Homework 7 Solutions

- Section 12 (p.67) # 4
- Section 13 (p.72) # 7 (explain why relevant)
- Section 22 (p.126) # 7
- Section 24 (p.141) # 1

Section 12 #4. Prove that the long roots in G_2 form a root system in E of type A_2.

Remark: One could also say that the *short* roots of G_2 form a root system of type A_2. But there is a difference: If α, β are long roots in $\Phi = \Phi_{G_2}$ and $\alpha + \beta \in \Phi$, then $\alpha + \beta$ is a long root. This implies that the x_α (α long) generate a Lie subalgebra of the Lie algebra $\mathtt{G}_2$. This would fail for short roots.

Solution. Humphreys describes $\mathtt{G}_2$ as a subalgebra of $\mathfrak{so}(7)$. Referring to the notation on pages 103-104, he names the 12 root vectors, and we repeat the description. Here if $1 \leqslant i, j \leqslant 7$ the matrix e_{ij} is the elementary matrix with a 1 in the i, j position, 0's elsewhere. The root vectors corresponding to the long roots are:

$$g_{1,-2} = e_{23} - e_{65}, \qquad g_{2,-1} = e_{32} - e_{56},$$
$$g_{1,-3} = e_{24} - e_{75}, \qquad g_{3,-1} = e_{42} - e_{75},$$
$$g_{2,-3} = e_{34} - e_{76}, \qquad g_{3,-2} = e_{43} - e_{76}.$$

He also gives the short roots but we don't need them for the solution to this problem. The Cartan subalgebra of B_3 is spanned by d_1, d_2, d_3 with $d_i = e_{i+1,i+1} - e_{i+4,i+4}$ and the Cartan subalgebra of G_2 is the codimension 1 subspace $H = \{a_1 d_1 + a_2 d_2 + a_3 d_3 | \sum a_i = 0\}$. In other words:

$$h := a_1 d_1 + a_2 d_2 + a_3 d_3 = \begin{pmatrix} 0 &&&&&& \\ & a_1 &&&&& \\ && a_2 &&&& \\ &&& a_3 &&& \\ &&&& -a_1 && \\ &&&&& -a_2 & \\ &&&&&& -a_3 \end{pmatrix}.$$

Now we compute easily with h as above

$$\mathrm{ad}(h) g_{1,-2} = (a_1 - a_2) g_{1,-2}, \qquad \mathrm{ad}(h) g_{2,-1} = (a_2 - a_1) g_{1,-2}$$
$$\mathrm{ad}(h) g_{1,-3} = (a_1 - a_3) g_{1,-2}, \qquad \mathrm{ad}(h) g_{3,-1} = (a_3 - a_1) g_{3,-1}$$
$$\mathrm{ad}(h) g_{2,-3} = (a_2 - a_3) g_{2,-3} \qquad \mathrm{ad}(h) g_{1,-2} = (a_3 - a_2) g_{3,-2}.$$

We may embed $\mathfrak{sl}_3$ as a Lie subalgebra of $\mathfrak{so}(7)$ as follows:

$$g \longmapsto \begin{pmatrix} 0 && \\ & g & \\ && -{}^t g \end{pmatrix}$$

and under this embedding, the image is contained in G_2. The above calculation shows that the root subgroups spanned by $g_{1,-2}$ etc. correspond to the roots $a_i - a_j$ of $\mathfrak{sl}_2$.

Section 13 #7. If $\varepsilon_1, \cdots, \varepsilon_\ell$ is an obtuse basis of the Euclidean space E (i.e., all $(\varepsilon_i, \varepsilon_j) \leqslant 0$ for $i \neq j$), prove that the dual basis is *acute* (i.e. $(\varepsilon_i^*, \varepsilon_j^*) \geqslant 0$ for $i \neq j$). [**Hint:** reduce to the case $\ell = 2$.]

Solution: We will not use Humphrey's hint. Let us construct a new basis from $\varepsilon_1, \cdots, \varepsilon_\ell$ by Gram-Schmidt orthogonalization. The new basis $\delta_1, \cdots, \delta_\ell$ is

$$\delta_1 = \varepsilon_1$$

$$\delta_2 = \varepsilon_2 - \frac{(\varepsilon_2, \delta_1)}{(\delta_1, \delta_1)}\delta_1$$

$$\vdots$$

$$\delta_k = \varepsilon_k - \frac{(\varepsilon_k, \delta_1)}{(\delta_1, \delta_1)}\delta_1 - \ldots - \frac{(\varepsilon_k, \delta_{k-1})}{(\delta_{k-1}, \delta_{k-1})}\delta_{k-1}.$$

$$\vdots$$

The Gram-Schmidt algorithm guarantees that $(\delta_i, \delta_j) = 0$ if $i \neq j$.

Lemma 1. *(i) The vector δ_k is a linear combination of $\varepsilon_1, \cdots, \varepsilon_k$ with nonnegative coefficients;*
(ii) If $i > k$ then $(\varepsilon_i, \delta_k) \leqslant 0$.

Proof. This is by induction on k. Assuming these assertions are true for smaller values of k, then using (ii) all of the coefficients in

$$\delta_k = \varepsilon_k + \left(-\frac{(\varepsilon_k, \delta_1)}{(\delta_1, \delta_1)}\right)\delta_1 + \ldots + \left(-\frac{(\varepsilon_k, \delta_{k-1})}{(\delta_{k-1}, \delta_{k-1})}\right)\delta_{k-1}$$

are nonnegative, and all the δ_i with $i \leqslant k-1$ are linear combinations of $\varepsilon_1, \cdots, \varepsilon_{k-1}$ with nonnegative coefficients, so δ_k is also a linear combination of $\varepsilon_1, \cdots, \varepsilon_k$ with nonnegative coefficients. Thus (i) is true. Now (ii) follows from (i) since if $i > k$ and $j \leqslant k$ then $(\varepsilon_i, \varepsilon_k) \leqslant 0$. □

Let C be the convex cone consisting of

$$\sum_{i=1}^{\ell} a_i \varepsilon_i, \qquad a_i \geqslant 0.$$

Let C' be the dual cone

$$C' = \{x \in E | (x, y) \geqslant 0 \text{ for } y \in C\}.$$

Proposition 2. *If $v, w \in C'$, then $(v, w) \geqslant 0$.*

Proof. By Lemma 1, the basis $\delta_1, \cdots, \delta_\ell \in C$, and so $(v, \delta_i) \geqslant 0$ and $(w, \delta_i) \geqslant 0$. The basis δ_i is orthogonal, so we can write

$$v = \sum a_i \delta_i, \qquad a_i = \frac{(v, \delta_i)}{(\delta_i, \delta_i)} \geqslant 0,$$

and

$$w = \sum b_i \delta_i, \qquad b_i = \frac{(w, \delta_i)}{(\delta_i, \delta_i)} \geqslant 0.$$

Now

$$(v, w) = \sum a_i b_i (\delta_i, \delta_i) \geqslant 0.$$

□

Since the $\varepsilon_i^* \in C'$, this solves the problem.

Application to root systems: We can take $\varepsilon_i = \alpha_i^\vee = \frac{2\alpha_i}{(\alpha_i, \alpha_i)}$. By Lemma 10.1 on page 47 of Humphreys, $(\varepsilon_i, \varepsilon_j) \leqslant 0$ for $i \neq j$. Then the dual cone C' consists of the positive Weyl chamber. Thus the Exercise implies that if λ, μ are dominant weights, then $(\lambda, \mu) \geqslant 0$.

Section 22 #7. Let $L = \mathfrak{sl}(2, F)$, and identify $m\lambda_1$ with the integer m. Use Propositions A and B of (22.5), along with Theorem 7.2 to derive the *Clebsch-Gordan formula*: if $n \leqslant m$, tnen

$$V(m) \otimes V(n) \cong V(m+n) \oplus V(m+n-2) \oplus \cdots \oplus V(m-n).$$

(Compare Exercise 7.6).

Solution. A good way to check this is to compute the characters of both sides. Please see the posted solution to Exercise 7.6, which solves this problem too.

Section 24 #1. Give a direct proof of Weyl's character formula (24.3) for type $\mathtt{A}_1$.

Solution. As Humphreys notes on page 124,

$$\mathrm{ch}_\lambda = e(\lambda) + e(\lambda - \alpha) + \ldots + e(\lambda - m\alpha), \qquad m = (\lambda, \alpha^\vee) = \frac{2(\lambda, \alpha)}{(\alpha, \alpha)}. \tag{1}$$

This just encodes the weights of the representation and their multiplicities:

$$V = \bigoplus_{k=0}^{m} V_{\lambda - k\alpha}, \qquad \dim(V_{\lambda - k\alpha}) = 1.$$

Later he changes notation so in Theorem 23.4, ε_λ is the same as e^λ. The Weyl character formula, which we are trying to check, then says

$$\mathrm{ch}_\lambda = \frac{\sum_{w \in W} (-1)^{\ell(w)} e^{w(\lambda+\rho)}}{\sum_{w \in W} (-1)^{\ell(w)} e^{w(\rho)}}.$$

There is only one root α and $\rho = \frac{\alpha}{2}$. And the Weyl group consists of $\{1, s\}$ where $s = s_\alpha$. Since there is only one root, and E is one-dimensional, $s : x \mapsto -x$ for $x \in E$. Thus

$$\mathrm{ch}_\lambda = \frac{e^{\lambda+\rho} - e^{-\lambda-\rho}}{e^\rho - e^{-\rho}}.$$

Writing $\lambda = m\rho$, so $m = (\lambda, \alpha^\vee)$, this equals

$$\frac{e^{(m+1)\rho} - e^{-(m+1)\rho}}{e^{\rho} - e^{-\rho}} = e^{m\rho} + e^{(m-2)\rho} + \ldots + e^{-m\rho} \tag{2}$$

as an application of the geometric series identity

$$\frac{x^{m+1} - x^{-(m+1)}}{x - x^{-1}} = x^m + x^{m-2} + \ldots + x^{-m}.$$

Note that since $\lambda = m\rho$ and $\alpha = 2\rho$, the two formulas (1) and (2) are the same.