Homework 6 Solutions

- Section 6 (p.30) # 4,
- Section 10 (p.54) # 9,12,
- Section 13 (p.72) # 9.

Note: I am using some slightly different notations from Humphreys in the lectures and in the statements of the homework problems. As Humphreys explains in a note at the end of the book, notations standardized to a large extent after the book was written. Here are some differences between my notation and his.

I am using the more standard notation s_α for the "simple reflection" r_α when $\alpha \in \Delta$.

I am using the notation w_0 for the "long element" of the Weyl group, which is now very standard.

I am denoting by ρ the Weyl vector

$$\rho = \frac{1}{2} \sum_{\alpha \in \Phi^+} \alpha.$$

This is denoted δ by Humphreys, but the notation ρ is now very standard.

Section 6 (p. 30) #4. Use Weyl's theorem to give another proof that if L is semisimple, then $\mathrm{ad}(L) = \mathrm{Der}(L)$. [**Hints:** If $\delta \in \mathrm{Der}(L)$, make the direct sum $F \oplus L$ into an L-module by the rule

$$x \cdot (a, z) = (0, a\delta(x) + [x, y]).$$

Then consider a complement to the submodule L.]

Solution. Let us check that this definition makes $F \oplus L$ into a module. We need to check that if $x, y \in L$ and $\xi \in F \oplus L$ then

$$[x, y] \cdot \xi = x \cdot (y \cdot \xi) - y \cdot (x \cdot \xi). \tag{1}$$

If $\xi = (a, z)$ then

$$x \cdot (y \cdot \xi) = x \cdot (0, a\delta(y) + [y, z]) = (0, a[x, \delta(y)] + [x, [y, z]]).$$

Interchangin x and y and subtracting,

$$x \cdot (y \cdot \xi) - y \cdot (x \cdot \xi) = (0, a[x, \delta(y)] - a[y, \delta(x)] + [x, [y, z]] - [y, [x, z]]) =$$

$$(0, a([\delta(x), y] + [x, \delta(y)]) + [x, [y, z]] - [y, [x, z]]).$$

Now we use the fact that δ is a derivation and the Jacobi triple product identity to write this as

$$(0, a\delta([x, y]) + [[x, y], z]) = [x, y] \cdot \xi,$$

proving (1). Thus we have an L-module on $F \oplus L$.

By Weyl's theorem, this module is completely reducible. Note that L is a submodule on which L acts via the adjoint representation. So by Weyl's theorem, L has a complementary submodule V in $F \oplus L$. Since L has codimension 1, this submodule is one-dimensional. Let $v = (a, z)$ be a basis vector. Note that $a \neq 0$ since $V \cap L = 0$. After multiplying by a nonzero constant, we can arrange that $a = 1$.

A 1-dimensional module over a semisimple Lie algebra is trivial, that is, $X \cdot v = 0$ for $X \in L$ and $v \in V$. So

$$0 = x \cdot (1, z) = (0, [x, z] + \delta(x)).$$

Therefore $\delta(x) = [z, x]$ for all $x \in L$, and $\delta = \mathrm{ad}(z)$ is an inner derivation. This proves that $\mathrm{ad}(L) = \mathrm{Der}(L)$.

Section 10 (p. 54) #9. Prove that there is a unique element $w_0 \in W$ sending Φ^+ to Φ^-. Prove that any reduced expression for w_0 must involve all simple reflections s_α ($\alpha \in \Delta$).

Solution. We will use the following:

Lemma 1. *Let $w \in W$ and let $\Phi_w = \Phi^+ \cap w^{-1}\Phi^+$. If Φ_w is nonempty then it contains a simple root.*

Proof. Let $\alpha \in \Phi_w$. Then $w(\alpha)$ is a positive root. Let us write α in terms of the simple roots:

$$\alpha = \sum_{\beta \in \Delta} n_\beta \cdot \beta, \qquad n_\beta \in \mathbb{Z}, \quad n_\beta \geqslant 0.$$

Now $w(\alpha) = \sum n_\beta w(\beta)$. This is in Φ^+ so at least one of the $w(\beta)$ must be in Φ^+. Then this $\beta \in \Phi_w \cap \Delta$, since it is a positive (even simple) root and $w(\beta) \in \Phi^+$. This contradiction proves that Φ_w contains a simple root β. □

Lemma 2. *There is a unique $w_0 \in W$ such that $\Phi^+ \cap w_0^{-1}\Phi^+$ is empty.*

Proof. First choose w_0 to minimize $|\Phi^+ \cap w_0^{-1}\Phi^+|$. We will prove that $\Phi^+ \cap w_0^{-1}\Phi^+ = \varnothing$. If this set is not empty, by the last Lemma it contains a simple reflection α. Then since $s_\alpha\Phi^+ = \Phi^+ - \{\alpha\} \cup \{-\alpha\}$ we see that

$$s_\alpha\Phi^+ \cap w_0^{-1}\Phi^+ = (\Phi^+ \cap w_0^{-1}\Phi^+) - \{\alpha\}.$$

This set is $s_\alpha(\Phi^+ \cap (w_0 s_\alpha)^{-1}\Phi^+)$ and its cardinality is the same as $\Phi^+ \cap (w_0 s_\alpha)^{-1}\Phi^+$ but less than that of $\Phi^+ \cap w_0^{-1}\Phi^+$, contradicting the minimality of w_0. This prove that $\Phi^+ \cap w_0^{-1}\Phi^+ = \varnothing$. □

We may now finish the solution. Since $w_0^{-1}\Phi^+$ is disjoint from Φ^+ we see that w_0 takes Φ^+ to Φ^-, as required. We must also show that w_0 is the unique element of W with this property. We must also show that w_0 is unique. If w is another element of W that takes Φ^+ to Φ^-, then $w^{-1}w_0$ fixes Φ^+, hence fixes Δ. It follows from Humphreys Theorem 10.3 (e) that $w = w_0$.

We are also asked to show that any reduced expression for w_0 involves every s_α ($\alpha \in \Delta$). If β is any root, we may write

$$\beta = \sum_{\alpha \in \Delta} n_\alpha(\beta)\alpha, \qquad n_\alpha(\beta) \in \mathbb{Z}.$$

Let us fix $\alpha \in \Delta$. The content of the assertion is that the group P_α generated by s_γ, $\gamma \in \Delta$, $\gamma \neq \alpha$ does not contain w_0. From the formula

$$s_\gamma(x) = x - \frac{2(\gamma, x)}{(\gamma, \gamma)}\gamma$$

it is clear that if $\alpha, \gamma \in \Delta$ and $\gamma \neq \alpha$ then $n_\alpha(s_\gamma(\beta)) = n_\alpha(\beta)$. Therefore $n_\alpha(w(\alpha)) = n_\alpha(\alpha) = 1$ for all $w \in P_\alpha$. Thus if w is a product of simple reflections s_γ with $\gamma \neq \alpha$ but no $\gamma = \alpha$ then $n_\alpha(w\alpha) = 1$. In particular $w(\alpha)$ can never be a negative root, so $w \neq w_0$.

Section 10 (p. 54) #12. Let $\lambda \in \mathfrak{C}(\Delta)$. If $w\lambda = \lambda$ for some $w \in W$, then $w = 1$.

Solution. For use in the following problem, we will prove a little more.

Lemma 3. *Let $\lambda \in \mathfrak{C}(\Delta)$. Suppose that $w\lambda \in \mathfrak{C}(\Delta)$. Then $w = 1$.*

Proof. The open positive Weyl chamber $\mathfrak{C}(\Delta)$ is by definition the set of λ in the Euclidean space E such that $(\lambda, \alpha) > 0$ for all $\alpha \in \Phi^+$. If also $w \neq 1$, then by Humphreys Theorem 10.3 (e) the set $w^{-1}\Delta$ contains a negative root. This means that there exists $\alpha \in \Delta$ such that $w^{-1}\alpha \in \Phi^-$. So $-w^{-1}\alpha \in \Phi^+$. Now $(\lambda, -w^{-1}\alpha) > 0$ since $\lambda \in \mathfrak{C}(\Delta)$. This equals $(w\lambda, -\alpha)$ and so $(w\lambda, \alpha) < 0$. Thus $w\lambda \notin \mathfrak{C}(\Delta)$. □

This clearly solves this problem.

Section 13 (p. 72) #9. Let $\lambda \in \Lambda^+$. Prove that for $w \in W$ the weight $w(\lambda + \rho) - \rho$ is dominant only if $w = 1$.

Solution. We will say that a λ is *strongly dominant* if $\lambda \in \mathfrak{C}(\Delta)$, that is, $(\lambda, \alpha) > 0$ for $\alpha \in \Delta$. To be *dominant* we only require that $(\lambda, \alpha) \geqslant 0$ for $\alpha \in \Delta$, i.e. that $\lambda \in \overline{\mathfrak{C}(\Delta)}$. The root λ is dominant, but ρ is strictly dominant, so $\lambda + \rho$ is strictly dominant. Now suppose that $w(\lambda+\rho)-\rho$ is dominant. Then $w(\lambda+\rho)$ is strictly dominant. Now by Lemma 3, $w = 1$.