The Weyl Character Formula

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Although we are mostly following Humphrey's book, in the later chapters a simplification is possible. The most important theorem in Lie theory is undoubtedly the Weyl Character formula for the character of an irreducible representation, which is Theorem 24.3 in Humphreys. In these notes we will explain Kac's proof of the Weyl character formula.

The proof in Humphreys follows an argument due to Bernstein, Gelfand and Gelfand (1971). Humphrey's book came out in 1972. In 1974, Victor Kac found a simplification of the BGG proof which appears in Chapter 10 of his book *Infinite-dimensional Lie algebras.* This proof was not available to Humphreys, who (following BGG) relied on a result of Harish-Chandra on the center $\mathfrak{Z} = Z(U(\mathfrak{g}))$ of the universal enveloping algebra. Kac' argument avoids the theorem of Harish-Chandra in place of a clever argument using only a single element of $\mathfrak{Z}$, the Casimir element.

We will slightly change the notation and terminology from Humphreys; as he notes in the Afterward to the 1994 edition, his δ is universally denoted ρ, and the terminology and notation for Verma modules differs from his usage. We will follow currently standard notation and terminology on these points.

1 Notation

Let $\mathfrak{g}$ be a semisimple Lie algebra, $\mathfrak{h}$ a maximal toral subalgebra, $\Phi \subset \mathfrak{h}^*$ the root system, which we partition into positive and negative roots. The Killing form κ restricted to $\mathfrak{h}$ is nondegenerate by Humphreys Corollary 8.2. As in Section 8.2 we will associate to $\phi \in \mathfrak{h}^*$ an element $t_\phi \in \mathfrak{h}$ such that

$$\kappa(t_\phi, h) = \phi(h) \quad \text{for } h \in \mathfrak{h}. \tag{1}$$

Then we will define an inner product on $\mathfrak{h}^*$ which we will denote

$$(\lambda|\mu) = \kappa(t_\lambda, t_\mu). \tag{2}$$

If $\alpha \in \Phi \cup \{0\}$ we will denote

$$\mathfrak{g}_\alpha = \{x \in \mathfrak{g} | [h, x] = \alpha(h)x \text{ for } h \in \mathfrak{h}\}.$$

If $\alpha = 0$, then $\mathfrak{g}_\alpha = \mathfrak{h}$. On the other hand, if $\alpha \in \Phi$, then $\mathfrak{g}_\alpha$ is one-dimensional. We have the decomposition

$$\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha \in \Phi} \mathfrak{g}_\alpha. \tag{3}$$

By Humphreys Proposition 8.3(c) we have, for $x \in \mathfrak{g}_\alpha$ and $y \in \mathfrak{g}_{-\alpha}$

$$[x, y] = \kappa(x, y) t_\alpha \tag{4}$$

As in Proposition 8.3 (g) of Humphreys we will denote by

$$h_\alpha = \frac{2t_\alpha}{(\alpha|\alpha)}. \tag{5}$$

Thus if $x_\alpha \in \mathfrak{g}_\alpha$ we have $[h_\alpha, x_\alpha] = 2x_\alpha$. If

$$\alpha^\vee = \frac{2\alpha}{(\alpha|\alpha)}$$

then for $\lambda \in \mathfrak{h}^*$ we have $(\alpha^\vee|\lambda) = \lambda(h_\alpha)$. Either $\alpha^\vee$ or h_α is called a *coroot*. They are really the same thing if we identify $\mathfrak{h}$ with its double dual $\mathfrak{h}^{**}$.

We will denote by ρ half the sum of the positive roots. Humphreys denotes this δ, but the notation ρ is now universally used by everyone. If $\alpha \in \Phi$ we will denote by r_α the reflection

$$r_\alpha(x) = x - (x|\alpha^\vee)\alpha.$$

If α is a simple root, we will also use the notation s_α for r_α. We have proved that s_α maps α to its negative and permutes the remaining positive roots. Therefore $s_\alpha(\rho) = \rho - \alpha$ and so

$$(\rho|\alpha^\vee) = 1 \tag{6}$$

for all simple roots α.

An element λ of $\mathfrak{h}^*$ is called an *integral weight* if $(\lambda|\alpha^\vee) \in \mathbb{Z}$ for all $\alpha \in \Phi^+$, or equivalently, for all simple roots α. The integral weights form a lattice

$$\Lambda = \left\{x \in V | (\alpha^\vee|x) \in \mathbb{Z} \text{ for } \alpha \in \Phi^+\right\},$$

called the *weight lattice*. We call $\lambda \in \mathfrak{h}^*$ *dominant* if $(\lambda|\alpha^\vee) \geqslant 0$ for all $\alpha \in \Phi^+$ (or equivalently for simple roots α). We call λ *strongly dominant* if $(\lambda|\alpha^\vee) > 0$. Thus by (6), the Weyl vector ρ is a strongly dominant integral weight.

Here are a couple of important properties of the Weyl group action. Let V be the $\mathbb{R}$-span of Φ in $\mathfrak{h}^*$. The inner product $(\ |\ )$ makes V into a Euclidean space, and $\mathfrak{h}^* = V + iV$. The set

$$\mathcal{C}^+ = \left\{x \in V | (\alpha^\vee|x) \geqslant 0 \text{ for } \alpha \in \Phi^+\right\}$$

is called the *positive Weyl chamber*. Thus the dominant weights are the ones in $\mathcal{C}^+$.

Proposition 1. *The positive Weyl chamber is a fundamental domain for the action of the Weyl group: if $x \in V$ there is a unique element of $\mathcal{C}^+$ in the W orbit of x.*

Proof. See Bump, *Lie Groups,* Second edition, Proposition 20.11. □

Proposition 2. *Let λ be a dominant, integral weight and let $w \in W$. Then $\lambda \succcurlyeq w\lambda$.*

Proof. See Bump, *Lie Groups,* Second edition, Proposition 22.3. □

2 Highest weight modules

Even though our real interest is in finite-dimensional modules, we will consider modules that are not necessarily finite-dimensional.

Let V be a $\mathfrak{g}$-module. For $\lambda \in \mathfrak{h}^*$ we denote the *weight space*

$$V_\lambda = \{v \in V | h \cdot v \in \lambda(h) v \text{ for } h \in \mathfrak{h}\}.$$

We will say that V is $\mathfrak{h}$-*diagonalizable* if V is the algebraic direct sum of the V_λ.

Proposition 3. *If V is $\mathfrak{h}$-diagonalizable, then so is any submodule or quotient module.*

Kac. Let $U \subseteq V$ be an submodule. We must show that an element of U may be expressed as a finite linear sum of $u_\lambda \in U_\lambda$. Since V has a weight space decomposition, we may write u as a sum of $u_\lambda \in V_\lambda$, and the problem is then to show that $u_\lambda \in U$. There exist a finite number of λ_i such that

$$u = \sum_{i=1}^{m} u_{\lambda_i},$$

and we choose $h \in \mathfrak{h}$ such that the values $\lambda_i(h)$ are all distinct. Then for $j = 0, \cdots, m-1$

$$h^j \cdot u = \sum \lambda_i(h)^j \, u_{\lambda_i} \in \mathfrak{h}.$$

The $m \times m$ matrix $\{\lambda_i(h)^j\}$ is invertible since its determinant is a Vandermonde determinant. Applying the inverse to this shows that each $u_{\lambda_i} \in U$, as required.

This proves that a submodule of a $\mathfrak{h}$-diagonalizable module is diagonalizable. It follows that the same is true for quotient modules, with $(V/U)_\lambda = V_\lambda / U_\lambda$. □

We will work exclusively with diagonalizable modules with $\dim(V_\lambda) < \infty$ for all $\lambda \in \mathfrak{h}^*$. We will define the *support* $\operatorname{supp}(V) = \{\lambda \in \mathfrak{h}^* | V_\lambda \neq 0\}$.

Let $U(\mathfrak{g})$ be the universal enveloping algebra. Let $\mathfrak{n}^+$ be the nilpotent subalgebra of $\mathfrak{g}$ generated by the $\mathfrak{g}_\alpha$ ($\alpha \in \Phi^+$), and let $\mathfrak{n}^-$ be the subalgebra generated by the $\mathfrak{g}_\alpha$ with $\alpha \in \Phi^-$. Then clearly we have the *triangular decomposition*

$$\mathfrak{g} = \mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}^+.$$

Lemma 4. *We have $U(\mathfrak{g}) \cong U(\mathfrak{n}^-) \otimes U(\mathfrak{h}) \otimes U(\mathfrak{n}^-)$ in the sense that the multiplication map*

$$U(\mathfrak{n}^-) \times U(\mathfrak{h}) \times U(\mathfrak{n}^+) \longrightarrow U(\mathfrak{g})$$

induces a vector space isomorphism $U(\mathfrak{n}^-) \otimes U(\mathfrak{h}) \otimes U(\mathfrak{n}^+) \longrightarrow U(\mathfrak{g})$.

Proof. This follows from the Poincaré-Birkhoff-Witt theorem (PBW) together with the triangular decomposition (2). Namely, if $\{x_i\}$ is a basis for $\mathfrak{g}$, then PBW asserts that a basis for $U(\mathfrak{g})$ consists of all elements of the form

$$x_1^{k_1} \cdots x_d^{k_d}, \qquad 0 \leqslant x_i \in \mathbb{Z}.$$

Now we take the basis in a particular way, where its first $\frac{1}{2}|\Phi|$ elements are a basis for $\mathfrak{n}^-$, the next ℓ elements are a basis for $\mathfrak{h}$, and the last $\frac{1}{2}|\Phi|$ elements are a basis for $\mathfrak{n}^+$. Then the element $x_1^{k_1} \cdots x_d^{k_d}$ factors uniquely as a product abc where a runs through a basis of $U(\mathfrak{n}^-)$, b runs through a basis of $U(\mathfrak{h})$ and c runs through a basis of $U(\mathfrak{n}^+)$. □

We will call a vector $v \in V$ a *highest weight vector* of weight λ if $v \in V_\lambda$ and if $x_\alpha v = 0$ for $\alpha \in \Phi^+$. (Humphreys calls such v a *maximal vector*.) We will call V a *highest weight module* of weight λ if it is generated by a highest weight vector $v \in V_\lambda$. (Humphreys calls a highest weight module a *standard cyclic module*.)

We recall the partial order $\succcurlyeq$ on $\mathfrak{h}^*$ introduced in Section 20.2 (page 108) of Humphreys: we write $\lambda \succcurlyeq \mu$ if $\lambda - \mu$ can be expressed as a sum of positive roots; that is

$$\lambda - \mu = \sum_{\alpha \in \Phi^+} k_\alpha \alpha, \qquad k_\alpha \in \mathbb{N}.$$

(Here $\mathbb{N} = \{0, 1, 2, 3, \cdots\}$.)

Proposition 5. *Suppose that $v \in V$ is a highest weight vector. Then the $\mathfrak{g}$-submodule $U(\mathfrak{g})v$ generated by v equals $U(\mathfrak{n}^-)v$. The weight space $V_\mu = 0$ unless $\mu \preccurlyeq \lambda$. We have* $\dim(V_\lambda) = 1$.

Proof. We note that any element of $U(\mathfrak{n}^+)$ may be written as a constant times an element of the left ideal $U(\mathfrak{n}^+)\mathfrak{n}^+$; but this ideal annihilates v so $U(\mathfrak{n}^+)v = \mathbb{C}v$. Similarly $U(\mathfrak{h})v = \mathbb{C}v$ since $v \in V_\lambda$. By Lemma 4, $U(\mathfrak{g})v = U(\mathfrak{n}^-)U(\mathfrak{h})U(\mathfrak{n}^+)v = U(\mathfrak{n}^-)v$.

Consider the basis $\{x_{-\alpha}\}$ $(\alpha \in \Phi^+)$ of $\mathfrak{n}^-$ with $x_{-\alpha} \in \mathfrak{g}_{-\alpha}$. Using a fixed order on Φ^+, the elements $\prod_{\alpha \in \Phi^+} x_{-\alpha}^{k_\alpha}$ are a PBW basis of $U(\mathfrak{n}^-)$. Since $x_{-\alpha}$ maps V_μ to $V_{\mu-\alpha}$,

$$\prod_{\alpha \in \Phi^+} x_{-\alpha}^{k_\alpha} v \in V_\mu, \qquad \mu = \lambda - \sum_{\alpha \in \Phi^+} k_\alpha \alpha,$$

so $\mu \preccurlyeq \lambda$. Unless all $k_\alpha = 0$, μ is strictly $\prec \lambda$, so V_λ is one-dimensional. □

Proposition 6. *Let V be a highest weight module with highest weight λ. A submodule U of V is proper if and only if $U \cap V_\lambda = 0$.*

Proof. Since $\dim(V_\lambda) = 1$, if $U \cap V_\lambda \neq 0$ then $V_\lambda \subseteq U$ and then since V_λ generates V, it is clear that $U = V$. On the other hand if $U \cap V_\lambda = 0$ then clearly U is proper. □

Proposition 7. *Let V be a highest weight module with highest weight λ. Then V has a unique maximal proper submodule. Moreover V has a unique irreducible quotient.*

Proof. Let Σ be the set of proper submodules of V, and let

$$W = \sum_{U \in \Sigma} U.$$

By Proposition 3 each $U \in \Sigma$ is diagonalizable, so evidently for $\mu \in \mathfrak{h}^*$

$$W_\mu = \sum_{U \in \Sigma} U_\mu.$$

We apply this with $\mu = \lambda$. Since $U \in \Sigma$ is proper, $U_\lambda = 0$ by Proposition 6, and so $W_\lambda = 0$. This shows that W is proper. We have proved that W is the unique maximal proper submodule of V, and consequently V/W is the unique irreducible quotient. □

Theorem 8. *Let $\lambda \in V^*$. There is a highest weight module $M = M(\lambda)$ with highest weight vector $m \in M_\lambda$ with the following universal property. If V is another highest weight module with highest weight λ and if $v \in V_\lambda$, then there is a unique $\mathfrak{g}$-module homomorphism $M \longrightarrow V$ mapping $m \longrightarrow v$. The map $\xi \longmapsto \xi \cdot v$ is vector space isomorphism $U(\mathfrak{n}^-) \longrightarrow M$.*

Proof. Note that since $\mathfrak{h}$ normalizes $\mathfrak{n}^+$, $\mathfrak{b} = \mathfrak{h} \oplus \mathfrak{n}^+$ is a subalgebra of $\mathfrak{g}$, the "Borel subalgebra." As in Lemma 4, $U(\mathfrak{g}) \cong U(\mathfrak{n}_-) \otimes U(\mathfrak{b})$, that is, the multiplication map $U(\mathfrak{n}_-) \times U(\mathfrak{b}) \longrightarrow U(\mathfrak{g})$ induces a vector space isomorphism $U(\mathfrak{n}_-) \otimes U(\mathfrak{b}) \to U(\mathfrak{g})$. This result is a simple consequence of this fact.

To elaborate, regarding $\mathbb{C}$ as a one-dimensional abelian Lie algebra, we have a Lie algebra homomorphism $\theta_\lambda : \mathfrak{b} \longrightarrow \mathbb{C}$ that maps $H \in \mathfrak{h}$ to $\lambda(\mathfrak{h})$, and $\mathfrak{n}^+$ to zero. Thus let $H_1, \cdots, H_\ell$ be a basis of $\mathfrak{h}$ and x_α ($\alpha \in \Phi^+$) be a basis of $\mathfrak{n}^+$. By the PBW theorem, the elements

$$H_1^{k_1} \cdots H_\ell^{k_\ell} \prod_{\alpha \in \Phi^+} x_\alpha^{k_\alpha}$$

with k_i and k_α nonnegative integers are a basis for $U(\mathfrak{b})$. It is understood that in the product $\prod x_\alpha^{k_\alpha}$ the roots $\alpha \in \Phi^+$ are taken in a fixed definite order. We then have

$$\theta_\lambda \left(H_1^{k_1} \cdots H_\ell^{k_\ell} \prod x_\alpha^{k_\alpha} \right) = \begin{cases} \prod \lambda(H_i)^{k_i} & \text{if all } k_\alpha = 0, \\ 0 & \text{if any } k_\alpha > 0. \end{cases}$$

Now let J_ψ be the left ideal generated by $\xi - \theta_\lambda(\xi)$ for $\xi \in \mathfrak{b}$. Let $M = M(\lambda) = U(\mathfrak{g})/J_\psi$, and let m be the image of $1 \in U(\mathfrak{g})$ in $M(\lambda)$.

It is clear from the PBW theorem that $Hv = \lambda(H)m$ for $H \in \mathfrak{h}$, while $\mathfrak{n}^+ v = 0$, and moreover from $U(\mathfrak{g}) \cong U(\mathfrak{n}^-) \otimes U(\mathfrak{b})$, it is clear that every element of $M(\lambda)$ may be written uniquely as $\eta \cdot v$ for $\eta \in U(\mathfrak{n}^-)$.

Now let us verify the universal property. Let V be a highest weight module with weight λ, and let $v_\lambda \in V_\lambda$ be a generator. Then we have a surjective $U(\mathfrak{g})$-module homomorphism $U(\mathfrak{g}) \longrightarrow V$ sending ξ to $\xi \cdot v$, and since $\beta \cdot v = \theta_\lambda(\beta) v$ for $\beta \in \mathfrak{b}$, J_ψ is in the kernel. Thus this homomorphism factors uniquely through $U(\mathfrak{g})/J_\psi = M(\lambda)$. □

Corollary 9. *Let $\lambda \in \mathfrak{h}^*$. Up to isomorphism, $\mathfrak{g}$ has a unique irreducible highest weight module $L(\lambda)$ with highest weight λ.*

Proof. Every highest weight module is a quotient of $M(\lambda)$. Since by Proposition 7 $M(\lambda)$ has a unique irreducible quotient, there is a unique irreducible highest weight module. □

Remark 1. *See Humphreys, bottom of page 109-110 for his introduction of the module $M(\lambda)$. Humphreys denotes this module $Z(\lambda)$ but the notation $M(\lambda)$ is now standard. The module $M(\lambda)$ is (nowadays) called a Verma module. The notation $L(\lambda)$ for the unique irreducible highest weight module is also standard.*

Remark 2. *The irreducible quotient $L(\lambda)$ might be finite or infinite dimensional. Recall that λ is called integral if $(\alpha^\vee|\lambda) \in \mathbb{Z}$ for all coroots $\alpha^\vee$, and dominant if $(\alpha^\vee|\lambda) \geqslant 0$. If λ is a dominant integral weight, then $L(\lambda)$ is finite-dimensional. On the other hand if λ is not integral, $L(\lambda)$ will be infinite dimensional, and unless $(\alpha^\vee|\lambda) \in \mathbb{Z}$ for some coroot $\alpha^\vee$, we will actually have $M(\lambda)$ irreducible, and $L(\lambda) = M(\lambda)$.*

Theorem 10. *Let V be a finite-dimensional irreducible module. Then $V \cong L(\lambda)$ where λ is a dominant integral weight. Conversely, if λ is a dominant integral weight, the $L(\lambda)$ is finite-dimensional.*

Proof. Assume that V is finite-dimensional. Choose a vector $v \in V_\lambda$ where λ is a weight of V that is maximal with respect to $\succcurlyeq$. If $\alpha \in \Phi^+$ then $x_\alpha v \in V_{\lambda+\alpha}$ so $x_\alpha v = 0$. Therefore v is a highest weight vector. Then $V = U(\mathfrak{g})v$ since V is irreducible. We have proved that V is a highest weight module; it is irreducible so $V \cong L(\lambda)$.

To show that λ is a dominant integral weight, let α be a simple positive root. The restriction of V to the $\mathfrak{sl}_2$ spanned by $x_\alpha, x_{-\alpha}$ and h_α is finite-dimensional, and $x_\alpha v = 0$. From the classification of finite-dimensional $\mathfrak{sl}_2$ modules, this means that $(\alpha^\vee|\lambda) = \lambda(h_\alpha) \in \mathbb{Z}$ is a nonnegative integer. Therefore λ is dominant and integral.

We will omit the slightly tedous proof of the converse, that if λ is a dominant integral weight then $L(\lambda)$ is finite-dimensional. For a proof of this see Humphreys, Theorem 21.2 or Kac, Lemma 10.1. □

Corollary 11 (Weyl)**.** *For V an irreducible finite-dimensional $\mathfrak{g}$-module, the highest weight λ is a dominant integral weight, and $V \longleftrightarrow \lambda$ is a bijection between the irreducible highest weight modules and the dominant integral weights.*

3 The Casimir element

As in Section 22.1 of Humphreys, the Casimir element of the universal enveloping algebra $U(\mathfrak{g})$ may be defined as follows. Let $\{\gamma_i\}$ be a basis of $\mathfrak{g}$ and $\{\gamma^i\}$ the dual basis with respect to the Killing form, so $\kappa(\gamma_i, \gamma^i) = \delta_{ij}$. Then

$$c_{\mathfrak{g}} = \sum_{i=1}^{\dim(\mathfrak{g})} \gamma_i \gamma^i$$

Proposition 12. *$c_{\mathfrak{g}}$ is is independent of the choice of basis $\{\gamma_i\}$. It lies in the center of $U(\mathfrak{g})$.*

Proof. See Exercise 2 in Section 22. □

Proposition 13. *Let h_i be a basis of $\mathfrak{h}$ and let h^i be the dual basis of $\mathfrak{h}$ with respect to the Killing form, so $\kappa(h_i, h^j) = \delta_{ij}$. Then if $\lambda, \mu \in \mathfrak{h}^*$ we have*

$$(\lambda|\mu) = \sum_i \lambda(h^i)\mu(h_i).$$

Proof. First let us show that

$$t_\mu = \sum_i \mu(h_i) h^i. \tag{7}$$

To check this, we pair both sides with h_j. We have

$$\kappa(t_\mu, h_j) = \mu(h_j) = \kappa\left(\sum_i \mu(h_i) h^i, h_j\right).$$

Since the h_j span $\mathfrak{h}$ and κ restricted to $\mathfrak{h}$ is nondegenerate, this proves (7).

Now (7) implies

$$(\lambda|\mu) = \kappa(t_\lambda, t_\mu) = \sum_i \mu(h_i)\kappa(t_\lambda, h^i) = \sum_i \mu(h_i)\lambda(h^i).$$

□

Proposition 14. *Let V be a highest weight module with highest weight λ. Then the Casimir element $c_\mathfrak{g}$ acts by the scalar*

$$|\lambda + \rho|^2 - |\rho|^2$$

on V.

Proof. Since $c_\mathfrak{g}$ is central in $U(\mathfrak{g})$ it commutes with the action of $\mathfrak{g}$ on any module V. Because V is generated by a highest weight vector $v_\lambda \in V_\lambda$ it is sufficient to show that

$$c_\mathfrak{g} v = (|\lambda + \rho|^2 - |\rho|^2)v. \tag{8}$$

We need to choose dual bases of $\mathfrak{g}$ with respect to the Killing form. For one basis, we choose a basis h_i of $\mathfrak{h} = \mathfrak{g}_0$, and vectors $x_\alpha \in \mathfrak{g}_\alpha$.

Now we describe the dual basis. We know that the Killing form is nondegenerate on $\mathfrak{h}$, so we find h^i such that $\kappa(h_i, h^j) = \delta_{ij}$. Then we define another set of representatives $y_\alpha \in \mathfrak{g}_\alpha$ so that

$$y_\alpha = \frac{x_\alpha}{\kappa(x_\alpha, x_{-\alpha})}$$

so that

$$\kappa(x_\alpha, y_{-\beta}) = \delta_{\alpha\beta}. \tag{9}$$

Thus we have dual bases $\{h_i, x_\alpha\}$ and $\{h^i, y_{-\alpha}\}$. Then

$$c_\mathfrak{g} = \sum_i h_i h^i + \sum_{\alpha\in\Phi} x_\alpha y_{-\alpha}.$$

We want to rewrite this slightly. We write this as

$$c_\mathfrak{g} = \sum_i h_i h^i + \sum_{\alpha\in\Phi^+} x_\alpha y_{-\alpha} + \sum_{\alpha\in\Phi^+} x_{-\alpha} y_\alpha.$$

By using (9) we have $[x_\alpha, y_{-\alpha}] = t_\alpha$, so in the enveloping algebra $x_\alpha y_{-\alpha} = t_\alpha + y_{-\alpha} x_\alpha$. Thus

$$c_\mathfrak{g} = \sum_i h_i h^i + \sum_{\alpha\in\Phi^+} t_\alpha + \sum_{\alpha\in\Phi^+} (y_{-\alpha} x_\alpha + x_{-\alpha} y_\alpha).$$

Since v_λ is a highest weight vector it is annihilated by x_α and y_α when $\alpha \in \Phi^+$. On the other hand $Hv_\lambda = \lambda(H)v_\lambda$ for $H \in \mathfrak{h}$, and so

$$c_\mathfrak{g} v_\lambda = \sum_i \lambda(h_i)\lambda(h^i) + \sum_{\alpha\in\Phi^+} \lambda(t_\alpha).$$

The first expression equals $(\lambda|\lambda)$ by Proposition 13, while

$$\sum_{\alpha\in\Phi^+}\lambda(t_\alpha)=\sum_{\alpha\in\Phi^+}(\lambda|\alpha)=2(\lambda|\rho).$$

Thus

$$c_{\mathfrak{g}}v_\lambda=((\lambda|\lambda)+2(\lambda|\rho))v_\lambda=((\lambda+\rho|\lambda+\rho)-(\rho|\rho))v_\lambda,$$

proving (8). □

4 Category $\mathcal{O}$ and the Weyl Character Formula

We will now prove the Weyl character formula following Kac. It will be useful to work in the following category of representations, Category $\mathcal{O}$, introduced by Bernstein, Gelfand and Gelfand.

Definition 1. *A module is in Category $\mathcal{O}$ if it is $\mathfrak{h}$-diagonalizable with finite dimensional weight spaces V_λ, such that there exists a finite set of weights $\{\lambda_1,\cdots,\lambda_N\}$ such that $V_\mu=0$ unless $\mu\preccurlyeq\lambda_i$ for some i.*

By Proposition 6, this category contains all highest weight modules, is closed under finite direct sums, and it contains all submodules and quotient modules of a Category $\mathcal{O}$ module. In particular it is an abelian category with enough projectives and injectives and a good homological theory. Humphreys wrote another book about it: Representations of Semisimple Lie Algebras in the BGG Category $\mathcal{O}$. We recommend this book, and also Chapters 9 and 10 of Kac, *Infinite-dimensional Lie algebras*.

The Verma modules $M(\lambda)$ may or may not be irreducible. We will say a module V is a *subquotient* of a module W if there are submodules $U\supset Q$ of W such that $U/Q\cong V$. Thus either a submodule or a quotient module is a subquotient.

Proposition 15. *Suppose that V is a highest weight module with weight μ and V is a subquotient of $M(\lambda)$ then*

$$|\lambda+\rho|^2=|\mu+\rho|^2.$$

Proof. Since c commutes with the action of $\mathfrak{g}$ it must act as a scalar on $M(\lambda)$, and by Proposition 14 that scalar is $|\lambda+\rho|^2-|\rho|^2$. So it acts by the same scalar on any submodule, quotient module or subquotient. Also by Proposition 14 c acts by the scalar $|\mu+\rho|^2-|\rho|^2$ on any highest weight module V with highest weight λ, so $|\lambda+\rho|^2-|\rho|^2=|\mu+\rho|^2-|\rho|^2$. □

Now let V be a module in Category $\mathcal{O}$. We define the *character* of V to be the formal expression

$$\chi_V=\sum_\lambda \dim(V_\lambda)e^\lambda$$

where e^λ is a formal symbol for $\lambda\in\mathfrak{h}^*$.

Proposition 16. *The character of $M(\lambda)$ is*

$$e^\lambda\prod_{\alpha\in\Phi^+}(1-e^{-\alpha})^{-1}.$$

Proof. Let v be the highest weight vector. We recall from Theorem 8 that the map $\xi \longmapsto \xi \cdot v$ from $U(\mathfrak{n}^-)$ to $M(\lambda)$ is a vector space isomorphism. So by the PBW theorem a basis of $M(\lambda)$ consists of the vectors

$$\left(\prod_{\alpha\in\Phi^+} x_{-\alpha}^{k_\alpha}\right) v, \qquad k_\alpha \geqslant 0,$$

where the positive roots Φ^+ are taken in some fixed definite order. The weight of this vector is $\lambda - \sum_{\alpha\in\Phi^+} k_\alpha\alpha$, so

$$\chi_V = e^\lambda \prod_{\alpha\in\Phi^+} e^{-k_\alpha\alpha} = e^\lambda \prod_{\alpha\in\Phi^+} (1-e^{-\alpha})^{-1}.$$

□

Let V be a module in Category $\mathcal{O}$. A nonzero vector $v \in V$ is called *primitive* if there exists a proper submodule $U \subset V$ such that $v \notin U$ but $x_\alpha v \in U$ for all $\alpha \in \Phi^+$ (or equivalently, for all simple roots). We can take $U = 0$, so if $x_\alpha v = 0$ then v is primitive. In other words, a highest weight vector is a primitive vector. More generally, v being primitive means that the image of v in V/U is a highest weight vector for some proper submodule U of V. We will call μ a *primitive weight* if V_μ contains a primitive vector.

Proposition 17. *Let V be a module in Category $\mathcal{O}$. Then V is generated by its primitive vectors.*

Proof. If not, consider the submodule U generated by the primitive vectors. Then $Q = V/U$ would be a nonzero submodule. If we choose a nonzero vector in Q whose weight is maximal with respect to $\preccurlyeq$, then its preimage in V would be a primitive vector, which is a contradiction. □

Proposition 18. *Let V be a module in Category $\mathcal{O}$. Assume that V has only a finite number weights. Then V has finite length. That is, it has a composition series*

$$V = V_m \supset V_{m-1} \supset \cdots \supset V_0 = 0$$

such that each quotient V_i/V_{i-1} is irreducible, isomorphic to $V(\mu)$, where μ is a primitive weight of V. (The quotients V_i/V_{i-1} are called composition factors, and they are independent of the composition series, by the Jordan-Hölder theorem.)

Proof. We argue by induction on the number of linearly independent primitive vectors.

Choose a primitive weight μ that is maximal with respect to $\succcurlyeq$. Then clearly a primitive vector v of weight μ must be a highest weight vector, so $W = U(\mathfrak{g})v = U(\mathfrak{n}^-)v$ is a highest weight module. By Proposition 7 it has a maximal submodule W' and the quotient $Q = W/W'$ is irreducible. Both V/W and W' have fewer independent primitive vectors than V, so by induction they have finite length. Since V/W, W' and the irreducible quotient W/W' all have finite length, it follows that V has finite length. □

Proposition 19. *Let $\lambda \in \mathfrak{h}^*$. Then the character $\chi_{L(\lambda)}$ is of the form*

$$\chi_{L(\lambda)} = \sum_{\substack{\mu \preccurlyeq \lambda \\ |\mu+\rho|^2 = |\lambda+\rho|^2}} c_\mu \chi_{M(\mu)}, \tag{10}$$

where $c_\lambda = 1$.

Proof. The weight μ of a primitive vector must satisfy $\mu \preccurlyeq \lambda$ and $|\mu+\rho|^2 = |\lambda+\rho|^2$ by Proposition 15. Since the inner product is positive definite, this implies that there are only a finite number of possible weights for primitive vectors, and so $M(\mu)$ has finite length. Also by Proposition 15 the composition factors of $M(\mu)$ must be $L(\nu)$ where $|\nu+\rho|^2 = |\mu+\rho|^2 = |\lambda+\rho|^2$. Let $d(\mu,\nu)$ be the multiplicity of such $L(\nu)$. Then

$$\chi_{M(\mu)} = \sum_{\substack{\nu \preccurlyeq \mu \\ |\nu+\rho|^2 = |\lambda+\rho|^2}} d(\mu,\nu) \chi_{L(\nu)}.$$

Now the matrix $d(\mu,\nu)$ indexed by pairs μ,ν is triangular since $d(\mu,\mu) = 1$ and $d(\mu,\nu) = 0$ unless $\nu \preccurlyeq \mu$. So it is invertible and we may write

$$\chi_{L(\mu)} = \sum_{\substack{\nu \preccurlyeq \mu \\ |\nu+\rho|^2 = |\lambda+\rho|^2}} d'(\mu,\nu) \chi_{M(\nu)}.$$

Applying this to $\mu = \lambda$ gives (10). □

Since the inner product is positive definite, (10) is a sum over only finitely many terms.

Remark 3. *As an alternative way of understanding (10), we can try to find a resolution of $L(\lambda)$ by Verma modules in Category $\mathcal{O}$. BGG showed that this can be done, and a resolution by Verma modules is called a BGG resolution. Then (10) and by extension the Weyl character formula are shadows of the BGG resolution.*

We will define the *Weyl denominator*

$$\Delta = e^\rho \prod_{\alpha \in \Phi^+} (1 - e^{-\alpha}).$$

Lemma 20. *Let $w \in W$ (the Weyl group). Then*

$$w(\Delta) = \operatorname{sgn}(w)\Delta.$$

Proof. It is sufficient to check this if $w = s_{\alpha_i}$ is a simple reflection. We recall that s_α maps the simple root α_i to $-\alpha_i$ and it permutes the remaining positive roots. Moreover $s_{\alpha_i}(\rho) = \rho - \alpha_i$. So if we write

$$\Delta = e^\rho (1 - e^{-\alpha_i}) \prod_{\substack{\alpha \in \Phi^+ \\ \alpha \neq \alpha_i}} (1 - e^{-\alpha})$$

then s_i maps $e^\rho(1 - e^{-\alpha_i})$ to $e^\rho e^{-\alpha_i}(1 - e^{\alpha_i}) = -e^\rho(1 - e^{-\alpha_i})$ and fixes the product. Hence $s_i(\Delta) = -\Delta$. □

Before we prove the Weyl character formula (postponing the proof of Proposition 19 to the next section) it will be useful to introduce the *dot action* of the Weyl group on $\mathfrak{h}^*$. This is just the usual action with the fixed point moved from the origin to $-\rho$. Thus

$$w \circ \lambda = w(\lambda + \rho) - \rho.$$

It is easy to check that $w_1 \circ (w_2 \circ \lambda) = (w_1 w_2) \circ \lambda$.

Theorem 21 (Weyl Character Formula)**.** *Let V be a finite-dimensional irreducible representation of $\mathfrak{g}$. Thus by Proposition 10 there is a dominant integral weight λ such that $V = L(\lambda)$. We have*

$$\chi_V = \Delta^{-1} \sum_{w \in W} \operatorname{sgn}(w) e^{w(\lambda+\rho)} \tag{11}$$

where W is the Weyl group and

$$\Delta = e^{\rho} \prod_{\alpha \in \Phi^+} (1 - e^{-\alpha}).$$

The following argument is due to Kac, improving the proof of BGG. As an application, Kac extended the applicability of the Weyl character formula for characters of integrable representations of infinite-dimensional Kac-Moody Lie algebras. See Chapter 10 of his book for this.

Proof. Using Proposition 16 we may rewrite (10) in the form

$$\chi_{L(\lambda)} = \sum_{\substack{\mu \preccurlyeq \lambda \\ |\mu+\rho|^2 = |\lambda+\rho|^2}} c_\mu e^{\mu+\rho} \Delta^{-1}.$$

It may be simpler to write this as

$$\chi_{L(\lambda)} = \sum_{\mu \in \mathfrak{h}^*} c_\mu e^{\mu+\rho} \Delta^{-1}$$

and remember that $c_\mu = 0$ unless $\mu \preccurlyeq \lambda$ and $|\mu + \rho|^2 = |\lambda + \rho|^2$. We claim that if $w \in W$, then

$$c_\mu = \operatorname{sgn}(w) c_{w \circ \mu}. \tag{12}$$

Indeed, since $\chi_{L(\lambda)}$ is invariant under the action of W, and since $w(\Delta) = \operatorname{sgn}(w)\Delta$, we have an identity

$$\sum_{\mu \in \mathfrak{h}^*} c_\mu e^{\mu+\rho} \Delta^{-1} = \sum_{\mu \in \mathfrak{h}^*} \operatorname{sgn}(w) c_\mu e^{w(\mu+\rho)} \Delta^{-1}$$

and comparing the coefficients of $e^{w \circ \mu} = e^{w(\mu+\rho)-\rho}$ on both sides of this equation gives (12).

We know that $c_\lambda = 1$, since this is part of Proposition 19. So by (12), we will have terms corresponding to μ of the form $w \circ \lambda$ and the sum of these terms is

$$\Delta^{-1} \sum_{w \in W} c_{w(\lambda+\rho)-\rho} e^{w(\lambda+\rho)} = \Delta^{-1} \sum_{w \in W} \operatorname{sgn}(w) e^{w(\lambda+\rho)}.$$

This is the right hand side of (11), so our task is to show that these are the *only terms.* That is, we must show that $c_\mu = 0$ unless μ is of the form $w \circ \lambda$ for some $w \in W$.

Therefore we start with μ such that $c_\mu \neq 0$. By Proposition 1, there exists $w \in W$ such that $w(\mu + \rho)$ is dominant. Let $\nu = w \circ \mu = w(\mu + \rho) - \rho$. We will show that $\nu = \lambda$. In any case by (12), $c_\nu \neq 0$ and so $\nu \preccurlyeq \lambda$ and $|\lambda + \rho|^2 = |\nu + \rho|^2$. We write

$$\lambda - \nu = \sum_{\alpha \in \Phi^+} k_\alpha \alpha,$$

where since $\nu \preccurlyeq \lambda$ we have $k_\alpha \geqslant 0$. We note the identity, for $a, b \in \mathfrak{h}^*$:

$$|a|^2 - |b|^2 = (a + b|a - b).$$

We apply this and learn that

$$|\lambda + \rho|^2 - |\nu + \rho|^2 = \left(\lambda + \nu + 2\rho \middle| \sum_{\alpha \in \Phi^+} k_\alpha \alpha \right).$$

Now λ and $\nu + \rho = w(\mu + \rho)$ are both dominant, so $\lambda + \nu + 2\rho$ is *strongly dominant* meaning

$$(\alpha^\vee | \lambda + \nu + 2\rho) > 0$$

for all positive roots α. Therefore $|\lambda + \rho|^2 = |\nu + \rho|^2$ implies that $k_\alpha = 0$ for all α and so $\nu = \lambda$.

□