^2$ for some $r$.

**Exercise 5.5.** Compute the Lie algebra of $Sp(2n, \mathbb{R})$ and the dimension of the group.

Let $\mathbb{H} = \mathbb{R} \oplus \mathbb{R} i \oplus \mathbb{R} j \oplus \mathbb{R} k$ be the ring of quaternions, where $i^2 = j^2 = k^2 = -1$ and $ij = -ji = k$, $jk = -kj = i$, $ki = -ik = j$. Then $\mathbb{H} = \mathbb{C} \oplus \mathbb{C}j$. If $x = a + bi + cj + dk \in \mathbb{H}$ with $a, b, c, d$ real, let $\overline{x} = a - bi - cj - dk$. We have $\overline{xy} = \overline{y}x$. If $u \in \mathbb{C}$, then $juj^{-1} = \pi$. The group $GL(n, \mathbb{H})$ consists of all $n \times n$ invertible quaternion matrices.

**Exercise 5.6.** Show that there is a ring isomorphism $Mat_n(\mathbb{H}) \rightarrow Mat_{2n}(\mathbb{C})$ with the following description. Any $A \in Mat_n(\mathbb{H})$ may be written uniquely as $A_1 + A_2j$ with $A_1, A_2 \in Mat_n(\mathbb{C})$. The isomorphism in question maps

$$A_1 + A_2j \mapsto \begin{pmatrix} A_1 & A_2 \\ -A_2 & A_1 \end{pmatrix}.$$

**Exercise 5.7.** Recall that the intersection of $Sp(2n, \mathbb{C})$ and $U(2n)$ is the group denoted $Sp(2n)$. This exercise gives a concrete realization of this group.

(i) Let $g \in GL(2n, \mathbb{C})$. Show that $-JgJ = \overline{g}$ if and only if $g$ is of the form

$$\begin{pmatrix} A_1 & A_2 \\ -A_2 & A_1 \end{pmatrix},$$

with $A_1, A_2$ complex, that is, if $g$ is in the image of the map from Exercise 5.6.

(ii) Show that $Sp(2n)$ is contained in the image of $Mat_n(\mathbb{H})$ under the map from Exercise 5.6. Moreover if $g \in GL(2n, \mathbb{C})$ is of the form (5.2) then $g \in Sp(2n)$ if and only if the quaternionic matrix $A$ satisfies $A^{-1}A = I$. This is equivalent to the conditions $A_1^t \overline{A_1} + A_2^t \overline{A_2} = I$ and $A_1^t A_2 = A_2^t A_1$. [Hint: Note that $g \in Sp(2n, \mathbb{C})$ if and only if $g^{-1} = -JgJ$ where $J$ is as in Example ??, while $g \in U(2n)$ if and only if $g^{-1} = \overline{g}$.]  

**Exercise 5.8.** Show that the groups $SO(2)$ and $SU(2)$ may be identified with the groups of matrices

$$\left\{ \begin{pmatrix} a & b \\ -\overline{b} & \overline{a} \end{pmatrix} \mid a, b \in F, |a|^2 + |b|^2 = 1 \right\},$$

where $F = \mathbb{R}$ and $\mathbb{C}$, respectively.
Exercise 5.9. The group SU(1, 1) is by definition the group of \( g \in \text{SL}(2, \mathbb{C}) \) such that
\[
g \cdot J \cdot \overline{g} = J, \quad J = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.
\]

(i) Show that SU(1, 1) consists of all elements of SL(2, \( \mathbb{C} \)) of the form
\[
\begin{pmatrix} a & b \\ \overline{b} & \overline{a} \end{pmatrix}, \quad |a|^2 - |b|^2 = 1.
\]

(ii) Show that the Lie algebra \( su(1, 1) \) of SU(1, 1) consists of all matrices of the form
\[
\begin{pmatrix} ai & b \\ \overline{b} & -ai \end{pmatrix}
\]
with \( a \) real.

(iii) Let \( C = \frac{1}{\sqrt{2i}} \begin{pmatrix} 1 & \overline{-i} \\ 1 & i \end{pmatrix} \in \text{SL}(2, \mathbb{C}) \). This element is sometimes called the Cayley transform. Show that \( C \cdot \text{SL}(2, \mathbb{R}) \cdot C^{-1} = \text{SU}(1, 1) \) and \( C \cdot \mathfrak{sl}(2, \mathbb{R}) \cdot C^{-1} = \mathfrak{su}(1, 1) \).